Bilipschitz and coarse embeddings into Banach spaces
Part I: Introduction

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Banach spaces

- Banach spaces ℓ_p^n, ℓ_p, $L_p(0,1)$, denoted just L_p, $L_p(\Omega,\Sigma,\mu)$, $1 \le p \le \infty$.

Metric spaces

- Important example: Graphs with graph distances.
  - Let $G = (V(G), E(G))$ be a graph, so V is a set of objects called *vertices* and E is some set of unordered pairs of vertices called *edges.* We denote an unordered pair consisting of vertices u and v by uv and say that u and v are *ends* of uv.

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  - A *walk* in G is a finite sequence of the form $W = v_0, e_1, v_1, e_2, \dots, e_k, v_k$ whose terms are alternately vertices and edges such that, for $1 \le i \le k$, the edge e_i has ends v_{i-1} and v_i. We say that W *starts* at v_0 and ends at v_k, and that W is a v_0v_k*-walk*. The number k is called the *lengths* of the walk.

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  - When we say "graph G with its graph distance" we mean the metric space $(V(G), d_G)$.

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- **Example.** Cut semimetrics. Let S be a subset of a set A, $\bar{S}$ be the complement of S. The pair $(S, \bar{S})$ is called a *cut* in A and $S, \bar{S}$ are called *parts* of this cut. The *cut semimetric* on A corresponding to the cut $(S, \bar{S})$ is defined by

$$d_S(u, v) = \begin{cases} 0 & \text{if } u \text{ and } v \text{ are in the same part} \\ 1 & \text{if } u \text{ and } v \text{ are in different parts} \end{cases}$$

Semimetrics and embeddings in combinatorial optimization

- We are going to describe the *sparsest cut problem*. In this problem we are given a connected graph $G = (V, E)$, with a positive weight (called a *capacity*) $c(e)$ associated to each edge $e \in E$, and a nonnegative weight (called a *demand*) $D(u, v)$ associated to each (unordered) pair of vertices $u, v \in V$.

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- By a *cut* of G we mean a partition of the vertex set V into two disjoint sets: S and its complement $\bar{S}$. The *sparsity* of the cut $(S, \bar{S})$ is defined as

$$\frac{\sum_{u\in S, v\in \bar{S}, uv\in E} c(uv)}{\sum_{u\in S, v\in \bar{S}} D(u, v)}, \qquad (1)$$

that is, the sparsity is the ratio between the capacities and the demands which "cross" the cut.

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- For this reason its approximate version is also of interest: to approximate the minimum sparsity.

Approximation of the sparsest cut

- One of the approaches to the approximate version of the problem comes from writing the quantity (1) in terms of a cut semimetric d_S:

$$\frac{\sum_{uv\in E} c(uv)d_S(u,v)}{\sum_{u,v\in V} D(u,v)d_S(u,v)}. \tag{2}$$

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- Obviously, the minimum decreases if instead of the minimum over cut metrics d_S we consider the minimum over all nontrivial semimetrics d on V (by a nontrivial semimetric here we mean a semimetric for which $\sum_{u,v\in V} D(u,v)d(u,v) \neq 0$).

- The point is that the problem of minimization of

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- In more detail, by homogeneity, we may restrict our attention to the metrics satisfying $\sum_{u,v\in V} D(u,v)d(u,v) = 1$. Then we can write the problem as: *Minimize the sum* $\sum_{uv\in E} c(uv)d(u,v)$ *over the set of all collections* $\{d(u,v)\}_{u,v\in V, u\ne v}$ *satisfying the conditions*

$$\begin{aligned} &\sum_{u,v\in V} D(u,v)d(u,v) = 1 \\ \forall u,v,w\in V \quad & d(u,w) \le d(u,v) + d(v,w) \qquad (4) \\ \forall u,v\in V \quad & d(u,v) = d(v,u) \\ \forall u,v\in V \quad & d(u,v) \ge 0. \end{aligned}$$

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- It turns out that the ratio between the minimum computed for this Linear Programming problem and the sparsest cut can be estimated from above in terms of the possible quality of embeddings of the semimetric space $(V, d_{\min})$ into the Banach space ℓ_1.
- This result and some other similar results led to a very strong interest of Computer Scientists to the theory of embeddings of metric spaces into Banach spaces and to a very active development of the area.

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- We start with embeddings which preserve distances.

Isometric embeddings

- A map $f : X \to Y$ between two metric spaces is called an *isometric embedding* if it preserves distances, that is $d_Y(f(u), f(v)) = d_X(u, v)$ for all $u, v \in X$.

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- If an isometric embedding of X into Y is a bijection of X and Y, we say that X and Y are *isometric*.

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- We prove only a very simple result of this theory. This result was proved in one of the first papers (if not the first paper) dealing with abstract metric spaces (Fréchet, Math. Ann., 1910).
- **Proposition** (Fréchet). Each countable metric space embeds isometrically into ℓ_∞. Each metric space with n elements embeds isometrically into ℓ_∞^n.

Proof of the Fréchet proposition

- Let $X = \{u_i\}_{i=0}^{\infty}$ be a countable metric space. We introduce a map $f : X \to \ell_\infty$ by
$$f(v) = \{d(v, u_i) - d(u_i, u_0)\}_{i=1}^{\infty}.$$
Observe that
$$||f(v) - f(w)|| = \sup_{i \in \mathbb{N}} |d(v, u_i) - d(w, u_i)|.$$

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$$\sup_{i \in \mathbb{N}} |d(v, u_i) - d(w, u_i)| \leq d(v, w).$$
- On the other hand, if $v \neq w$, then at least one of v, w is among $\{u_i\}_{i=1}^{\infty}$. Suppose that $v \in \{u_i\}_{i=1}^{\infty}$. We get
$$\sup_{i \in \mathbb{N}} |d(v, u_i) - d(w, u_i)| \geq |d(v, v) - d(w, v)| = d(v, w).$$
This proves the first statement.

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- We can make its proof simpler if we observe that for a bounded metric space $X = \{u_i\}$ (that is, for a space X for which $\sup_{u,v \in X} d(u, v)$ is finite) the definition of $f(v)$ can be simplified to $f(v) = \{d(v, u_i)\}_i$.

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- In particular, if $X = \{u_i\}_{i=1}^n$, then $f(v) = \{d(v,u_i)\}_{i=1}^n$ defines an isometric embedding into ℓ_∞^n.

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- Definition (More Graph Theory)

A complete graph with n vertices in which any two distinct vertices are joined by exactly one edge is denoted K_n. A *path* with n vertices is a graph whose vertices form a sequence $\{v_i\}_{i=1}^n$ and edges are determined by the following: v_k, $k = 2, \ldots, n-1$ is joined by exactly one edge with v_{k-1} and v_{k+1}. The vertex v_1 is joined with v_2 only and the vertex v_n is joined with v_{n-1} only. The path with n vertices is denoted P_n. If we add an edge joining v_1 and v_n we get a graph called a *cycle* of length n and denoted by C_n. A graph is called *simple* if any two vertices in it are joined by at most one edge and there are no loops. The *degree* of a vertex is the number of edges incident to it.

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- PROOF. It is easy to find isometric embeddings of K_n and P_n into ℓ_2. Let $\{e_k\}_{k=1}^{\infty}$ be the unit vector basis in ℓ_2. For K_n we map $v_k \mapsto \frac{e_k}{\sqrt{2}}$. For P_n we map $v_k \mapsto ke_1$. It is easy to see that both maps are isometric embeddings.

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- To prove the "only if" part of the statement we assume that G is a finite simple connected graph, which is not a path, but is such that $(V(G), d_G)$ is isometric to a subset of ℓ_2, denote the isometric embedding by f. Our goal is to show that these conditions imply that G is a complete graph.

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- Since vertices v_{k-1}, v_k, v_{k+1} in a cycle satisfy $d_G(v_{i-1}, v_{i+1}) = d_G(v_{i-1}, v_i) + d_G(v_i, v_{i+1})$, we get that the images of v_{k-1}, v_k, v_{k+1} should be on the same line, with the image of v_k being a midpoint.

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- Since this observation is applicable also for v_n, v_1, v_2, we get a contradiction.

- Now let $v \in V(G)$ be a vertex of degree ≥ 3, and let u_1, u_2, u_3 be its neighbors. We show that u_i are pairwise adjacent. If two pairs of them (say u_1, u_2 and u_2, u_3) are not adjacent, we get a contradiction because $f(v)$ should be simultaneously a midpoint of the line segment $f(u_1)$ and $f(u_2)$ and a midpoint of the line segment joining $f(u_2)$ and $f(u_3)$.

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- If only one edge, say u_1u_3, is missing then both $f(u_2)$ and $f(v)$ should be midpoints of the line segment joining $f(u_1)$ and $f(u_3)$.
- Therefore v and all of its neighbors should form a complete subgraph in G. Since the same should hold for each of the neighbors of v, we get that G should be a complete graph.

A wider class of embeddings

- Definition

  Let $C < \infty$. A map $f : (X, d_X) \to (Y, d_Y)$ between two metric spaces is called *C-Lipschitz* if

  $$\forall u, v \in X \quad d_Y(f(u), f(v)) \leq C d_X(u, v).$$

  A map f is called *Lipschitz* if it is C-Lipschitz for some $C < \infty$. For a Lipschitz map f we define its *Lipschitz constant* by

  $$\mathrm{Lip} f := \sup_{d_X(u,v) \neq 0} \frac{d_Y(f(u), f(v))}{d_X(u, v)}.$$

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  A map $f : X \to Y$ is called a *C-bilipschitz embedding* if there exists $r > 0$ such that

$$\forall u, v \in X \quad r d_X(u, v) \leq d_Y(f(u), f(v)) \leq rC d_X(u, v). \qquad (5)$$

  A *bilipschitz embedding* is an embedding which is C-bilipschitz for some $C < \infty$. The smallest constant C for which there exist $r > 0$ such that (5) is satisfied is called the *distortion* of f. (It is easy to see that such smallest constant exists.)

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  - It is easy to see that each bijective embedding of a finite metric space is bilipschitz (possibly with very large distortion). So for bilipschitz embeddings of finite spaces the main focus is shifted to either finding low-distortion embeddings or finding bilipschitz embeddings of families of spaces with uniformly bounded distortions.

Exercise of the day

- As we can derive from our proposition on isometric embeddings of graphs into ℓ_2, an isometric image of a graph in ℓ_2 is either a set of points on one line or a set of vertices of a simplex.

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- The point of this exercise is to show that if we allow larger distortions, we can get images of graphs which are ε-nets in balls of finite-dimensional Hilbert spaces.
- A standard argument implies that for each $\varepsilon > 0$ we can find a finite set $\{v_i\}_{i=1}^k$ in the unit ball $B_2^n = \{x \in \ell_2^n : \ ||x|| \le 1\}$ satisfying the conditions

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- A standard argument implies that for each $\varepsilon > 0$ we can find a finite set $\{v_i\}_{i=1}^k$ in the unit ball $B_2^n = \{x \in \ell_2^n : \ ||x|| \leq 1\}$ satisfying the conditions
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- **Exercise.** Show that the identical embedding of this graph into ℓ_2^n has distortion ≤ 3.

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- Thus, it is interesting to study the distortion.

Open problem of the day

- Is it possible to find an (infinite) metric space M such that a Banach space X is nonreflexive if and only if M admits a bilipschitz embeddings into X?

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  - A Banach space is nonsuperreflexive if and only if it contains a bilipschitz image of an infinite binary tree.

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- A Banach space does not have type > 1 if and only if it contains uniformly bilipschitz images of Hamming cubes of all sizes.
- A Banach space X has James tree property (whatever this means) if and only if there exists a mapping of the metric space called *infinite diamond* which has the bilipschitz property on certain set of distances (explicitly describable).

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- **Question:** Is it possible to give a general definition of a metric characterization which is (1) Short; (2) Includes all examples given above; (3) Excludes trivial characterizations?

Bilipschitz and coarse embeddings into Banach spaces
Part II: Poincaré inequalities and expanders

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January 2011, Winter School, Kácov

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- We start with a simple example: consider a 4-cycle C_4 and label its vertices in the cyclic order: v_1, v_2, v_3, v_4. We are going to show that the Euclidean distortion of C_4 can be estimated using the following inequality

$$\begin{aligned} &||f(v_1) - f(v_3)||^2 + ||f(v_2) - f(v_4)||^2 \\ &\leq ||f(v_1) - f(v_2)||^2 + ||f(v_2) - f(v_3)||^2 \\ &+ ||f(v_3) - f(v_4)||^2 + ||f(v_4) - f(v_1)||^2, \end{aligned} \tag{1}$$

which holds for an arbitrary collection $f(v_1), f(v_2), f(v_3), f(v_4)$ of elements of a Hilbert space.

- To prove (1) we use the identity $||a-b||^2 = ||a||^2 - 2\langle a, b\rangle + ||b||^2$ for each of the terms in (1). Then we move everything to the right-hand side and observe that the obtained inequality can be written in the form

$$0 \le ||f(v_1) - f(v_2) + f(v_3) - f(v_4)||^2.$$

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- We postpone the computation of $c_2(C_4)$ slightly, introducing some terminology first. Inequality (1) can be considered as one of the simplest Poincaré inequalities for embeddings of metric spaces.

Poincaré inequalities

- Definition

  Let (X, d_X) and (Y, d_Y) be a metric space, $\Psi : [0,\infty) \to [0,\infty)$ be a non-decreasing function, $a_{u,v}$, $b_{u,v}$, $u, v \in X$ be arrays of nonnegative real numbers. If for an arbitrary function $f : X \to Y$ the inequality

$$\sum_{u,v\in X} a_{u,v}\Psi(d_Y(f(u), f(v))) \geq \sum_{u,v\in X} b_{u,v}\Psi(d_Y(f(u), f(v))) \quad (2)$$

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  holds, we say that Y-valued functions on X satisfy the *Poincaré inequality* (2).
  - Observe that in this inequality the structure of X plays no role, we use X just as a set of labels for elements $f(u) \in Y$.

- The inequality (2) is useful for the theory of embeddings only if a similar inequality does not hold for the identical map on X, that is, if

$$\sum_{u,v\in X} a_{u,v}\Psi(d_X(u,v)) < \sum_{u,v\in X} b_{u,v}\Psi(d_X(u,v)). \qquad (3)$$

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- Definition

We call the quotient

$$\frac{\sum_{u,v\in X} b_{u,v}\Psi(d_X(u,v))}{\sum_{u,v\in X} a_{u,v}\Psi(d_X(u,v))}$$

the *Poincaré ratio* of the metric space X corresponding to the Poincaré inequality (2) and denote it $P_{a,b,\Psi(t)}(X)$.

- Having more information on the values of sides of (3) and on the function Ψ, we can get an estimate for the distortion $c_Y(X)$. The corresponding estimate of $c_Y(X)$ is quite simple if $\Psi(t) = t^p$ for some $p > 0$.

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- In fact, the following can be obtained by simple manipulations with the definitions:
  Proposition. If Y-valued functions on X satisfy the Poincaré inequality (2) with $\Psi(t) = t^p$, then

$$c_Y(X) \geq \left(P_{a,b,t^p}(X)\right)^{1/p}. \quad (4)$$

- Now we are ready to estimate $c_2(C_4)$. It is clear that $||f(v_1) - f(v_3)||^2 + ||f(v_2) - f(v_4)||^2 \leq ||f(v_1) - f(v_2)||^2 + ||f(v_2) - f(v_3)||^2 + ||f(v_3) - f(v_4)||^2 + ||f(v_4) - f(v_1)||^2$ is a Poincaré inequality for ℓ_2-valued functions on C_4 (more precisely: for ℓ_2-valued functions on $V(C_4)$).

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- By the Proposition from the previous slide we get $c_2(C_4)\ge\sqrt{2}$.
- This estimate is sharp, this can be shown by an embedding whose image is the set of all points in $\mathbb{R}^2$ with coordinates 0 and 1.

Expanders

- Definition
  For a graph G with vertex set V and a subset $F \subset V$ by ∂F we denote the set of edges connecting F and $V \backslash F$. The *expanding constant* (a.k.a. *Cheeger constant*) of G is

$$h(G) = \inf \left\{ \frac{|\partial F|}{\min\{|F|, |V \backslash F|\}} : \ F \subset V, \ 0 < |F| < +\infty \right\}$$

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  (where $|A|$ denotes the cardinality of a set A.)
- Definition
  A sequence $\{G_n\}$ of graphs is called a *family of expanders* if all of G_n are finite, connected, *k-regular* for some $k \in \mathbf{N}$ (this means that each vertex is incident with exactly k edges), their expanding constants $h(G_n)$ are bounded away from 0 (that is, there exists $\varepsilon > 0$ such that $h(G_n) \geq \varepsilon$ for all n), and their sizes (numbers of vertices) tend to ∞ as $n \to \infty$.

Examples of expanders

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- If we do not mind or graphs to have *parallel edges*, that is, edges with the same pairs of ends, we can get expanders using the following simple construction.
- Consider a set A of cardinality $2n$. Let $A = A_1 \cup A_2$ be a partition of A into two equal parts of cardinality n each. Let $\pi_1, \pi_2, \pi_3, \pi_4, \pi_5$ be 5 bijections of A_1 onto A_2. Consider the graph with the vertex set A and the edge set defined by the rule: each edge uv has one end vertex in A_1, say $u \in A_1$, the other vertex in A_2 ($v \in A_2$) and is such that $v = \pi_i(u)$ for some i in the set $\{1, 2, 3, 4, 5\}$

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- **Remark.** You are expected to use the Stirling formula.
- Many explicit constructions of expanders are also known, but their expanding properties are more difficult to prove.

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- We denote the *adjacency matrix* of a graph $G = (V, E)$ by $\{a_{u,v}\}_{u,v \in V}$, that is

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- Theorem

  The following Poincaré inequality holds for L_1-valued functions on V:

$$\sum_{u,v \in V} a_{u,v} ||f(u) - f(v)|| \geq \sum_{u,v \in V} \frac{h}{|V|} ||f(u) - f(v)||. \qquad (5)$$

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- The Poincaré inequality for expanders is very powerful, it and its versions for L_p-spaces have shown that expanders produce classes of metric spaces which are the most resistant to embeddings into 'reasonably good' Banach spaces.

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- The Poincaré inequality (5) can be used to get an estimate for L_1-distortion of a k-regular graph with expansion constant h. In fact, to estimate such distortion from below we need to estimate from below the corresponding Poincaré ratio:

$$\frac{\sum_{u,v\in V} \frac{h}{|V|} d_G(u,v)}{\sum_{u,v\in V} a_{u,v} d_G(u,v)}. \qquad (6)$$

- The denominator $\sum_{u,v\in V} a_{u,v} d_G(u,v)$ of the ratio is equal to $2|E|$, where $|E|$ is the number of edges in G. Since the graph is k-regular, we have $2|E| = k|V|$.

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- On the other hand, the number of vertices at distance D to a given vertex in a k-regular graph is at most

$$1 + k + k(k-1) + \cdots + k(k-1)^{D-1} \leq k^D + 1.$$

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- Let $D = \log_k\left(\frac{|V|}{2} - 1\right)$. Then there are at most $\frac{|V|}{2}$ vertices with distance $\leq D$ to a given vertex. Therefore

$$\sum_{u,v\in V} \frac{h}{|V|} d_G(u,v) \geq \frac{h}{|V|} \cdot \frac{|V|^2}{2} \cdot \log_k\left(\frac{|V|}{2} - 1\right)$$

and the Poincaré quotient (6) is

$$\geq \frac{h}{2k} \log_k\left(\frac{|V|}{2} - 1\right) \geq c \ln |V|.$$

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- We get that distortions of members of a family of expanders grow as logarithms of their sizes.

- **Remark.** It is known that the logarithmic distortion is the largest possible. Bourgain (1985) proved that there exists an absolute constant C such $c_1(X) \le c_2(X) \le C \ln n$ for each n-element set X.

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- Expanders were also used to answer an important for applications in Algebraic Topology question of Gromov.
- Gromov introduced the following class of embeddings:

Definition

A map $f : (X, d_X) \to (Y, d_Y)$ between two metric spaces is called a *coarse embedding* if there exist non-decreasing functions $\rho_1, \rho_2 : [0, \infty) \to [0, \infty)$ (observe that this condition implies that ρ_2 has finite values) such that $\lim_{t\to\infty} \rho_1(t) = \infty$ and

$$\forall u, v \in X \ \rho_1(d_X(u, v)) \leq d_Y(f(u), f(v)) \leq \rho_2(d_X(u, v)).$$

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  - It is well known that L_2 is isometric to a subspace of L_1 and it is not difficult to see how to construct a space with bounded geometry containing isometric copies of all elements of a family of expanders. Therefore this observation of Gromov answers the question above.

Proof of Gromov's observation

- In fact, suppose that there is an embedding $f : V \to L_1$ satisfying

$$\forall u, v \in V \quad \rho_1(d_G(u, v)) \leq ||f(u) - f(v)|| \leq \rho_2(d_G(u, v))$$

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- Combining this inequality with the Poincaré inequality for expanders (5) we get
$$\sum_{u,v \in V} \frac{h}{|V|}\rho_1(d_G(u,v)) \le \sum_{u,v \in V} a_{u,v}||f(u) - f(v)|| \le k|V|\rho_2(1).$$

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- Now we recall that ρ_1 is nondecreasing and that we have already proved that at least for $\frac{|V|^2}{2}$ out of $|V|^2$ terms in the left-hand side of the last inequality we have $d_G(u,v) \ge \log_k\left(\frac{|V|}{2} - 1\right)$. We get

$$\frac{|V|^2}{2} \cdot \frac{h}{|V|} \cdot \rho_1(\log_k\left(\frac{|V|}{2} - 1\right)) \le k|V|\rho_2(1)$$

- We can rewrite the last inequality (I repeat it)

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- It is clear that a function ρ_1 satisfying $\lim_{t\to\infty}\rho_1(t) = \infty$ cannot satisfy the inequality (7) for a sequence $\{|V_k|\}_{k=1}^{\infty}$ with $|V_k| \to \infty$ (if we plug each $|V_k|$ instead of $|V|$).

Open problem of the day

- We say that a Banach space X has a *nontrivial type* if there exists $\varepsilon \in (0,1)$ and $k \in \mathbb{N}$ such that for each set $\{x_i\}_{i=1}^k$ of vectors in X

$$\inf_{\omega_i=\pm 1} \left\| \sum_{i=1}^k \omega_i x_i \right\| \leq k(1-\varepsilon) \sup_i ||x_i||.$$

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- The existence of nonreflexive spaces with nontrivial type was a well-known open problem in the period 1964-1974. After that several examples were constructed. The first example is due to James (1974). An example with the simplest (in my opinion) formula for the norm is due to Pisier–Xu (1987).

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- **Problem.** Let X be a nonreflexive space with nontrivial type and M be a metric space containing isometric copies of all elements of some family of expanders. Does it always follow that M does not admit a coarse embedding into X? (The problem is open for all known examples of X.)

Proof of the Poincaré inequality for expanders

- Lemma
  Let $G = (V, E)$ be a connected graph with the expanding constant h, and $f : V \to \mathbb{R}$ be a real-valued function on V. Then

$$\sum_{v \in V} |f(v) - M| \leq \frac{1}{h} \sum_{uv \in E} |f(u) - f(v)|, \qquad (8)$$

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  - Replacing f by $\widetilde{f} = f - M$, we may assume that $M = 0$. Also we assume (for simplicity) that the number of vertices is odd. (Only a slight modification is needed in the even case.)
  - Let $f_1 \le f_2 \le \cdots \le f_k \le 0 = f_{k+1} \le f_{k+2} \le \cdots \le f_{2k+1}$ be the values of the function. Then

    $$\sum_{v\in V} |f(v)| = \sum_{i=1}^{2k+1} |f_i|.$$

- We introduce *level sets* of the function f as

$$L_i^- := \{v : \ f(v) \le f_i, \ i = 1, \dots, k\}$$

and

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- We have

$$\sum_{i=1}^{2k+1} |f_i| = \sum_{i=1}^{k} |L_i^-| f_i^\Delta + \sum_{i=k+2}^{2k+1} |L_i^+| f_i^\nabla. \qquad (9)$$

- Cardinalities of the sets L_i^- and L_i^+ do not exceed k. Observe that the definition of the expanding constant implies $|\partial F| \geq h(G)|F|$ for each F with $|F| \leq |V|/2$.

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- Hence we have

$$\sum_{i=1}^{k}|L_i^-|f_i^{\Delta} + \sum_{i=k+2}^{2k+1}|L_i^+|f_i^{\nabla}$$

$$\le \sum_{i=1}^{k}\frac{1}{h}|\partial(L_i^-)|f_i^{\Delta} + \sum_{i=k+2}^{2k+1}\frac{1}{h}|\partial(L_i^+)|f_i^{\nabla} \qquad (10)$$

$$= \frac{1}{h}\left(\sum_{i=1}^{k}|\partial(L_i^-)|f_i^{\Delta} + \sum_{i=k+2}^{2k+1}|\partial(L_i^+)|f_i^{\nabla}\right).$$

- Observe that the contribution of the edge uv to the sum

$$\left(\sum_{i=1}^{k}|\partial(L_i^-)|f_i^{\Delta}+\sum_{i=k+2}^{2k+1}|\partial(L_i^+)|f_i^{\nabla}\right)$$

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- Therefore we have

$$\begin{aligned}&\frac{1}{h}\left(\sum_{i=1}^{k}|\partial(L_i^-)|f_i^{\Delta}+\sum_{i=k+2}^{2k+1}|\partial(L_i^+)|f_i^{\nabla}\right)\\&=\frac{1}{h}\sum_{uv\in E(G)}|f(u)-f(v)|.\end{aligned}\tag{11}$$

- This proves the lemma.

Proof of the theorem

- Since continuous functions are dense in $L_1(0,1)$, it suffices to prove the inequality (5) in the case when the functions $f(u,t)$ are continuous as functions of t, and so $f(u,t)$ is well-defined for all $t \in [0,1]$. For each $t \in [0,1]$ we let $M(t)$ be a median of the set $\{f(u,t)\}_{u \in V}$. It is easy to show that the medians can be selected in such a way that $M(t)$ is a continuous function on $[0,1]$.

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- Applying Lemma 8 for each value of t, we get

$$\sum_{uv \in E} |f(u,t) - f(v,t)| \geq h \sum_{v \in V} |f(v,t) - M(t)|.$$

Integrating this inequality over $[0,1]$ we get

$$\sum_{uv \in E} ||f(u) - f(v)|| \geq h \sum_{v \in V} ||f(v) - M||. \qquad (12)$$

- By the triangle inequality we have

$$||f(u) - f(v)|| \le ||f(u) - M|| + ||f(v) - M||.$$

Therefore

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- Combining this inequality with (12) and the definition of the adjacency matrix we get (5).

Bilipschitz and coarse embeddings into Banach spaces
Part III: Obstructions for coarse embeddability of discrete metric spaces into L_2

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Definitions, Examples

- Definition

  Let $\rho_1, \rho_2 : [0,\infty) \to [0,\infty)$ be two non-decreasing functions (important: ρ_2 has finite values), and let $F : (X, d_X) \to (Y, d_Y)$ be a mapping between two metric spaces such that
  $\forall u, v \in X \;\; \rho_1(d_X(u,v)) \le d_Y(F(u), F(v)) \le \rho_2(d_X(u,v))$.
  The mapping F is called a *coarse embedding* if ρ_1 can be chosen to satisfy $\lim_{t\to\infty} \rho_1(t) = \infty$.

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    - **Example 1.** The mapping $F : \mathbb{R} \to \mathbb{N}$ given by $F(x) = \lfloor x \rfloor$ is a coarse embedding.

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  - **Example 1.** The mapping $F : \mathbb{R} \to \mathbb{N}$ given by $F(x) = \lfloor x \rfloor$ is a coarse embedding.
  - **Example 2.** The vertex set V of an infinite dyadic tree T with its graph distance can be coarsely embedded into ℓ_2 in the following way: we consider a bijection between the set of all edges of T and vectors of an orthonormal basis $\{e_i\}$ in ℓ_2, and map each vertex from V onto the sum of those vectors from $\{e_i\}$ which correspond to a path from a root O of T to the vertex, O is mapped to 0.

Applications of coarse embeddings

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- We need to recall that a discrete metric space A is said to have a *bounded geometry* if for each $r > 0$ there exist a positive integer $M(r)$ such that each ball in A of radius r contains at most $M(r)$ elements.

Exercise and Open Problem of the day

- Definition
  Let X be a space with bounded geometry and $\{Y_n\}_{n=1}^{\infty}$ be a family of expanders. We say that X *weakly contains* $\{Y_n\}$ if there are maps $f_n : Y_n \to X$ satisfying (with some abuse of notation we use Y_n to denote the vertex set of Y_n)
    - Lipschitz constants $\mathrm{Lip}(f_n)$ are uniformly bounded
    - $\displaystyle \lim_{n\to\infty} \max_{y\in Y_n} \frac{|f_n^{-1}(f_n(y))|}{|Y_n|} = 0.$

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  - **Open Problem.** Suppose that a metric space M with bounded geometry is not coarsely embeddable into ℓ_2. Does it follow that M weakly contains a family of expanders?
  - **Exercise.** Suppose that a metric space M weakly contains a family of expanders. Show that M does not embed coarsely into L_1 (and therefore into ℓ_2).

- The following more vague problem is also of interest: *Find some expander-like structures in a metric space which is not coarsely embeddable into a Hilbert space*.

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- The purpose of this lecture is to present some results on this problem.
- We are going to work with L_1 instead of L_2. Let me explain why.

Some remarks before presenting the example: L_2 vs L_1

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- We define the embedding in the following way: we map each function from $L_1(\mathbb{R})$ to the indicator function of the set between the graph of the function and the x-axis. This indicator function is considered as an element of $L_2(\mathbb{R}^2)$. One can check that this mapping has the desired properties. (The observation is due to Schoenberg, the presented proof was suggested by Naor.)

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- These results show that to prove coarse embeddability/non-embeddability results for a Hilbert space it suffices to prove similar results for L_1.

- The following result is the first attempt to find expander-like structures in spaces which do not admit coarse embeddings into ℓ_2. Recall that a metric space is called *locally finite* is all balls in it have finitely many elements.

- The following result is the first attempt to find expander-like structures in spaces which do not admit coarse embeddings into ℓ_2. Recall that a metric space is called *locally finite* is all balls in it have finitely many elements.

- Theorem (MO (2009), Tessera (2009))

  Let M be a locally finite metric space which is not coarsely embeddable into L_1. Then there exists a constant D, depending on M only, such that for each $n \in \mathbb{N}$ there exists a finite set $B_n \subset M \times M$ and a probability measure μ on B_n such that

  - $d_M(u, v) \geq n$ *for each* $(u, v) \in B_n$.
  - *For each Lipschitz function* $f : M \to L_1$ *we have*

$$\int_{B_n} ||f(u) - f(v)||_{L_1} d\mu(u, v) \leq D\mathrm{Lip}(f). \qquad (1)$$

- **Lemma**

  Let M be a locally finite metric space which is not coarsely embeddable into L_1. There exists a constant C depending on M only such that for each Lipschitz function $f : M \to L_1$ there exists a subset $B_f \subset M \times M$ such that $\sup\limits_{(x,y)\in B_f} d_M(x,y) = \infty$, *but*

  $$\sup_{(x,y)\in B_f} ||f(x) - f(y)||_{L_1} \leq C\,\mathrm{Lip}(f).$$

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  - PROOF. Assume the contrary. Then, for each $n \in \mathbb{N}$, the number n^3 cannot serve as C. This means, that for each $n \in \mathbb{N}$ there exists a Lipschitz mapping $f_n : M \to L_1$ such that for each subset $U \subset M \times M$ with

$$\sup_{(x,y)\in U} d_M(x,y) = \infty,$$

    we have

$$\sup_{(x,y)\in U} ||f_n(x) - f_n(y)|| > n^3\mathrm{Lip}(f_n).$$

- We choose a point in M and denote it by O. Without loss of generality we may assume that $f_n(O) = 0$.

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- Consider the mapping

$$f : M \to \left(\sum_{k=1}^{\infty} \oplus L_1 \right)_1 \subset L_1$$

given by

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{Kk^2} \cdot \frac{f_k(x)}{\mathrm{Lip}(f_k)},$$

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- It is clear that the series converges and $\mathrm{Lip}(f) \leq 1$.

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- The assumption implies that for each $n \in \mathbb{N}$ there is $N \in \mathbb{N}$ such that

$$d_M(x, y) \geq N \Rightarrow ||f_n(x) - f_n(y)|| > n^3 \mathrm{Lip}(f_n).$$

On the other hand

$$||f_n(x) - f_n(y)|| > n^3 \mathrm{Lip}(f_n) \Rightarrow$$
$$||f(x) - f(y)|| = \sum_{k=1}^{\infty} \frac{1}{Kk^2} \cdot \frac{||f_k(x) - f_k(y)||}{\mathrm{Lip}(f_k)} > \frac{n}{K}.$$

Hence $f : M \to L_1$ is a coarse embedding and we get a contradiction.

- Lemma

  Let C be the constant whose existence is proved in the previous Lemma and let $\varepsilon >$ be arbitrary. For each $n \in \mathbb{N}$ we can find a finite subset $M_n \subset M$ such that for each Lipschitz mapping $f : M \to L_1$ there is a pair $(u_{f,n}, v_{f,n}) \in M_n \times M_n$ such that

  - $d_M(u_{f,n}, v_{f,n}) \geq n$.
  - $||f(u_{f,n}) - f(v_{f,n})|| \leq (C + \varepsilon)\mathrm{Lip}(f)$.

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  - Proof. The ball in M of radius R centered at O will be denoted by $B(R)$. It is clear that it suffices to prove the result for 1-Lipschitz mappings satisfying $f(O) = 0$.

- Lemma

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  - PROOF. The ball in M of radius R centered at O will be denoted by $B(R)$. It is clear that it suffices to prove the result for 1-Lipschitz mappings satisfying $f(O) = 0$.
  - Assume the contrary. Since M is locally finite, this implies that for each $R \in \mathbb{N}$ there is a 1-Lipschitz mapping $f_R : M \to L_1$ such that $f_R(O) = 0$ and, for $u, v \in B(R)$, the inequality $d_M(u, v) \geq n$ implies $||f_R(u) - f_R(v)||_{L_1} > C + \varepsilon$.

- We form an ultraproduct of the mappings $\{f_R\}_{R=1}^{\infty}$, that is, a mapping $f : M \to (L_1)^{\mathcal{U}}$, given by $f(m) = \{f_R(m)\}_{R=1}^{\infty}$, where $\mathcal{U}$ is a non-trivial ultrafilter on $\mathbb{N}$ and $(L_1)^{\mathcal{U}}$ is the corresponding ultrapower.

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- It is well-known that each ultrapower of L_1 is isometric to an L_1 space on some measure space, and its separable subspaces are isometric to subspaces of $L_1(0,1)$. Therefore we can consider f as a mapping into $L_1(0,1)$. It is easy to verify that $\mathrm{Lip}(f) \le 1$ and that f satisfies the condition

$$d_M(u,v) \ge n \Rightarrow ||f(u) - f(v)||_{L_1} \ge (C + \varepsilon).$$

We get a contradiction with the definition of C.

- Proof of the Theorem. Let D be a number satisfying $D > C$, and let B be a number satisfying $C < B < D$.

- PROOF OF THE THEOREM. Let D be a number satisfying $D > C$, and let B be a number satisfying $C < B < D$.
- According to the second Lemma, there is a finite subset $M_n \subset M$ such that for each 1-Lipschitz function f on M there is a pair (u, v) in M_n such that $d_M(u, v) \geq n$ and $||f(u) - f(v)|| \leq B$.

- PROOF OF THE THEOREM. Let D be a number satisfying $D > C$, and let B be a number satisfying $C < B < D$.
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- Let α_n be the cardinality of M_n, we choose a point in M_n and denote it by O. Proving the theorem it is enough to consider 1-Lipschitz functions $f : M_n \to L_1$ satisfying $f(O) = 0$. Each α_n-element subset of L_1 is isometric to a subset in $\ell_1^{\alpha_n(\alpha_n-1)/2}$ (Witsenhausen (1986), Ball (1990)). Therefore it suffices to prove the result for 1-Lipschitz embeddings into $\ell_1^{\alpha_n(\alpha_n-1)/2}$.

- It is clear that it suffices to prove the inequality

$$\int_{B_n} ||f(u) - f(v)|| d\mu(u, v) \leq B$$

for a $\left(\frac{D-B}{2}\right)$-net in the set of all functions satisfying the conditions mentioned above, endowed with the metric

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- We are going to use the *minimax theorem*.

- Let A be the matrix whose columns are labelled by functions belonging to N, whose rows are labelled by pairs (u, v) of elements of M_n satisfying $d_M(u, v) \geq n$, and whose entry on the intersection of the column corresponding to f, and the row corresponding to (u, v) is $||f(u) - f(v)||$.

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- Then, for each column vector $x = \{x_f\}_{f \in N}$ with $x_f \geq 0$ and $\sum_{f \in N} x_f = 1$, the entries of the product Ax are the differences $||F(u) - F(v)||$, where $F : M \to \left(\sum_{f \in N} \oplus \ell_1^{\alpha_n(\alpha_n - 1)/2}\right)_1$ is given by $F(m) = \sum_{f \in N} x_f f(m)$. The function F can be considered as a function into L_1. It satisfies $\mathrm{Lip}(F) \leq 1$. Hence there is a pair (u, v) in M_n satisfying $d_M(u, v) \geq n$ and $||F(u) - F(v)|| \leq B$. Therefore we have $\max_x \min_\mu \mu A x \leq B$, where the minimum is taken over all vectors $\mu = \{\mu(u, v)\}$, indexed by $u, v \in M_n$, $d_M(u, v) \geq n$, and satisfying the conditions $\mu(u, v) \geq 0$ and $\sum \mu(u, v) = 1$.

- By the von Neumann minimax theorem we have

$$\min_{\mu} \max_{x} \mu A x \le B,$$

which is exactly the inequality we need to prove because μ can be regarded as a probability measure on the set of pairs from M_n with distance $\geq n$.

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- The proof above can be summarized in the following way: we can consider our situation as a kind of a two-person game: one person picks a 1-Lipschitz function and the other picks a pair of points in M_n at distance $\geq n$. The second person wins if $||f(u) - f(v)|| \leq B$.

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- By the minimax theorem the second person has always a winning weighted strategy.

- This is still far from the desired result. In fact, one can prove an analogue of the Poincaré inequality (introduced in the previous lecture) for L_p-valued functions on expander graphs, and show that metric spaces containing families of expanders do not embed coarsely into L_p for $1 \leq p < \infty$ (the same is true for weak expanders).

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- On the other hand, using the result of Johnson-Randrianarivony (2006) (or its strengthening due to Mendel-Naor (2008)) one can construct a locally finite metric space which is a subset of ℓ_p, p is some number satisfying $p > 2$, which is not coarsely embeddable into ℓ_2, and thus contains structures described above.

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- Therefore properties of the structures whose existence we proved today are quite different from properties of *real expanders*.
- The following result was proved with the purpose to get from the previous result some more satisfactory expander-like structures.

Expansion properties of sets M_n.

- Let s be a positive integer. We consider graphs $G(n,s)=(M_n,E(M_n,s))$, where the edge set $E(M_n,s)$ is obtained by joining those pairs of vertices of M_n which are at distance $\le s$. The graphs $G(n,s)$ have uniformly bounded degrees if the metric space M has bounded geometry.

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- (*) For some $s \in \mathbb{N}$ there is a number $h_s > 0$ and subgraphs H_n of $G(n, s)$ of indefinitely growing sizes (as $n \to \infty$) such that the expansion constants of $\{H_n\}$ are uniformly bounded from below by h_s.

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- (*) For some $s \in \mathbb{N}$ there is a number $h_s > 0$ and subgraphs H_n of $G(n, s)$ of indefinitely growing sizes (as $n \to \infty$) such that the expansion constants of $\{H_n\}$ are uniformly bounded from below by h_s.
- If we would prove that in the bounded geometry case the condition (*) is satisfied, it would solve the problem mentioned at the beginning of the talk: whether each metric space with bounded geometry which does not embed coarsely into a Hilbert space contains weak expanders?

- At this point we are able to prove only the following weaker expansion property of the graphs $G(n,s)$. We introduce the measure ν_n on M_n by $\nu_n(A) = \mu_n(A \times M_n)$. Let F be an induced subgraph of $G(n,s)$. We denote the vertex boundary of a set A of vertices in F by $\delta_F A$. (The *vertex boundary* of A is the set of vertices which are not in A but are adjacent to some vertices of A.)

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- Theorem (MO (2009))

  Let s and n be such that $2n > s > 8D$. Let $\varphi(D,s) = \frac{s}{4D} - 2$. Then $G(n,s)$ contains an induced subgraph F with d_M-diameter $\geq n - \frac{s}{2}$, such that each subset $A \subset F$ of d_M-diameter $< n - \frac{s}{2}$ satisfies the condition: $\nu_n(\delta_F A) > \varphi(D,s)\nu_n(A)$.

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  - The proof uses the exhaustion process similar to the one used by Linial-Saks (1993) and "random" signing of functions similar to the way it was used by Rao (1999) in his work on Lipschitz embeddings of planar graphs into ℓ_2.

Final comment

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- Recall that the *girth* of a graph is the length of the shortest cycle in it.