Monotone assignments in compact and function spaces

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1 Introduction

The method of elementary submodels is a set-theoretical method which can be used in various branches of mathematics. A. Dow in [1] illustrated the use of this method in topology, P. Koszmider in [5] used it in functional analysis. Later, inspired by [5], W. Kubiś in [6] used it to construct retractional (resp. projectional) skeleton in certain compact (resp. Banach) spaces. The notion of monotone assignment, considered along this text, is implicit in the use of elementary submodels technique and appears naturally in several contexts. This concept is very simple, natural, and at the same time strength considerably some topological structures.

The purpose of this work is to show some applications of the use of ω-monotone assignments. We will deal with monotone ω-stability, a concept which result to be useful to study retractional skeletons in general and in function spaces. These assignments also are used to provide a proof of the characterisations of Corson and Valdivia compact spaces by some special retractional skeletons. Finally, ω-monotone assignments are used to define c-skeletons, which results to be an useful tool to detect Corson compact spaces inside function spaces.

2 Monotone assignments

Recall that a partially ordered set Γ is *up-directed* if for every $s_0, s_1 \in \Gamma$ there is $t \in \Gamma$ such that $s_0 \leq t$ and $s_1 \leq t$. Γ is *σ-complete* if every sequence $s_0 < s_1 < \cdots$ has the least upper bound in Γ.

In what follows, Γ and Σ will always be used to denote up-directed σ-closed posets. The following definition come from [9].

Definition 2.1. A function $\phi : \Sigma \to \Gamma$ is ω-*monotone* if satisfy:

1. $s \leq t \in \Sigma$ imply $\phi(s) \leq \phi(t)$;
2. if $\{s_n\}_{n \in \omega} \subset \Sigma$ is increasing, then $\phi(\sup_{n < \omega} s_n) = \sup_{n < \omega} \phi(s_n)$.

Remark 2.2. A function $\phi : [X]^{\leq \omega} \to [Y]^{\leq \omega}$ is ω-monotone if and only if $\phi(A) = \bigcup_{F \in [A]^{< \omega}} \phi(F)$ for all $A \in [X]^{\leq \omega}$.

Consider an ω-monotone function $\phi : [X]^{\leq \omega} \to [X]^{\leq \omega}$. Say that $C \subset X$ is closed under ϕ whenever $\phi([C]^{\leq \omega}) \subset [C]^{\leq \omega}$. For each $A \in [X]^{\leq \omega}$, the *closure* $\overline{\phi}(A)$ *of A under ϕ* is the smallest, with respect to inclusion, set $C \subset X$ such that $A \subset C$ and C is closed under ϕ. Note that there is a least C, namely

$$C = \bigcap \{D : A \subset D \subset X \text{ and } D \text{ is closed under } \phi\}.$$

Theorem 2.3. *If $\phi : [X]^{\leq \omega} \to [X]^{\leq \omega}$ is ω-monotone, then the function $\overline{\phi} : [X]^{\leq \omega} \to [X]^{\leq \omega}$ is well defined and ω-monotone.*

Proof. Recursively define $\mathcal{C}_n : [X]^{\leq \omega} \to [X]^{\leq \omega}$ as follows. Set $\mathcal{C}_0(A) = A$ for each $A \in [X]^{\leq \omega}$. Besides, let $\mathcal{C}_{n+1}(A) = \mathcal{C}_n(A) \cup \bigcup_{F \in [\mathcal{C}_n(A)]^{< \omega}} \phi(F)$ for all $A \in [X]^{\leq \omega}$. By induction on n each function $\mathcal{C}_n$ is ω-monotone. The equality $\mathcal{C}(A) = \bigcup_{n \in \omega} \mathcal{C}_n(A)$ for every $A \in [X]^{\leq \omega}$ follows from Remark 2.2 and implies that $\mathcal{C}$ is well defined and ω-monotone. $\qquad\square$

The following Lemma is easy to prove and will be an useful tool.

Lemma 2.4. *If $f : X \to \Gamma$ is an arbitrary map, then we can construct an ω-monotone map $\phi : [X]^{\leq \omega} \to \Gamma$ such that $\phi(\{x\}) = f(x)$ for each $x \in X$.*

3   Monotonically ω-stable spaces

Monotone assignments are naturally associated to some topological structures. Stable spaces, introduced by Arhangel'skii, proved to be very useful both for the theory of cardinal invariants and C_p-theory. We will consider a monotone version of this property which was introduced in [8]. Before let us recall some notation.

Set $\mathcal{N} \subset \mathcal{P}(X)$. Given $\mathcal{C} \in \mathcal{P}(X)$ we say that $\mathcal{N}$ is a *network for $\mathcal{C}$ in X* if whenever $C \in \mathcal{C}$ and $C \subset U$ with U open, then $C \subset N \subset U$ for some

$N \in \mathcal{N}$. If $A \subset X$, then $\mathcal{N}$ is said to be a network for A in X whenever it is a network for $\{\{x\}\}_{x \in A}$ in X. A network for X in X is called simply a network for X. Given $M \subset C(X)$, we say that $\mathcal{N}$ is a *network for M in X* if it is a network for X endowed with the weak topology generated by M. Finally, for $f \in C(X)$, a network for f is simply a network for $\{f\}$.

Definition 3.1. We say that a space X is *monotonically ω-stable* if there exists an ω-monotone map $\mathcal{N} : [C_p(X)]^{\leq \omega} \to [\mathcal{P}(X)]^{\leq \omega}$ such that $\mathcal{N}(A)$ is a network for $\overline{A}$ for all $A \in [C_p(X)]^{\leq \omega}$.

The above property seems to be artificial, however it appears naturally in a wide class of spaces and will be a useful tool along this work.

Proposition 3.2. *Assume that a space X has a cover $\mathcal{C}$ by pseudocompact spaces X and a countable network $\mathcal{O}$ for the cover $\mathcal{C}$. Then X is monotonically ω-stable.*

Proof. Fix a countable base $\mathcal{B}_{\mathbb{R}}$ for the real line $\mathbb{R}$. Given $\mathcal{E} \in [\mathcal{P}(X)]^{\leq \omega}$ let $\mathcal{F}(E) = \{\bigcup \mathcal{J} : \mathcal{J} \in [\mathcal{E}]^{<\omega}\}$ and $\mathcal{C}(E) = \{O \setminus E : O \in \mathcal{O}, E \in \mathcal{E}\}$. Clearly $\mathcal{F}$ and $\mathcal{C}$ are ω-monotone. For each $A \in [C_p(X)]^{\leq \omega}$ let $\mathcal{I}(A) = \{g^{-1}(B) : g \in A, B \in \mathcal{B}_{\mathbb{R}}\}$ and $\mathcal{N}(A) = \mathcal{C}(\mathcal{F}(\mathcal{I}(A)))$. Note that $\mathcal{N}$ is ω-monotone.

Given $A \subset X$, we shall prove that $\mathcal{N}(A)$ is a network for $\overline{A}$. Assume that $x \in U$ where $U = \bigcap_{f \in F} f^{-1}(B_f)$ for some $F \in [\overline{A}]^{<\omega}$ and $B_f \in \mathcal{B}_{\mathbb{R}}$ for each $f \in F$. Pick $P \in \mathcal{C}$ containing x. Choose $y \in P \setminus U$. Fix $f_y \in F$ and $B_y \in \mathcal{B}_{\mathbb{R}}$ so that $f_y(y) \in B_y$ and $f_y(x) \in B_{f_y} \setminus \overline{B}_y$. Select $g_y \in A$ such that $g_y(y) \in B_y$ and $g_y(x) \in B_{f_y} \setminus \overline{B}_y$. Let $\mathcal{U} = \{g_y^{-1}(B_y) : y \in P \setminus U\}$. Note $P \setminus U \subset \bigcup \mathcal{U}$ and $x \notin \bigcup \mathcal{U}$. Choose $\mathcal{V} \subset [\mathcal{U}]^{<\omega}$ such that $P \subset U \cup \bigcup \mathcal{V}$. Select a set $O \in \mathcal{O}$ such that $P \subset O \subset U \cup \bigcup \mathcal{V}$. Observe that if $N = O \setminus \bigcup \mathcal{V}$, then $N \in \mathcal{N}(A)$ and $x \in N \subset U$. $\qquad \square$

Corollary 3.3. *Every pseudocompact and every Σ-space is monotonically ω-stable*

4   Retractional skeletons

Now we will consider an optimal notion of an indexed system of retractions introduced in [7].

Definition 4.1. An *r-skeleton* in a space X is a family of retractions $\{r_s\}_{s \in \Gamma}$, satisfying the following conditions:

1. $r_s(X)$ is has a countable network for each $s \in \Gamma$.
2. $s \leq t$ implies $r_s = r_s \circ r_t = r_t \circ r_s$.
3. For every $x \in X$, $x = \lim_{s \in \Gamma} r_s(x)$.
4. Given $s_0 < s_1 < \cdots$ in Γ, if $t = \sup_{n \in \omega} s_n$, then $r_t(x) = \lim_{n \to \infty} r_{s_n}(x)$ for every $x \in X$.

Let $\{r_s\}_{s \in \Gamma}$ be a family of retractions on X. We say that $\{r_s\}_{s \in \Gamma}$ is a *weak r-skeleton* in X if it only satisfies conditions 1,2 and 4 in the above definition. Besides, the set $\bigcup_{s \in \Gamma} r_s(X)$ will be called *the set induced by* $\{r_s\}_{s \in \Gamma}$.

Theorem 4.2. *Let X be monotonically ω-stable and let Y be the set induced by a weak r-skeleton $\{r_s\}_{s \in \Gamma}$ in X. If $n \in \omega$, $F \subset Y^n$ and $s_0 \in \Gamma$, then there exist $t \in \Gamma$ and $D \in [F]^{\leq \omega}$ such that $s_0 \leq t$ and $r_t^n(\overline{F}) \subset \overline{D}$.*

Proof. Let $\mathcal{N}$ be a monotonically stable operator in X. Select an ω-monotone map $\mathcal{E} : [C_p(X)]^{\leq \omega} \to [F]^{\leq \omega}$ such that $\mathcal{E}(A) \cap N \neq \emptyset$ whenever $N \in \mathcal{N}(A)^n$ and $N \cap F \neq \emptyset$. Using Lemma 2.4 we can construct an ω-monotone map $s : [F]^{\leq \omega} \to \Gamma$ such that $s_0 \leq s(P)$ and $P \subset r_{s(P)}^n(X)$ for each $P \in [F]^{\leq \omega}$. For each $G \in [F]^{<\omega}$ fix a countable dense subset $A_{s(G)}$ of $r_{s(G)}^*(C_p(r_{s(G)}(X)))$. Define $\mathcal{M} : [F]^{\leq \omega} \to [C_p(X)]^{\leq \omega}$ as $\mathcal{M}(P) = \bigcup_{G \in [P]^{<\omega}} A_{s(G)}$. Then $\mathcal{M}$ is ω-monotone. Note that $\mathcal{M}(P)$ is dense in $r_{s(P)}^*(C_p(r_{s(P)}(X)))$. Finally set $\mathcal{A} = \overline{\mathcal{M} \circ \mathcal{E}} : [C_p(X)]^{\leq \omega} \to [C_p(X)]^{\leq \omega}$ the closure of $\mathcal{M} \circ \mathcal{E}$.

Set $A = \mathcal{A}(A_{s(\emptyset)})$, $D = \mathcal{E}(A)$ and $t = s(D)$. Clearly $s_0 \leq t$. By continuity it is enough to show that $r_t^n(F) \subset \overline{D}$. Assume on the contrary that $r_t^n(x) \notin \overline{D}$ for some $x \in F$. Choose $B \in \tau(X)^n$ such that $r_t^n(x) \in U = \prod_{i \in n} B(i) \subset X^n \setminus D$. Choose $f \in C_p(r_t^n(X))$ such that $f(r_t^n(x)) = 0$ and $f(\overline{D} \cap r_t^n(X)) \subset \{1\}$. We know that $r_t^*(C_p(r_t(X))) \subset \overline{\mathcal{M}(D)} = \overline{\mathcal{M}(\mathcal{E}(A))} \subset \overline{A}$. Since $\mathcal{N}(A)$ is a network for $\overline{A}$, it is a network for $r_t^*(C_p(r_t(X)))$. It follows that $\mathcal{N}(A)$ is a network for r_t. Then we can find $N \in \mathcal{N}(A)^n$ such that $x \in N \subset (r_t^n)^{-1}(U)$. Then there exists $y \in D$ such that $y \in N \cap F$. We know that $y \in r_t^n(X)$. We then have that $y = r_t^n(y) \in U \cap D$, which is not possible. $\square$

Corollary 4.3. *Let X be monotonically ω-stable and let Y be a set induced by a weak r-skeleton $\{r_s\}_{s \in \Gamma}$ in X. Then:*

1. *$t(Y) \leq \omega$.*
2. *$x = \lim_{s \in \Gamma} r_s(x)$ for each $x \in \overline{Y}$.*

Proof. 1. Set $A \subset Y$ and $x \in \overline{A}$. Choose $s_0 \in \Gamma$ so that $r_{s_0}(x) = x$. By Theorem 4.2, we can find $D \in [A]^{\leq \omega}$ and $s \in \Gamma$ such that $s_0 \leq s$ and $r_s(\overline{A}) = \overline{D}$. This implies that $x = r_s(x) \in r_s(\overline{A}) \subset \overline{D}$.

2. Fix $x \in \overline{Y}$. Let U be an open neighborhood of x. Choose an open set V in X such that $x \in V \subset \overline{V} \subset U$. Set $F_1 = V \cap Y$ and $F_2 = (X \setminus U) \cap Y$. From Theorem 4.2 and condition 4, we can find $s \in \Gamma$ such that $r_s(\overline{F_i})) \subset \overline{F_i}$ for $i = 1, 2$. Then $r_s(x) \in \overline{F}_1$. Choose $t \in \Gamma$ such that $s \leq t$. If $r_t(x) \notin U$, then $r_t(x) \in F_2$ and so $r_s(x) = r_s(r_t(x)) \in \overline{F}_2$, which is not possible. $\qquad \square$

5 r-skeletons and Σ-subsets

Different characterisations of Corson and Valdivia compacta have appeared in the literature. In [7], it is proved that a compact K is Valdivia if and only if it admits a commutative r-skeleton. Here we present a proof of this fact.

We say that an r-skeleton $\{r_s\}_{s \in \Gamma}$ in X is *full* if $X = \bigcup_{s \in \Gamma} r_s(X)$, and *commutative* if $r_s \circ r_t = r_t \circ r_s$ whenever $s, t \in \Gamma$.

Lemma 5.1. *Let Y be induced by an r-skeleton $\{r_s\}_{s \in \Gamma}$ in a compact X. Then there is a family of retractions $\{r_A\}_{A \in \mathcal{P}(Y)}$ in X such that for every $A \in \mathcal{P}(Y)$ we have:*

1. *$\{r_B \upharpoonright_{r_A(X)}\}_{B \in [A]^{\leq \omega}}$ is an r-skeleton on $r_A(X)$ and induces $r_A(X) \cap Y$;*
2. *$A \subset r_A(X)$ and $d(r_A(X) \cap Y) \leq |A| + \omega$;*
3. *$r_B \circ r_A = r_A \circ r_B = r_B$ whenever $B \subset A$;*
4. *If $A = \bigcup_{\alpha < \gamma} A_\alpha$ for some increasing family $\{A_\alpha\}_{\alpha < \gamma}$, then $r_A(x) = \lim_{\alpha < \gamma} r_{A_\alpha}(x)$ for every $x \in X$.*

If in addition the r-skeleton $\{r_s\}_{s \in \Gamma}$ is commutative, then we also can get $r_B \circ r_A = r_A \circ r_B$ for every $A, B \in \mathcal{P}(Y)$.

Proof. Using Lemma 2.4 we can construct an ω-monotone map $s : [Y]^{\leq \omega} \to \Gamma$ such that $A \subset r_{s(A)}(Y)$ for each $A \in [Y]^{\leq \omega}$. For each $F \in [Y]^{<\omega}$ fix a countable dense subset $D_{s(F)}$ of $r_{s(F)}(X)$ containing F. For each $A \in [Y]^{\leq \omega}$ set $r_A = r_{s(A)}$ and $D_A = \bigcup_{F \in [A]^{<\omega}} D_{s(F)}$. Then $r_A(X) = \overline{D}_A$ for each $A \in [Y]^{\leq \omega}$.

Now, choose an arbitrary $A \subset Y$. Let $\Gamma_A = [A]^{\leq \omega}$ and $D_A = \bigcup_{B \in \Gamma_A} D_B$. It is easy to verify, using Corollary 4.3, that $\{r_B \upharpoonright_{\overline{D}_A}\}_{B \in \Gamma_A}$ is an r-skeleton on $\overline{D}_A$. The countable tightness of Y implies that the set induced by that

r-skeleton is $\bigcup_{B\in\Gamma_A} r_B(X) = \bigcup_{B\in\Gamma_A} \overline{D}_B = \overline{D}_A \cap Y$. Consider the retraction $r_A : X \to \overline{D}_A$ which assign to each $x \in X$ the only point in $\overline{D}_A \cap \bigcap_{B\in\Gamma_A} r_B^{-1}(r_B(x))$. Note that $r_B \circ r_A = r_B$ holds for every $B \in \Gamma_A$. Besides $r_A(x) = \lim_{B\in\Gamma_A} r_B(r_A(x)) = \lim_{B\in\Gamma_A} r_B(x)$ for every $x \in X$. We have finished the construction.

Fix $A \in \mathcal{P}(Y)$. Conditions 1 and 2 are clear from the construction. To verify 3 note that: if $B \subset A \in [Y]^{\leq\omega}$, then $r_B(r_A(x)) = \lim_{D\in\Gamma_B} r_D(r_A(x)) = \lim_{D\in\Gamma_A} r_D(x) = r_B(x) = r_A(r_B(x))$ for each $x \in X$. Now assume that $A = \bigcup_{\alpha<\gamma} A_\alpha$ is as in 4. Fix $x \in X$. Let U be a neighborhood of $r_A(x)$. Choose $B \in \Gamma_A$ such that $r_D(x) \in U$ whenever $B \subset D \in \Gamma_A$. If γ has uncountable cofinality, then select $\alpha < \gamma$ such that $B \subset A_\alpha$. In this case $r_{A_\beta}(x) \in \overline{U}$ whenever $\alpha \leq \beta < \gamma$. If $\{\alpha_n\}_{n\in\omega}$ is a cofinal subset of γ, then there exists $n \in \omega$ such that $r_{A_\beta}(x) \in \overline{U}$ whenever $\alpha_n \leq \beta < \gamma$ (otherwise we can get $r_D(x) \notin U$ for some $D \in \Gamma_A$ containing B). Since U is arbitrary, we conclude that $r_A(x) = \lim_{\alpha<\gamma} r_{A_\alpha}(x)$.

Finally, assume that $\{r_s\}_{s\in\Gamma}$ is commutative. Proceeding as in item 3 we can verify that $r_B \circ r_A = r_A \circ r_B$ for every $A, B \in \mathcal{P}(Y)$. $\square$

Recall that a set $Y \subset X$ as a Σ-*subset of* X if there exists an embedding $\phi : X \to I^T$, for some set T, such that $Y = \phi^{-1}(\Sigma I^T)$.

Theorem 5.2. *Let Y be dense in a compact X. If Y is induced by a commutative or full r-skeleton on X, then Y is a Σ-subset of X.*

Proof. By induction on the density of Y. For $d(Y) = \omega$ the result is clear. Assume that $d(Y) = \kappa > \omega$ and the result holds for spaces of density smaller than κ. Choose the family $\{r_A\}_{A\in\mathcal{P}(Y)}$ of retractions on X as in Lemma 5.1. Let $\{y_\alpha\}_{\alpha<\kappa}$ be a dense subspace of Y. For each $\alpha \leq \kappa$, set $A_\alpha = \{y_\beta\}_{\beta<\alpha}$ and $r_\alpha = r_{A_\alpha}$. Given $\alpha < \kappa$, we may apply the inductive hypothesis to find a set T_α and an embedding $\phi_\alpha : r_\alpha(X) \to I^{T_\alpha}$ such that $Y \cap r_\alpha(X) = \phi_\alpha^{-1}(\Sigma I^{T_\alpha})$. We can assume that $T_\alpha \cap T_\beta = \emptyset$ whenever $\alpha < \beta < \kappa$. Consider the set $T = \bigcup_{\alpha<\kappa} T_\alpha$. We identify I^T with $\prod_{\alpha<\kappa} I^{T_\alpha}$. Define $\phi : X \to I^T$ as follows:

$$\phi(x)(\alpha) = \begin{cases} \phi_{\alpha+1}(r_{\alpha+1}(x)) - \phi_{\alpha+1}(r_\alpha(x)) & \text{if } \alpha > 0; \\ \phi_0(r_0(x)) & \text{if } \alpha = 0. \end{cases}$$

For each $x \in X$ and $\alpha < \kappa$. To see that ϕ is an embedding we only need to show that ϕ is one-to-one. Fix distinct $x, y \in X$. The fact that $\{r_A\}_{A\in[Y]^{\leq\omega}}$ is an r-skeleton on X implies that $r_F(x) \neq r_F(y)$ for some $F \in [Y]^{\leq\omega}$. Set

$\beta = \min\{\alpha < \kappa : r_\alpha(x) \neq r_\alpha(y)\}$. If $\beta = 0$, then $\phi(x)(0) \neq \phi(y)(0)$. Otherwise, $\beta = \alpha + 1$ for some $\alpha < \kappa$ and so $\phi(x)(\alpha) \neq \phi(y)(\alpha)$. Hence $\phi(x) \neq \phi(y)$. We shall verify that $Y = \phi^{-1}(\Sigma I^T)$. To see that $Y \subset \phi^{-1}(\Sigma I^T)$, fix $x \in Y$. Note that $r_\alpha(x) \in Y$ for each $\alpha < \kappa$ (in the commutative case this allows from $r_\alpha(r_{\{x\}}(x)) = r_{\{x\}}(r_\alpha(x))$). The countable tightness of Y implies that $\{r_\alpha(x)\}_{\alpha < \kappa}$ is countable. Hence $\phi(x) \subset \Sigma I^T$. Besides, the countable tightness of ΣI^T and the properties of Y imply that $\phi^{-1}(\Sigma I^T) \subset Y$. $\qquad\square$

Corollary 5.3. *A compact space is Valdivia (Corson) if and only if it admits a commutative (full) r-skeleton.*

Proof. Given a set T, by Corollary 3.3 the space I^T (ΣI^T) is monotonically ω-stable. Besides it is easy to verify that I^T (ΣI^T) admits a commutative (full) r-skeleton. Then we may apply Theorem 4.2 and 4.3 to see that each Valdivia (Corson) compact space admits a commutative (full) r-skeleton. The converse follows from Theorem 5.2. $\qquad\square$

Corollary 5.4. *If a countably compact space admits a full commutative retractional skeleton iff X can be embedded in ΣI^T for some set T.*

Proof. If X admits a a full commutative retractional skeleton, then it is easy to see that X is induced by a commutative retractional skeleton in βX. It follows from Theorem 5.2 that X is a Σ-subset of βX and hence X can be embedded in ΣI^T for some set T. Now, if X can be embedded in ΣI^T for some set T, must be embedded as a closed subspace. Therefore the result follows from Theorem 4.2 since ΣI^T admits a full commutative r-skeleton. $\qquad\square$

6   Strong r-skeletons in C_p-duality

An r-skeleton $\{r_s\}_{s \in \Gamma}$ in a space X is said to be *strong* if whenever $s_0 \in \Gamma$ and F is closed in X^n, for some $n \in \mathbb{N}$, there exists $s \in \Gamma$ such that $s_0 \leq s$ and $r_s^n(F) \subset F$. It is easy to verify that the property of admit a strong r-skeleton is invariant under countable topological sums, countable products and closed subspaces. We will show that this property is also preserved under the formation of C_p-spaces, and as a consequence by $\mathbb{R}$-quotient maps.

The following fact follows from Theorem 4.2

Corollary 6.1. *Let X be a monotonically ω-stable space. Then an r-skeleton on X is strong if and only if it is full.*

The following Lemma implies that every space which admits a strong r-skeleton has the Sokolov property and, in particular, is collectionwise normal, ω-stable, ω-monolithic and has countable extent (see [11]). Besides we also can verify that these spaces are countably paracompact.

Lemma 6.2. *Let $\{r_s\}_{s\in\Gamma}$ be a strong r-skeleton on X. If $s_0 \in \Gamma$ and $\mathcal{F}_n$ is a countably family of closed subsets of X^n for each $n \in \mathbb{N}$, then there exists $s \in \Gamma$ such that $s_0 \leq s$ and $r_s^n(F) \subset F$ whenever $F \in \mathcal{F}_n$.*

Proof. Let $\{N_n\}_{n\in\mathbb{N}}$ be a partition of $\mathbb{N}$ in infinite sets. For each $n \in \mathbb{N}$ enumerate $\mathcal{F}_n$ as $\{F_n\}_{n\in N_n}$, where each element appears infinitely many times. For each $m > 0$ choose recursively $s_m \in \Gamma$ such that $s_{m-1} \leq s_m$ and $r_{s_m}^n(F_m) \subset F_m$ if $F_m \in \mathcal{F}_n$. Note that $s = \sup\{s_m\}_{m\in\omega}$ is as required. $\qquad\square$

In order to prove our main result related to C_p-spaces, let us introduce some notation. Fix a countable base $\mathcal{B}_\mathbb{R}$ for the real line $\mathbb{R}$. Let X be a space. Given $n \in \mathbb{N}$, $x \in X^n$ and $B \in \mathcal{B}_\mathbb{R}^n$, set

$$U_{x,B} = \{f \in C(X) : \forall i \in n\big(f(x(i)) \in B(i)\big)\}.$$

For each $A \subset X$ let $\mathcal{B}(A) = \{U_{x,B} : x \in X^n, B \in \mathcal{B}_\mathbb{R}^n, n \in \omega\}$. Then $\mathcal{B}(X)$ is a base for the topology of $C_p(X)$. It is straightforward to verify that the assignment $A \to \mathcal{B}(A)$, for $A \in [X]^{\leq\omega}$, is ω-monotone.

Theorem 6.3. *A space X has a strong r-skeleton if and only if $C_p(X)$ has a strong r-skeleton.*

Proof. Assume that $\{r_s\}_{s\in\Gamma}$ is a strong r-skeleton on X. For each $s \in \Gamma$ consider the retraction $\hat{r}_s = r_s^* \circ \pi_{r_s(X)} : C_p(X) \to C_p(X)$. It is standard to verify that $\{\hat{r}_s\}_{s\in\Gamma}$ is an r-skeleton in $C_p(X)$. In order to prove that this r-skeleton is strong choose $s_0 \in \Gamma$ and a closed subset G of $C_p(X)^n$ for some $n \in \mathbb{N}$. Let nX be the disjoint topological union of n copies of X and let nr_s denote the natural retraction induced by r_s on nX. It is standard to verify that $\{nr_s\}_{s\in\Gamma}$ is a strong r-skeleton on nX. We canonically identify $C_p(X)^n$ with $C_p(nX)$. Pick $B \in \mathcal{B}_\mathbb{R}^m$ for some $m \in \mathbb{N}$. Consider the set

$$F_B = \{x \in (nX)^m : U_{x,B} \cap G = \emptyset\}.$$

Note that F_B is closed in $(nX)^m$. By Lemma 6.2 we can find $s \in \Gamma$ such that $s_0 \leq s$ and $(nr_s)^m(F_B) \subset F_B$ for each $m \in \mathbb{N}$ and $B \in \mathcal{B}_\mathbb{R}^m$. We assert that s is as promised. Assume on the contrary that there exists $f \in G$ such that $\hat{r}_s^n(f) \notin G$. We identify the maps $\hat{r}_s^n$ and $(nr_s)^* \circ \pi_{nr_s(nX)}$. Then

$f \circ nr_s = \hat{r}_s^n(f) \notin G$. Select $x \in (nX)^m$ and $B \in \mathcal{B}_{\mathbb{R}}^m$, for some $m \in \mathbb{N}$, such that $f \circ nr_s \in U_{x,B}$ and $U_{x,B} \cap G = \emptyset$. Then $f \in U_{nr_s(x),B}$ and $x \in F_B$. The election of nr_s implies that $nr_s(x) \in F_B$; which is not possible since $f \in U_{nr_s(x),B} \cap G$.

Now assume that $C_p(X)$ admits a strong r-skeleton. By the above, $C_p(C_p(X))$ has a strong r-skeleton. Since $C_p(C_p(X))$ has a closed copy of X, we conclude that X admits a stron r-skeleton. $\qquad\square$

7   Networks and C_p-duality

In order to establish some technical duality results for networks in C_p-spaces, let us introduce some notation. Let X be a space. Given $n \in \mathbb{N}$, $N \in \mathcal{P}(X)^n$ and $B \in \mathcal{B}_{\mathbb{R}}^n$, set

$$W_{N,B} = \{f \in C_p(X) : \forall i \in n\big(f(N(i)) \subset B(i)\big)\}.$$

For each $\mathcal{N} \subset \mathcal{P}(X)$ let $\mathcal{W}(\mathcal{N}) = \{W_{N,B} : N \in \mathcal{N}^n, B \in \mathcal{B}_{\mathbb{R}}^n, n \in \omega\}$. It is straightforward to verify that the assignment $\mathcal{N} \to \mathcal{W}(\mathcal{N})$, from $[\mathcal{P}(X)]^{\leq\omega}$ to $[\mathcal{P}(C_p(X))]^{\leq\omega}$, is ω-monotone.

Theorem 7.1. *Let $\mathcal{N}$ be a family of subsets of X.*

1. *If $f : X \to Y$ and $\mathcal{N}$ is a network for f, then $\mathcal{W}(\mathcal{N})$ is a network for $f^*(C_p(Y))$ in $C_p(X)$.*

2. *If $A \subset X$ and $\mathcal{N}$ is a network for A in X, then the family $\mathcal{W}(\mathcal{N})$ is a network for π_A on $C_p(X)$.*

Proof. 1. Take $f^*(g) \in C_p(Y)$ and assume that $f^*(g) = g \circ f \in U_{x,B}$ for some $U_{x,B} \in \mathcal{B}(X)$. Then $x \in X^n$ and $B \in \mathcal{B}_{\mathbb{R}}^n$ for some $n \in \omega$. Since $x(i) \in f^{-1}(g^{-1}(B(i)))$ for each $i \in n$, there exists $N \in \mathcal{N}^n$ satisfying $x(i) \in N(i) \subset f^{-1}(g^{-1}(B(i)))$ for every $i \in n$. Note that $W_{N,B} \in \mathcal{W}(\mathcal{N})$ and $f^*(g) \in W_{N,B} \subset U_{x,B}$.

2. Let $f \in C_p(X)$ and assume that $f \in \pi_A^{-1}(U_{x,B})$ for some $U_{x,B} \in \mathcal{B}(A)$. Then $x \in X^n$ and $B \in \mathcal{B}_{\mathbb{R}}^n$ for some $n \in \omega$. Since $x(i) \in f^{-1}(B(i))$ for each $i \in n$, we can find $N \in \mathcal{N}^n$ such that $x(i) \in N(i) \subset f^{-1}(B(i))$ for each $i \in n$. Observe that $W_{N,B} \in \mathcal{W}(\mathcal{N})$ and $f \in W_{N,B} \subset \pi_A^{-1}(U_{x,B})$. $\qquad\square$

8 Monotone stability and C_p-duality

It happens that stability has a two way C_p-dual property. Monolithic spaces introduced by Arhangelskii satisfy that: monolithicity of either X or $C_p(X)$ is equivalent to stability of the other. In this section we establish a monotone version of these results.

Monotonically ω-monolithic spaces were introduced by Tkachuk [10] in order to study the D-property in spaces of continuous functions $C_p(X)$.

Definition 8.1. We say that a space X is *monotonically ω-monolithic* if there exists an ω-monotone assignment $\mathcal{N} : [X]^{\leq\omega} \to [\mathcal{P}(X)]^{\leq\omega}$ such that $\mathcal{N}(A)$ is a network for $\overline{A}$ in X for all $A \in [X]^{\leq\omega}$.

The next lemma is easy to prove.

Lemma 8.2. *If X is monotonically ω-stable, $Y \subset X$ and there exists $Z \subset C_p(Z)$ such that $\pi_Y \upharpoonright_Z : Z \to C_p(Y)$ is closed and onto, then $C_p(Y)$ is monotonically ω-stable.*

Theorem 8.3. *A space X is monotonically ω-monolithic (ω-stable) if and only if the space $C_p(X)$ is monotonically ω-stable (ω-monolithic).*

Proof. First, suppose that X is a monotonically ω-monolithic space. Take an assignment $\mathcal{N}$ that witnesses this fact. By the factorisation theorem, for each $f \in [C_p(C_p(X))]^{<\omega}$ we can fix fix $S_f \in [X]^{\leq\omega}$ such that $f \in \pi_{S_f}^*(C_p(\pi_{S_f}(C_p(X))))$. For each $A \in [C_p(C_p(X))]^{<\omega}$ let $\mathcal{S}(A) = \bigcup_{f\in A} \mathcal{S}(A)$. Then the assignment $A \to \mathcal{S}(A)$ is ω-monotone. Consider the ω-monotone map $\mathcal{O} = \mathcal{W} \circ \mathcal{N} \circ \mathcal{S}$. Since $\mathcal{N}(\mathcal{S}(A))$ is a network for $F_A = \overline{\mathcal{S}(A)}$ in X, Theorem 7.1 implies that $\mathcal{O}(A)$ is a network for the open map π_{F_A}. Hence $\mathcal{O}(A)$ is a network for the family of maps $\pi_{F_A}^* C_p(\pi_{F_A}(C_p(X)))$. Since $A \subset \pi_{F_A}^* C_p(\pi_{F_A}(C_p(X)))$ and the last set is closed, we conclude that $\mathcal{O}(A)$ is a network for $\overline{A}$. In this way $C_p(X)$ is monotonically ω-stable.

Secondly, assume that X is monotonically ω-stable and take an assignment $\mathcal{N}$ that witnesses this fact. Consider the ω-monotone map $\mathcal{O} = \mathcal{W} \circ \mathcal{N}$. Select $A \in [C_p(X)]^{<\omega}$. Let $Y_A = \Delta_{\overline{A}}(X)$. Since $\mathcal{N}(A)$ is a network for $\Delta_{\overline{A}}$ in X, Theorem 7.1 implies that $\mathcal{W}(\mathcal{N}(A))$ is a network for $(\Delta_{\overline{A}})^*(C_p(Y_A))$. Observe that $\overline{A} \subset (\Delta_{\overline{A}})^*(C_p(Y_A))$ and hence $\mathcal{W}(\mathcal{N}(A))$ is a network for $\overline{A}$. Thus $C_p(X)$ is monotonically ω-monolithic.

Now, assume that $C_p(X)$ is monotonically ω-stable (ω-monolithic). By the above $C_p(C_p(X))$ is monotonically ω-monolithic (ω-stable). Since the

space $C_p(C_p(X))$ contains a copy of X (satisfying the conditions in Lemma 8.2), the space X is is monotonically ω-monolithic (ω-stable). $\square$

Theorem 8.3 let us deduce properties of monotonically ω-stable spaces from monotonically monotonically ω-monolithic spaces and vice versa.

9 Strong r-skeletons and D-spaces

The notion of D-space introduced in [2] have been intensively studied in many topological contexts (see [3]).

A *neighborhood assignment* for a space (X, τ) is a function $\phi : X \to \tau$ with $x \in \phi(x)$ for every $x \in X$. A *kernel* for ϕ is a subset $D \subset X$ such that $\phi(D) := \bigcup_{x \in D} \phi(x) = X$. X is said to be a *D-space* if every neighborhood assignment ϕ for X has a closed discrete kernel.

Theorem 9.1. *If X is has a strong r-skeleton and is monotonically ω-monolithic, then X is a Lindelöf D-space.*

Proof. It is enough to show that every neighborhood assignment has a countable closed discrete kernel. Let $\{r_s\}_{s \in \Gamma}$ be a strong r-skeleton in X and fix an operator $\mathcal{N}$ which witnesses monotone ω-monolithicity of X. Given $s \in \Gamma$ fix a countable dense subset A_s of $r_s(X)$. For each $A \in [X]^{\leq \omega}$ let

$$\mathcal{C}(A) = \{x \in X : x \in N \subset \phi(x) \text{ for some } N \in \mathcal{N}(A)\}.$$

Note that $\mathcal{C}$ satisfies conditions in Definition 2.1.

Let $\{\Omega_n\}_{n \in \omega}$ be a partition of ω in infinite sets. Choose $x_0 \in X$ and $s_0 \in \Gamma$. Assume that $x_i \in X$ and $s_i \in \Gamma$ have been constructed and that if $A_i = \bigcup_{j < i} A_{s_j}$ then the family $\mathcal{N}(A_i)$ has been enumerated as $\{N_k : k \in \Omega_i\}$, for $i < n$. Set $A_n = \bigcup_{i < n} A_{s_i}$, $D_n = \{x_i\}_{i < n}$ and enumerate $\mathcal{N}(A_n)$ as $\{N_k : k \in \Omega_n\}$. If $\phi(D_n) = X$ take $D = D_n$ and stop the construction. In the other case, look at

$$X_n = \mathcal{C}(A_n) \setminus \phi(D_n)$$

If $X_n \neq \emptyset$, choose $x_n \in X_n$ such that the corresponding $N \in \mathcal{N}(A_n)$ is the least possible in the above enumeration. Otherwise select a point $x_n \in X \setminus \phi(D_n)$ arbitrarily. After that, choose $s_n \in \Gamma$ such that $s_{n-1} \leq s_n$ and $r_{s_n}(X \setminus \phi(x_i)) \subset X \setminus \phi(x_i)$ for each $i < n$.

If the process does not finish in any finite step, let us show that $D = \{x_n : n \in \omega\}$ works. Clearly D is closed discrete in $\phi(D)$. So, it is enough to verify

that $\phi(D) = X$. Fix $x \in X$. Set $s = \sup_{n \in \omega} s_n$ and $A = \bigcup_{n \in \omega} A_n$. Note that $r_s(x) = \lim_{n \to \infty} r_{s_n}(x) \in \overline{A} \subset \mathcal{C}(A)$. If $r_s(x) \notin \phi(D)$, the properties of $\mathcal{C}$ imply that $r_s(x) \in X_n$ for sufficiently large n. Fix $k \in \omega$ and $O \in \mathcal{N}(A_k)$ such that $r_s(x) \in O \subset \phi(r_s(x))$. Since $\phi(x_n)$ always contains the least N corresponding to some $y \in X_n$, eventually we choose x_n with $x_n \in O \subset \phi(x_n)$, which puts $r_s(x) \in \phi(x_n)$, which is not possible. Thus $r_s(x) \in \phi(D)$. Pick $k \in \omega$ such that $r_s(x) \in \phi(x_k)$. Since $r_{s_n}(X \setminus \phi(x_k)) \subset X \setminus \phi(x_k)$ for n large enough, we must have $r_s(X \setminus \phi(x_k)) \subset X \setminus \phi(x_k)$. Therefore $x \in \phi(x_k)$. $\qquad\square$

10 r-skeletons and W-sets

The notion of W-set was introduced by Gruenhage in [4]. Given a space X let $FS(X) = \bigcup_{n \in \mathbb{N}} X^n$ be the set of all finite sequences in X. For a map $O : FS(X) \to \tau(X)$, an element $x \in X^\omega$ is called an O-sequence if $x(n) \in O(x \restriction_n)$ for each $n \in \mathbb{N}$. Besides, we say that a set $H \subset X$ is a W-set in X if there is a map $O : FS(X) \to \tau(H, X)$ such that every O-sequence converges to H.

Theorem 10.1. *Let X be a countably compact with a full r-skeleton $\{r_s\}_{s \in \Gamma}$. If H is nonempty and closed in X, then H is a W-set in X.*

Proof. It follows from Corollary 6.1 that the r-skeleton $\{r_s\}_{s \in \Gamma}$ is strong. Define recursively an order preserving function $s : FS(X) \to \Gamma$ satisfying $r_{s(a)}(a(k)) = a(k)$ and $r_{s(a)}(H) \subset H$, whenever $a \in X^n$ and $k \leq n$. Given $a \in FS(X)$, since $r_{s(a)}(H)$ is closed in the metrizable space $r_{s(a)}(X)$, we can fix a decreasing base of open neighborhoods $\{U_{a,n}\}_{n \in \omega}$ for $r_{s(a)}(H)$ in $r_{s(a)}(X)$. Define $O : FS(X) \to \tau(H, X)$ as

$$O(a) = \bigcap_{k \leq n} r_{s(a \restriction k)}^{-1}(U_{a \restriction k, n})$$

for each $a \in X^n$ and $n \in \mathbb{N}$. We assert that every O-sequence converges to H. Let $x \in X^\omega$ be an O-sequence. Note that, since X is countably compact and Frechét-Urysohn, it is enough tho show that the limit of each convergent subsequence of x belongs to H. Let $m \in \omega^\omega$ be strictly increasing and assume that $x \circ m$ converges to some $y \in X$. Choose $k \in \omega$. For each $n > k$, since $m(n) > k$ we have that $x(m(n)) \in O(x \restriction_{m(n)}) \subset r_{s(x \restriction k)}^{-1}(U_{x \restriction k, m(n)})$ and in this way $r_{s(x \restriction k)}(x(m(n))) \in U_{x \restriction k, m(n)}$. Since m is strictly increasing and $\{U_{x \restriction k, n}\}_{n \in \omega}$ is a decreasing base for $r_{s(x \restriction k)}(H)$ in $r_{s(x \restriction k)}(X)$ we must have that $r_{s(x \restriction k)}(y) = \lim_{n \to \infty} r_{s(x \restriction k)}(x(n)) \in r_{s(x \restriction k)}(H) \subset H$. Therefore, for

$t = \sup\{s(x \restriction_k)\}_{k\in\mathbb{N}}$, we must have $y = \lim_{n\to\infty} x(n) = \lim_{n\to\infty} r_t(x(n)) = r_t(y) = \lim_{n\to\infty} r_{s(x\restriction k)}(y) \in H$. $\qquad\square$

11 c-skeletons

Definition 11.1. Given a space X say that $\{(F_s, \mathcal{B}_s)\}_{s\in\Gamma} \subset \mathcal{CL}(X) \times [\tau(X)]^{\leq\omega}$ is a *c-skeleton* on X if:

1. for each $s \in \Gamma$, $\mathcal{B}_s$ is a base for a topology τ_s on X and there exist a Tychonoff space Z_s and a continuous map $g_s : (X, \tau_s) \to Z_s$ which separates the points of F_s;
2. if $s, t \in \Gamma$ and $s \leq t$, then $F_s \subset F_t$;
3. $X = \overline{\bigcup_{s\in\Gamma} F_s}$;
4. the assignment $s \to \mathcal{B}_s$ is ω-monotone.

In addition, if $X = \bigcup_{s\in\Gamma} F_s$, then we say that the c-skeleton is *full*.

Theorem 11.2. *If X is countably compact and has a (full) c-skeleton, then X has a (full) r-skeleton.*

Proof. Let $\{(F_s, \mathcal{B}_s)\}_{s\in\Gamma}$ be a full c-skeleton in X. Let $Y = \bigcup_{s\in\Gamma} F_s$ and let $\Sigma = [Y]^{\leq\omega}$. By Lemma 2.4 we can construct an ω-monotone function $s : \Sigma \to \Gamma$ such that $M \subset F_{s(M)}$ for each $M \in \Sigma$. Select an ω-monotone function $\mathcal{E} : \Sigma \to \Sigma$ such that $\mathcal{E}(M) \cap B \neq \emptyset$ for each $B \in \mathcal{B}_{s(M)}$ and put $\mathcal{C} = \overline{\mathcal{E}}$. Set $\mathcal{B}_M = \mathcal{B}_{s(\mathcal{C}(M))}, F_M = F_{s(\mathcal{C}(M))}, \ldots$ and so on.

We claim that $g_M(X) = g_M(F_M)$. Fix $x \in X$. Set $\mathcal{B}_M^x = \{B \in \mathcal{B}_M : x \in B\}$. Select a decreasing base $\{B_n : n \in \omega\} \subset \mathcal{B}_M$ for x in (X, τ_M). For each $n \in \omega$ choose $y_n \in B_n \cap \mathcal{E}(\mathcal{C}(M)) = B_n \cap \mathcal{C}(M)$. Let y be an accumulation point of $\{y_n\}_{n\in\omega}$. Note that $y \in \overline{B \cap F_M}$ for each $B \in \mathcal{B}_M^x$. The continuity of the map $g_M : (X, \tau_M) \to Z_M$ imply that $g_M(y) = g_M(x)$. Since $y \in F_M$ the claim have been proved.

Every continuous one-to-one map from a countably compact space onto a Frechét-Urysohn space is a homeomorphism, so the topologies on F_M inherited from X and (X, τ_M) coincide. The compactness of F_M, the equality $g_M(X) = g_M(F_M)$ and the fact that g_M separates the points of F_M imply that $g_M \restriction_{F_M}$ is a homeomorphism onto its image. Then $r_M = (g_M \restriction_{F_M})^{-1} \circ g_M : X \to F_M$ is a retraction. Given $x \in X$, if y is as above, then $r_M(x) = y$ and so $r_M(x) \in \overline{B \cap F_M}$ for each $B \in \mathcal{B}_M^x$.

We shall prove that $\{r_M\}_{M\in\Sigma}$ is an r-skeleton in X. Condition 1 is immediate. Item 3 follows from Corollary 4.3 once we prove 2 and 4.

2. Choose $M, N \in \Sigma$ with $M \subset N$. From $F_M \subset F_N$ we deduce that $r_M = r_N \circ r_M$. If $g_M(x) \neq g_M(r_N(x))$ for some $x \in X$, then we can find disjoint sets $B_1 \in \mathcal{B}_M^x \subset \mathcal{B}_N^x$ and $B_2 \in \mathcal{B}_M^{r_N(x)} \subset \mathcal{B}_N^{r_N(x)}$, which is not possible since $r_N(x) \in \overline{B_1 \cap F_N}$. Thus $r_M = r_M \circ r_N$.

4. Select $M_0 \subset M_1 \subset \cdots$ in Σ and $N = \bigcup_{n\in\omega} M_n$. Pick $x \in X$. Suppose that $r_N(x) \neq \lim_{n\to\infty} r_{M_n}(x)$ and choose $B \in \mathcal{B}_N^{r_N(x)}$ such that $r_{M_n}(x) \notin \overline{B \cap F_N}$ for infinitely many $n \in \omega$. However $r_{M_n}(x) = r_{M_n}(r_N(x)) \in \overline{B \cap F_{M_n}} \subset \overline{B \cap F_N}$ because $B \in \mathcal{B}_{M_n}^{r_N(x)}$, for n large enough, a contradiction.

Finally, its is clear that $\{r_M\}_{M\in\Sigma}$ is full whenever $\{(F_s, \mathcal{B}_s)\}_{s\in\Gamma}$ is full. $\square$

12 c-skeletons in C_p-duality

In order to get a C_p-dual concept to c-skeleton, we introduce the following notion.

Definition 12.1. Let X be a space. Consider a family $\{(q_s, D_s)\}_{s\in\Gamma}$, where $q_s : X \to X_s$ is an $\mathbb{R}$-quotient map and D_s is a countable subset of X for each $s \in \Gamma$. We say that $\{(q_s, D_s)\}_{s\in\Gamma}$ is a q-*skeleton* on X if:

1. the set $q_s(D_s)$ is dense in X_s;
2. if $s, t \in \Gamma$ and $s \leq t$, then there exists a continuous onto map $p_{t,s} : X_t \to X_s$ such that $q_s = p_{t,s} \circ q_t$;
3. the assignment $s \to D_s$ is ω-monotone;
4. $C_p(X) = \overline{\bigcup_{s\in\Gamma} q_s^*(C_p(X_s))}$.

We say that the q-skeleton is *full* whenever $C_p(X) = \bigcup_{s\in\Gamma} q_s^*(C_p(X_s))$.

Proposition 12.2. *If X has a (full) c-skeleton, then $C_p(X)$ has a (full) q-skeleton.*

Proof. Let $\{(F_s, \mathcal{B}_s)\}_{s\in\Gamma}$ be a c-skeleton in X and set $\mathcal{B} = \bigcup\{\mathcal{B}_s\}_{s\in\Gamma}$. For each nonempty set $W \in \mathcal{W}(\mathcal{B})$ fix a map $d_W \in W$. For each $s \in \Gamma$ we set $X_s = \pi_{F_s}(C_p(X))$, $q_s = \pi_{F_s} : C_p(X) \to X_s$ and $D_s = \{d_N : N \in \mathcal{W}(\mathcal{B}_s)\}$. We assert that $\{(q_s, D_s)\}_{s\in\Gamma}$ is a q-skeleton on $C_p(X)$. Conditions 2 and 3 are easy to see. Let us verify 1 and 4.

1. Fix $s \in \Gamma$. The fact that $\mathcal{B}_s$ is a network for the topology generated by g_s, implies that $\mathcal{W}(\mathcal{B}_s)$ is a network for $g_s^*(C_p(Z_s))$ in $C_p(X)$. Then $q_s(\mathcal{W}(\mathcal{B}_s))$ is a network for $q_s(g_s^*(C_p(Z_s))) = (g \upharpoonright_{F_s})^*(C_p(Z_s))$ in X_s. Since $g_s \upharpoonright_{F_s}$ is a condensation, $(g \upharpoonright_{F_s})^*(C_p(Z_s))$ is dense in $C_p(F_s)$ and hence in X_s. It follows that $q_s(D_s)$ is dense X_s.

4. Set $Y = \bigcup_{s \in \Gamma} F_s$. It is enough to show that $\pi_Y^*(C_p(\pi_Y(C_p(X)))) \subset \bigcup_{s \in \Gamma} q_s^*(C_p(X_s))$. Select $\phi \in \pi_Y^*(C_p(\pi_Y(C_p(X))))$. Then $\phi = \psi \circ \pi_Y$ for some $\psi \in C_p(\pi_Y(C_p(X)))$. By applying the Factorisation Theorem, we can find $A \in [Y]^{\leq \omega}$ and a continuous map $\psi_A \in C_p(\pi_A(C_p(X)))$ such that $\psi = \psi_A \circ \pi_{Y,A} \upharpoonright_{\pi_Y(C_p(X))}$. Fix $s \in \Gamma$ such that $A \subseteq F_s$. Then

$$\phi = \psi \circ \pi_Y = \psi_A \circ \pi_{Y,A} \circ \pi_Y = \psi_A \circ \pi_{F_s,A} \circ q_s = q_s^*(\psi_A \circ \pi_{F_s,A} \upharpoonright_{X_s}) \in q_s^*(C_p(X_s)).$$

Finally, note that $\{(q_s, D_s)\}_{s \in \Gamma}$ is full whenever $\{(F_s, \mathcal{B}_s)\}_{s \in \Gamma}$ is full. $\qquad\square$

Proposition 12.3. *If X has a (full) q-skeleton, then $C_p(X)$ has a (full) c-skeleton.*

Proof. Let $\{(q_s, D_s)\}_{s \in \Gamma}$ be a q-skeleton in X. Fix $s \in \Gamma$. Since q_s is a quotient map, the set $F_s = q_s^*(C_p(X_s))$ is closed in $C_p(X)$. Besides, $\mathcal{B}_s = \mathcal{B}(D_s)$ is a countable family of canonical open subsets of $C_p(X)$. We shall prove that $\{(F_s, \mathcal{B}_s)\}_{s \in \Gamma}$ is a c-skeleton on $C_p(X)$. Items 3 and 4 are immediate. We will verify 1 and 2.

1. Fix $s \in \Gamma$. It is clear that $\mathcal{B}_s$ is a base for a topology τ_s on $C_p(X)$. Consider the Tychonoff space $Z_s = \pi_{D_s}(C_p(X))$. Then $g_s = \pi_{D_s} : (C_p(X), \tau_s) \to Z_s$ is a continuous and onto map. Note that the map $g_s \upharpoonright_{F_s} = \pi_{D_s} \circ q_s^* = (q_s \upharpoonright_{D_s})^* \circ \pi_{q_s(D_s)}$ is one to one. Thus g_s separates the points of F_s.

2. Fix $s, t \in \Gamma$ with $s \leq t$. The equality $q_s = p_{t,s} \circ q_t$ implies that

$$F_s = q_s^*(C_p(X_s)) = (p_{t,s} \circ q_t)^*(C_p(X_s)) = q_t^* \circ p_{t,s}^*(C_p(X_s)) \subseteq q_t^*(C_p(X_t)) = F_t.$$

Finally, observe that if the q-skeleton $\{(q_s, D_s)\}_{s \in \Gamma}$ is full, then the c-skeleton $\{(F_s, \mathcal{B}_s)\}_{s \in \Gamma}$ is full. $\qquad\square$

13   Generating q-skeletons

Lemma 13.1. *Let $f \in C(X, Y)$, let $\mathcal{B}_Y$ be a base for Y and consider the family $\mathcal{F} = \{f^{-1}(\overline{B}) : B \in \mathcal{B}_Y\}$. If $r \in C(X, X)$ and $r(F) \subset F$ for all $F \in \mathcal{F}$, then $f = f \circ r \in r^*(C_p(r(X)))$.*

Proof. Take any point $x \in X$; we must show that $f(x) = f(r(x))$. Given an arbitrary set $B \in \mathcal{B}_Y$ with $f(x) \in B$, observe that $x \in F = f^{-1}(\overline{B}) \in \mathcal{F}$. By our hypothesis, we have $r(x) \in F$, so $f(r(x)) \in \overline{B}$. Since B is arbitrary, we conclude that $f(x) = f(r(x))$. $\qquad\qquad\square$

Theorem 13.2. *If X has a (strong) full r-skeleton, then X has a (full) q-skeleton.*

Proof. We consider the case of strong r-skeletons. The proof for full r-skeletons is similar. Let $\{r_s\}_{s\in\Gamma}$ be a strong r-skeleton in X. Let $\Sigma = [\mathcal{CL}(X)]^{\leq\omega}$ the up-directed and σ-complete partially ordered set consisting of all countable collections of closed subsets of X. Using Lemma 2.4 we can construct an ω-monotone map $s : \Sigma \to \Gamma$ such that $r_{s(\mathcal{F})}(F) \subset F$ whenever $F \in \mathcal{F}$. For each $\mathcal{G} \in [\mathcal{CL}(X)]^{<\omega}$ fix a countable dense subset $D_{s(\mathcal{G})}$ of $r_{s(\mathcal{G})}(X)$. Given $\mathcal{F} \in \Sigma$ set $D_\mathcal{F} = \bigcup_{\mathcal{G}\in[\mathcal{F}]^{<\omega}} D_{s(\mathcal{G})}$ and $q_\mathcal{F} = r_{s(\mathcal{F})}$. Observe that $q_\mathcal{F}(X) = \overline{D}_\mathcal{F}$. It is standard to verify that $\{(q_\mathcal{F}, D_\mathcal{F})\}_{\mathcal{F}\in\Sigma}$ is a q-skeleton in X.

To prove that the q-skeleton is full, fix $f \in C_p(X)$. Consider the family $\mathcal{F} = \{f^{-1}(\overline{B}) : B \in \mathcal{B}_\mathbb{R}\} \in \Sigma$. By the above and Lemma 13.1, the map $q_\mathcal{F}$ satisfies $f = f \circ q_\mathcal{F} \in q_\mathcal{F}^*(C_p(q_\mathcal{F}(X)))$. $\qquad\qquad\square$

Lemma 13.3. *Let $A \subset C_p(X)$ for some space X. If A is a dense subset of $\Delta_A^*(C_p(\Delta_A(X)))$, then the map $\Delta_{\overline{A}}$ is $\mathbb{R}$-quotient.*

Proof. Given $x, y \in X$ note that $\Delta_{\overline{A}}(x) = \Delta_{\overline{A}}(y)$ if and only if $\Delta_A(x) = \Delta_A(y)$. So, the projection $p_A : \Delta_{\overline{A}}(X) \to \Delta_A(X)$ is a condensation. It follows that $p_A^*(C_p(\Delta_A(X)))$ is dense in $C_p(\Delta_{\overline{A}}(X))$. The equality $\Delta_A = p_A \circ \Delta_{\overline{A}}$ implies that $\Delta_A^*(C_p(\Delta_A(X)))$ is dense in $\Delta_{\overline{A}}^*(C_p(\Delta_{\overline{A}}(X)))$. By our hypothesis, we conclude that A is dense in $\Delta_{\overline{A}}^*(C_p(\Delta_{\overline{A}}(X)))$. In this way $\Delta_{\overline{A}}^*(C_p(\Delta_{\overline{A}}(X))) = \overline{A}$. Therefore $\Delta_{\overline{A}}$ is $\mathbb{R}$-quotient.

Theorem 13.4. *Every monotonically ω-stable space has a full q-skeleton.*

Proof. Let $\mathcal{N}$ be a monotonically ω-stable operator in X. Put $\Gamma = [C_p(X)]^{\leq\omega}$. Let $\mathcal{D} : \Gamma \to [X]^{\leq\omega}$ be an ω-monotone function such that $\mathcal{D}(A) \cap N \neq \emptyset$ for each nonempty set $N \in \mathcal{N}(A)$. For each $F \in [C_p(X)]^{<\omega}$ fix a countable dense subset E_F of $\Delta_F^*(C_p(\Delta_F(X)))$ containing F. For each $A \in \Gamma$ let $\mathcal{E}(A) = \bigcup_{F\in[A]^{<\omega}} E_F$. Then $\mathcal{E} : \Gamma \to \Gamma$ is ω-monotone. Let $\mathcal{A} = \overline{\mathcal{E}}$. Given $A \in \Gamma$ set $q_A = \Delta_{\overline{\mathcal{A}(A)}}$ and $D_A = \mathcal{D}(\mathcal{A}(A))$.

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Select $A \in \Gamma$. Since D_A intersects each nonempty element of $\mathcal{N}(\mathcal{A}(A))$, the set D_A is a dense subset of X endowed with the weak topology generated by $\overline{\mathcal{A}(A)}$. It follows that $q_A(D_A)$ is dense in $q_A(X)$.

Given $A \in \Gamma$, to see that q_A is an $\mathbb{R}$-quotient map, by Lemma 13.3 it is enough to show that $\mathcal{A}(A)$ is a dense in $\Delta_{\mathcal{A}(A)}^*(C_p(\Delta_{\mathcal{A}(A)}(X)))$. Select a nonempty open set $U = U_{x,B}$ in $C_p(X)$, for some $x \in X^n$, $B \in \mathcal{B}_{\mathbb{R}}^n$ and $n \in \omega$. Pick a function $f \in U \cap \Delta_{\mathcal{A}(A)}^*(C_p(\Delta_{\mathcal{A}(A)}(X)))$. Select a map $g \in C_p(\Delta_{\mathcal{A}(A)}(X))$ satisfying $f = g \circ \Delta_{\mathcal{A}(A)}$. Choose $F \in [\mathcal{A}(A)]^{<\omega}$ so that $\Delta_F(x(i)) = \Delta_F(x_j)$ if and only if $\Delta_{\mathcal{A}(A)}(x_i) = \Delta_{\mathcal{A}(A)}(x_j)$ for all $i, j \in n$. Select $h \in C_p(\Delta_F(X))$ such that $h(\Delta_F(x(i))) = g(\Delta_{\mathcal{A}(A)}(x_i)) = f(x(i))$ for all $i \in n$. Then $h \circ \Delta_F \in U \cap \Delta_F^*(C_p(\Delta_F(X)))$. Since the set E_F is dense in $\Delta_F^*(C_p(\Delta_F(X)))$ and $E_F \subset \mathcal{E}(\mathcal{A}(A)) \subset \mathcal{A}(A)$, we must have that $\emptyset \neq U \cap E_F \subset U \cap \mathcal{A}(A)$. Therefore $\mathcal{A}(A)$ is a dense in $\Delta_{\mathcal{A}(A)}^*(C_p(\Delta_{\mathcal{A}(A)}(X)))$.

Finally, it is easy to verify that $\{(q_A, D_A)\}_{A \in \Gamma}$ is a full q-skeleton in X. $\square$

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