Isometry groups of separable reflexive spaces and the existence of maximal norms, joint with V. Ferenczi

Christian Rosendal,
University of Illinois at Chicago

39th Winter School in Abstract Analysis 2011

First lecture

Bounded subgroups of $GL(X)$

To fix our attention, we will assume that all spaces considered are *separable*, real or complex Banach spaces.

We let $GL(X)$ denote the group of all automorphisms of X, i.e., by the open mapping theorem, the group of all continuous bijective linear operators $T\colon X \to X$.

The group $GL(X)$ comes equipped with several natural topologies, each of independent interest.

The norm topology is that induced from the norm

$$\|T\| = \sup_{x \in B_X} \|Tx\|.$$

The strong operator topology is given by pointwise convergence on X, i.e.,

$$T_i \to T \quad \Leftrightarrow \quad \|T_i x - Tx\| \to 0 \quad \text{for all } x \in X,$$

while the weak operator topology is given by weak convergence, i.e.,

$$T_i \to T \quad \Leftrightarrow \quad \phi(T_i x) \to \phi(Tx) \quad \text{for all } x \in X \text{ and } \phi \in X^*.$$

Note that if $\|\!| \cdot |\!\|$ is an *equivalent* norm on X, i.e., such that

$$\mathrm{Id}\colon (X, \|\cdot\|) \to (X, \|\!|\cdot|\!\|)$$

is an isomorphism, then $GL(X, \|\cdot\|) = GL(X, \|\!|\cdot|\!\|)$ and the three topologies remain unaltered, although the norm of course changes.

Suppose $G \leqslant GL(X)$ is a weakly bounded subgroup, i.e., such that for any $x \in X$ and $\phi \in X^*$,

$$\sup_{T \in G} |\phi(Tx)| < \infty.$$

Then, by the uniform boundedness principle, G is actually norm bounded, that is,

$$\sup_{T \in G} \|T\| < \infty.$$

So we can simply talk about bounded subgroups of $GL(X)$.

Note that if G is bounded, then

$$|||x||| = \sup_{T \in G} \|Tx\|$$

is an equivalent norm on X such that G acts by *isometries* on $(X, |||\cdot|||)$.

So bounded subgroups of $GL(X)$ are simply groups of isometries for equivalent norms.

Now, it follows from a theorem of Gelfand that $GL(X)$ has no small subgroups in the norm topology, that is, there is a norm neighbourhood $\mathcal{U}$ of Id not containing any non-trivial subgroup.

So if $G \leqslant GL(X)$ is locally compact second countable in the norm topology, then by Gleason, Montgomery, Yamabe and Zippin's solution to Hilbert's 5th problem, G is a Lie group.

Maximal norms

The *Mazur rotation problem* asks if every separable Banach space, whose isometry group acts transitively on the unit sphere, must be a Hilbert space.

In connection with this, A. Pełczyński and S. Rolewicz (1962) introduced the notion of a maximal norm on a Banach space.

Here a norm $\|\cdot\|$ on X is maximal if

$$\mathrm{Isom}(X, \|\cdot\|)$$

is a maximal bounded subgroup of $GL(X)$.

Equivalently, if $|||\cdot|||$ is an equivalent norm on X such that

$$\mathrm{Isom}(X, \|\cdot\|) \leqslant \mathrm{Isom}(X, |||\cdot|||),$$

then actually

$$\mathrm{Isom}(X, \|\cdot\|) = \mathrm{Isom}(X, |||\cdot|||).$$

Thus, if we think of $\mathrm{Isom}(X, \|\cdot\|)$ as the set of symmetries of X, then the norm is maximal if it attains a maximal degree of symmetry on X.

Second lecture

Maximal norms

Recall that a norm $\|\cdot\|$ on X is maximal if

$$\mathrm{Isom}(X, \|\cdot\|)$$

is a maximal bounded subgroup of $GL(X)$.

For example, the standard norms on

- ℓ_p (Rolewicz),
- $L_p([0,1])$ (Rolewicz),
- $C(K,\mathbb{C})$ for K a compact manifold (Kalton, Wood)

are all maximal, but the uniform norm on $C([0,1],\mathbb{R})$ is not (Partington).

Wood's problem

Given a space X, one is tempted to *smooth* the norm to weed out irregularities and perhaps eventually arrive at a maximally symmetric norm.

For example, as shown by Auerbach '30, any finite-dimensional Banach space $(X, \|\cdot\|)$ admits an isometry-invariant inner product $\langle\cdot|\cdot\rangle$.

Thus, the induced norm

$$\|\cdot\|_{\langle\cdot|\cdot\rangle} = \sqrt{\langle\cdot|\cdot\rangle}$$

is a transitive and hence maximal norm on X such that

$$\mathrm{Isom}(X, \|\cdot\|) \leqslant \mathrm{Isom}(X, \|\cdot\|_{\langle\cdot|\cdot\rangle}).$$

This was extended by Szőkefalvi-Nagy, Day, and Dixmier '50, who noticed that if G is a bounded subgroup of $B(\mathcal{H})$, that is locally compact and amenable in the strong operator topology, then it is unitarisable.

That is, there is an equivalent inner product on $\mathcal{H}$ with respect to which G is a group of unitary operators.

To see this, suppose for simplicity that G is countable and let μ be a finitely-additive right-invariant probability measure on G.

We set

$$\langle x|y\rangle_G = \int_G \langle gx|gy\rangle d\mu(g).$$

Then $\langle\cdot|\cdot\rangle_G$ is a G-invariant equivalent inner product on $\mathcal{H}$ and hence G is contained in the maximal bounded subgroup

$$U(\mathcal{H}, \langle\cdot|\cdot\rangle_G).$$

However, the fundamental question of whether every space X admits an equivalent maximal norm has remained open. In other words:

- Does $GL(X)$ always contain a maximal bounded subgroup? (Wood 1982)

In fact, even more restrictive questions have been open so far:

- Is every bounded subgroup contained in a maximal bounded subgroup? (Wood 2006)

- Do super-reflexive spaces admit equivalent transitive norms? (Deville, Godefroy, Zizler 1993)

Isometries of hereditarily indecomposable spaces

Definition

A Banach space X is said to be hereditarily indecomposable or just HI if no two infinite-dimensional subspaces form a direct sum in X.

In other words, if $Y, Z \subseteq X$ are infinite-dimensional subspaces, then

$$d(S_Y, S_Z) = 0.$$

The first hereditarily indecomposable space was constructed by Gowers and Maurey (1992) as an example of a space not containing any infinite unconditional basic sequence.

Operators on HI spaces

In their paper, Gowers and Maurey showed that any bounded operator $T\colon X \to X$ on an HI space has the form,

$$\lambda \mathrm{Id} + S,$$

where S is strictly singular, i.e., such that there is no infinite-dimensional subspace $Y \subseteq X$ such that $S|_Y$ is an embedding.

It follows from this that any embedding $T\colon X \to X$ from X into X is actually surjective.

However, isometries are even more restrictive, namely, F. Räbiger and W. J. Ricker (1998) showed that any isometry has the form

$$\lambda \mathrm{Id} + K,$$

where K is compact.

But this result can be improved.

Theorem (V. Ferenczi, C.R.)

Let X be a complex HI space and $T\colon X \to X$ an isometry. Then

$$T = \lambda\mathrm{Id} + F,$$

where F is a finite-dimensional operator.

Moreover, X admits a T-invariant decomposition,

$$X = H_T \oplus F_T,$$

where $T|_{H_T} = \lambda\mathrm{Id}$ and $F_T = \mathrm{im}(\lambda\mathrm{Id} - T)$ is finite-dimensional.

Since T is then an isometry of the finite-dimensional space F_T, it can be diagonalised.

So we can write X as

$$\begin{aligned} X &= \ker(\lambda \mathrm{Id} - T) \oplus [x_1, \ldots, x_n] \\ &= H_T \oplus F_T, \end{aligned}$$

where $x_1, \ldots, x_n$ are eigenvectors for T.

So this fully explains the structure of a single isometry of X, but what about a group of isometries?

First, we say that an isometry $T\colon X \to X$ on a complex HI space is positive if

$$T = \mathrm{Id} + F,$$

for F finite-dimensional.

Letting $\mathrm{Isom}_+(X)$ denote the subgroup of positive isometries of X, we see that

$$\mathrm{Isom}(X) = \mathbb{T} \times \mathrm{Isom}_+(X),$$

where $\mathbb{T}$ is the circle group.

It is slightly easier to work with $\mathrm{Isom}_+(X)$ instead of $\mathrm{Isom}(X)$.

Groups of isometries

Proposition

Let $T_1, \ldots, T_m$ be a finite set of positive isometries of a complex HI space X. Then X admits a decomposition

$$X = E \oplus Y,$$

where $E \subseteq X$ is finite-dimensional and T_i-invariant, and moreover $T_i|_Y = \mathrm{Id}$ for every i.

Since the isometry group of a finite-dimensional space is compact, it follows that the subgroup

$$\langle T_1, \ldots, T_n \rangle \leqslant \mathrm{Isom}_+(X)$$

is relatively compact in the norm topology.

When X^* is separable, the finite-dimensional operators form a norm-separable subset of $\mathcal{L}(X)$. So in this case, $\mathrm{Isom}(X)$ is separable and complete in the norm topology, i.e.,

$$(\mathrm{Isom}(X), \|\cdot\|)$$

is a Polish group.

Also, whenever Z is a separable Banach space, $\mathrm{Isom}(Z)$ is a Polish group in the strong operator topology (sot).

So since the norm topology refines the strong operator topology, the identity map

$$\mathrm{id}\colon (\mathrm{Isom}(X), \|\cdot\|) \;\to\; (\mathrm{Isom}(X), \mathrm{sot})$$

is continuous.

Therefore, by the open mapping theorem for Polish groups, i.e., Pettis' Theorem, this is an isomorphism of topological groups.

Corollary

If X is a complex HI space with separable dual, then the norm and strong operator topologies coincide on $\mathrm{Isom}(X)$.

Since $\mathrm{Isom}_+(X)$ has no small subgroups in the norm topology, it also has no small subgroups in the strong operator topology.

So we can find an sot-neighbourhood of Id, say

$$\mathcal{U} = \{T \in \mathrm{Isom}_+(X) \mid \|Tx_i - x_i\| < \epsilon \text{ for } i = 1, \ldots n\}$$

not containing any non-trivial subgroup.

In particular, if $Tx_i = x_i$ for all $i = 1, \ldots, x_n$, then $T = \mathrm{Id}$.

It follows that the mapping

$$T \in \mathrm{Isom}_+(X) \mapsto (Tx_1, \ldots, Tx_n) \in X^n$$

is continuous and injective.

Now, assume moreover that $\mathrm{Isom}_+(X)$ is compact.

By a result of K. Shiga (1955), we have the following two conditions.

- Every irreducible subspace, i.e., minimal isometry-invariant, is finite-dimensional,
- the linear span of irreducible subspaces is dense in X.

So, perturbing the x_i a bit and expanding them as finite linear combinations, we can suppose that each x_i belongs to some finite-dimensional isometry-invariant subspace E_i.

Therefore,

$$T \in \mathrm{Isom}_+(X) \;\mapsto\; T|_{E_1+\ldots+E_n} \in \mathrm{Isom}(E_1 + \ldots + E_n)$$

is a continuous injective homomorphism from a compact group and therefore a topological embedding.

In particular, we can recover the Gleason, Montgomery, Yamabe and Zippin Theorem for our special case.

Theorem (V. Ferenczi, C.R.)

Suppose X is a complex HI space with separable dual such that $\mathrm{Isom}_+(X)$ is compact in the strong operator topology.
Then there is a finite-dimensional isometry-invariant subspace $E \subseteq X$ such that

$$T \in \mathrm{Isom}_+(X) \;\mapsto\; T|_E \in \mathrm{Isom}(E)$$

is a topological group embedding.

So in this case $\mathrm{Isom}(X)$ is a compact Lie group.

For deeper results, we need assumptions of reflexivity.

LUR renormings and duality mappings

While the structure result for isometries of HI spaces is obtained by spectral theory, the key instrument for groups of isometries is renorming theory. In particular, we shall use a variation of a result due to G. Lancien (1993).

Proposition

Let $(X, \|\cdot\|)$ be a separable reflexive space. Then X admits an equivalent locally uniformly rotund (LUR) norm $|\!|\!|\cdot|\!|\!|$, whose dual norm is LUR, and such that

$$\mathrm{Isom}(X, \|\cdot\|) \leqslant \mathrm{Isom}(X, |\!|\!|\cdot|\!|\!|).$$

Definition

Suppose X is a Banach space and $x \in S_X$. A support functional for x is a functional $\phi \in S_{X^*}$ such that $\phi(x) = 1$.

Support functionals always exist and moreover are unique when the norm on X is Gâteaux differentiable, e.g., when the dual is LUR.

In this case, for $x \in S_X$, we let Jx denote the unique support functional for x and extend J to all of X by positive homogeneity, i.e.,

$$J(tx) = t \cdot Jx$$

for $t \geqslant 0$.

Elementary properties of the duality map

Lemma

Suppose the norm of X is Gâteaux differentiable and that $T\colon X \to X$ is an isometry. Then

$$JT^{-1} = T^*J.$$

Lemma

Suppose X is reflexive with a Gâteaux differentiable norm. Then if $Y \subseteq X$ is a subspace such that also $J[Y] \subseteq X^*$ is a subspace, we have

$$X^* = J[Y] \oplus Y^\perp,$$

where the projection onto $J[Y]$ has norm 1.

Lemma

Suppose X is reflexive space such that both $\|\cdot\|$ and $\|\cdot\|^*$ are LUR. Then $J\colon X \to X^*$ is a homeomorphism whose inverse is the duality mapping J_* from X^* to X.

Now, assume X is a Banach space and $G \leqslant \mathrm{Isom}(X)$. We set

$$X_G = \{x \in X \mid \text{ the orbit } G \cdot x \text{ is relatively compact }\},$$

$$X_G^* = \{\phi \in X^* \mid \text{ the orbit } G \cdot \phi \text{ is relatively compact }\},$$

and note that X_G and X_G^* are subspaces of X and X^* respectively.

Combining the lemmas and the renorming of Lancien, we have

Proposition

Let X be a separable reflexive Banach space and let G be a group of isometries of X. Then $J[X_G] = X_G^*$ and so we have a G-invariant decomposition

$$X = X_G \oplus (X_G^*)^\perp.$$

Moreover, either $X_G = X$ or X_G has infinite codimension in X.

By this result, the study of the isometry group of a separable reflexive space splits into two cases:

- all orbits are relatively compact,
- no orbit is relatively compact.

The first case corresponds to the case when $\mathrm{Isom}(X)$ is compact by the following lemma.

Lemma

Suppose X is a Banach space and $G \leqslant \mathrm{Isom}(X)$ a subgroup. Then $X = X_G$ if and only if G is precompact in the strong operator topology.

Third lecture

Results from yesterday

Theorem (V. Ferenczi, C.R.)

Suppose X is a complex HI space with separable dual such that $\mathrm{Isom}_+(X)$ is compact in the strong operator topology.
Then there is a finite-dimensional isometry-invariant subspace $E \subseteq X$ such that

$$T \in \mathrm{Isom}_+(X) \;\mapsto\; T|_E \in \mathrm{Isom}(E)$$

is a topological group embedding.

So in this case

$$\mathrm{Isom}(X) \cong \mathbb{T} \times \mathrm{Isom}_+(X)$$

is a compact Lie group.

We therefore need to decide when the group of positive isometries is compact in the strong operator topology.

Lemma

Suppose X is a Banach space and $G \leqslant \mathrm{Isom}(X)$ a subgroup. Then the following are equivalent.

- *G is precompact in the strong operator topology,*
- *every G-orbit on X is relatively compact.*

Thus, letting

$$X_G = \{x \in X \mid \text{ the orbit } G \cdot x \text{ is relatively compact } \},$$

which is a closed G-invariant subspace of X, we see that G is precompact if and only if $X = X_G$.

Decomposing X according to compactness of orbits

Also, if we similarly set

$$X_G^* = \{\phi \in X^* \mid \text{ the orbit } G \cdot \phi \text{ is relatively compact } \},$$

for reflexive X, we obtain the following decomposition.

Proposition

Let X be a separable reflexive Banach space and let G be a group of isometries of X.
Then X splits into G-invariant subspaces

$$X = X_G \oplus (X_G^*)^\perp.$$

By this result, the study of the isometry group of a separable reflexive space splits into two cases:

- all orbits are relatively compact and hence the isometry group is compact,
- no orbit is relatively compact.

Let us now consider the case when no orbit is relatively compact.

No relatively compact orbit

The first tool is a variant of Neumann's Lemma in group theory.

Lemma

Suppose G is a group acting by isometries on a complete metric space M. Then the following conditions are equivalent.

1. *no orbit is relatively compact,*
2. *for every compact $C \subseteq M$, there is a $g \in G$ such that*
$$g[C] \cap C = \emptyset,$$
3. *for any $g \in G$ and compact $C \subseteq M$, there are $f_1, f_2 \in G$ such that $g = f_1 f_2$ and*
$$f_i[C] \cap C = \emptyset.$$

Now if $E \subseteq X$ is a finite-dimensional subspace, the unit sphere S_E is compact, so we can apply the lemma to this case.

Lemma

Let X be a Banach space and $G \leqslant \mathrm{Isom}(X)$ a subgroup such that no G-orbit on $X \setminus \{0\}$ is relatively compact.
Then for any finite-dimensional subspace $E \subseteq X$ there is $T \in G$ such that

$$T[E] \cap E = \{0\}.$$

Moreover, any $S \in G$ can be written as a product of two such T's.

Decompositions along a sequence of isometries

If $T\colon X \to X$ is an operator on a Banach space X, we let

$$H_T = \ker(T - \mathrm{Id})$$

and

$$F_T = \mathrm{im}(T - \mathrm{Id}).$$

So, in particular, if T is a positive isometry of a complex HI space X, then

$$X = H_T \oplus F_T$$

is a decomposition into T-invariant subspaces with $\dim F_T < \infty$.

But this result generalises to reflexive spaces as follows.

Theorem (V. Ferenczi, C.R.)

Let X be a separable, reflexive Banach space and suppose $T_1, \ldots, T_n$ are isometries of X. Then

$$X = \left(H_{T_1} \cap \ldots \cap H_{T_n}\right) \oplus \overline{F_{T_1} + \cdots + F_{T_n}}.$$

It follows that if $T_1, T_2, \ldots$ is an infinite sequence of isometries, then the spaces

$$E_0 = \bigcap_{n=1}^{\infty} H_{T_n}$$

and

$$E_n = \overline{F_{T_1} + \cdots + F_{T_n}} \cap H_1 \cap \ldots \cap H_{n-1},$$

for $n \geqslant 1$, form a Schauder decomposition of X.

Now suppose X is a separable reflexive complex HI space and G is a group of positive isometries of X.

Assume also that no G-orbit on $X \setminus \{0\}$ is relatively compact and let $T_1 \in G$ be a non-trivial isometry.

Then we can inductively choose $S_2, S_3, \ldots$ in G such that setting $T_n = S_n T_1 S_n^{-1}$, we have that $F_{T_n} = S_n[F_{T_1}]$ and so

$$\begin{aligned} F_{T_n} \cap & \big(F_{T_1} + F_{T_2} + \ldots + F_{T_{n-1}}\big) \\ &= S_n[F_{T_1}] \cap \big(F_{T_1} + S_2[F_{T_1}] + \ldots + S_{n-1}[F_{T_1}]\big) \\ &= \{0\}. \end{aligned}$$

In particular,

$$\dim\left(F_{T_1} + \ldots + F_{T_n} = n \cdot \dim F_{T_1}\right)$$

and so

$$\dim E_n = \mathrm{codim}_{F_{T_1}+\ldots+F_{T_n}}\left(F_{T_1} + \ldots + F_{T_{n-1}}\right) = \dim F_{T_1}.$$

Now, by the preceding theorem

$$X = \Big(\bigcap_n H_{T_n}\Big) \oplus E_1 \oplus E_2 \oplus E_3 \oplus \ldots$$

is a Schauder decomposition of X.

Moreover, since the E_n all have the same finite dimension, the decomposition

$$E_1 \oplus E_2 \oplus E_3 \oplus \ldots$$

can be refined to an infinite Schauder basis for the complemented subspace $E_1 \oplus E_2 \oplus E_3 \oplus \ldots$.

Since there are no non-trivial complemented subspaces in an HI space, we get the following result.

Theorem (V. Ferenczi, C.R.)

Let X be a separable reflexive complex HI space.
Assume that no point of $X \setminus \{0\}$ has a relatively compact orbit under the group of isometries of X.
Then X has a Schauder basis.

Continuity of spectral theory

For further results, we need some continuity of the spectral theory on $\mathrm{Isom}_+(X)$.

So assume X is a complex HI space and consider the map δ on $\mathrm{Isom}_+(X)$ given by

$$\delta(T) = \dim F_T.$$

Then, in the norm topology, every T is a local minimum for δ and the set of local maxima is dense in $\mathrm{Isom}_+(X)$.

Moreover, there are $\delta > 0$ and N such that for all $T, S \in \mathrm{Isom}_+(X)$,

$$\|T - S\| < \delta \;\Rightarrow\; |\delta(T) - \delta(S)| \leqslant N.$$

Using the density of the set of points of local maximality for δ, we can prove the following result.

Theorem (V. Ferenczi, C.R)

Let X be a complex HI space with separable dual.
Assume that no point of $X \setminus \{0\}$ has a relatively compact orbit under the group of isometries of X.

Then $\mathrm{Isom}_+(X)$ is a discrete locally finite group in the strong operator topology.
Therefore,

$$\mathrm{Isom}(X) \cong \mathbb{T} \times \mathrm{Isom}_+(X)$$

is an amenable Lie group.

The structure of the isometry group of a reflexive HI space

Theorem (V. Ferenczi, C.R.)

Let X be a separable reflexive complex HI space. Then one of the following holds.

- *There is a finite-dimensional isometry invariant $E \subseteq X$ such that*

  $$\mathrm{Isom}_+(X) \mapsto T|_E \in \mathrm{Isom}(E)$$

  is an embedding.
- *There is an isometry invariant decomposition $X = E \oplus Y$ with E finite-dimensional such that $\mathrm{Isom}_+(Y)$ discrete locally finite in the strong operator topology.*

Corollary

Let X be a separable reflexive complex HI space. Then $\mathrm{Isom}(X)$ is an amenable Lie group.

Solution to Wood's problem

The structure theorem is not quite enough to rule out the existence of maximal norms on reflexive complex HI spaces, but with a bit more work we get

Theorem (V. Ferenczi, C.R.)

Suppose that X is a separable, reflexive, complex HI space without a Schauder basis. Then there is an isometry-invariant decomposition

$$X = F \oplus H,$$

where $F \subseteq X$ is finite-dimensional and $T|_H = \lambda \mathrm{Id}_H$ for any isometry T.

Now, Ferenczi constructed a uniformly convex HI.

So it follows, by Szankowski's refinements of Enflo's solution to the Approximation Problem, that Ferenczi's space contains a subspace without a Schauder basis. So our theorem is not vacuous.

What is more important however is that we get a negative answer to Wood's question and its variations.

Corollary

If X is a separable, reflexive, complex HI space without a Schauder basis, then there is no maximal bounded subgroup of $GL(X)$ and hence no maximal equivalent norm on X.

Corollary

There exists a separable, super-reflexive, complex Banach space which does not admit an equivalent maximal norm.

Corollary

There exists a real, separable, reflexive Banach space which does not admit an equivalent maximal norm.

- F. Cabello-Sanchez, *Regards sur le problème des rotations de Mazur*, Extracta Math. **12** (1997), 97–116.
- R. Fleming and J. Jamison, *Isometries on Banach spaces. Vol. 2. Vector-valued function spaces*, Chapman and Hall/CRC.
- A. Pełczyński and S. Rolewicz, *Best norms with respect to isometry groups in normed linear spaces*, Short communication on International Mathematical Congress in Stockholm (1964), 104.
- S. Rolewicz, *Metric linear spaces*, second. ed., Polish Scientific Publishers, Warszawa, 1984.
- G. Wood, *Maximal symmetry in Banach spaces*, Proc. Roy. Irish Acad. **82** (1982), 177–186.