Introduction to linear dynamics

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Part 1: basic facts

What it is about

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Definition.

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Starting point:

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Starting point: PhD thesis of C. Kitai (1982).

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- No *compact* operator can be hypercyclic.

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Remark.

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Remark. hypercyclic $\Longrightarrow$ topologically transitive is always true.

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Then T is hypercyclic.

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Corollary. *If B is the unweighted backward shift on X, then λB is hypercyclic whenever $|\lambda| > 1$* (Rolewicz 1969).

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Godefroy-Shapiro

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Lemma. *Let $T \in \mathcal{L}(X)$, where the tvs X is Polish and complex. Put $E^+(T) := \bigcup_{|\lambda|>1} \ker(T - \lambda I)$ and $E^-(T) := \bigcup_{|\lambda|<1} \ker(T - \lambda I)$.*

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Corollary.

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Corollary. *Assume that there exists an analytic or anti-analytic map $s \mapsto e_s$*

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Corollary. *Assume that there exists an analytic or anti-analytic map $s \mapsto e_s$ from some open set $\Omega \subset \mathbb{C}$ into X*

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Corollary. *Assume that there exists an analytic or anti-analytic map* $s \mapsto e_s$ *from some open set* $\Omega \subset \mathbb{C}$ *into* X *such that*

- $\overline{\mathrm{span}}\,\{e_s;\ s \in \Omega\} = X$*;*

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Introduction to linear dynamics

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Part 2: weakly mixing operators

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Frequently hypercyclic operators and chaotic operators are weakly mixing.

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Corollary. *For* T *to be non-weakly mixing, it is enough to have a nonzero linear functional* $\phi : \mathbb{K}[T]e_0 \to \mathbb{K}$

How to ensure non-weak mixing

$$T \in \mathcal{L}(X) \text{ hypercyclic}$$
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Corollary. *For* T *to be non-weakly mixing, it is enough to have a nonzero linear functional* $\phi : \mathbb{K}[T]e_0 \to \mathbb{K}$ *such that the bilinear functional* $(x, y) \mapsto \phi(x \cdot y)$ *is continuous on* $\mathbb{K}[T]e_0 \times \mathbb{K}[T]e_0$.

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Formulas

$$Te_{b_n-1} := \varepsilon_n e_{b_n} + f_n$$

$$\varepsilon_n = \frac{a_{n-1}}{a_n\, w(b_{n-1}+1)\cdots w(b_n-1)}$$

$$f_n = \frac{a_{n-1}}{w(b_{n-1}+1)\cdots w(b_n-1)} \left(P_n(T)e_0 - T^{b_n-b_{n-1}} P_{n-1}(T)e_0\right)$$

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Introduction to linear dynamics

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Part 3: ergodic measures

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Remark. T is weakly mixing wrt μ iff $T \times T$ is ergodic wrt $\mu \otimes \mu$.

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If there exists a Borel probability measure μ on X such that μ has full support and T is ergodic wrt μ, then T is hypercyclic, and even *frequently* hypercyclic.

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Fact. T has no unimodular eigenvalue.

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Let T be a *chaotic* operator on X, i.e. T is hypercyclic with a dense set of periodic points. Does there exist a Gaussian measure μ with full support such that T is weakly mixing wrt μ?