Strictly convex norms and topology

41st Winter School in Abstract Analysis, Kácov

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Strictly convex norms

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Problem 1.3 (Lindenstrauss 1975/6)

Characterize those Banach spaces X which admit an equivalent strictly convex norm.

Strictly convex *dual* norms

Problem 1.4

When does a dual Banach space admit a strictly convex *dual* norm?

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Theorem 1.5 (Šmulyan 1940)

If $||\cdot||$ is a strictly convex *dual* on X^*, then the predual norm on X is Gâteaux smooth.

Definition of $(*)$

Definition 2.1 (Orihuela, Troyanski, S 2012)

A topological space X has $(*)$ if there are families $\mathscr{U}_n$, $n \in \mathbb{N}$, of open sets, such that for any $x, y \in X$, there is $n \in \mathbb{N}$ satisfying

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Example 2.2

1. $X = \mathbb{R}$, $\mathscr{U}_n = \{$open intervals of length $n^{-1}\}$.
2. Spaces having G_δ-diagonals have (∗).
3. There are many compact non-metrizable spaces having (∗).

Motivation for (∗)

Proposition 2.3 (OTS 2012)

If a dual Banach space X^* admits a strictly convex *dual* norm, then (B_{X^*}, w^*) has (∗).

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Theorem 2.4 (OTS 2012)

The space X^* admits a strictly convex dual norm if and only if (B_{X^*}, w^*) has $(*)$ *with slices*.

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Given *compact* (B_{X^*}, w^*), to what extent can we do without the geometry, i.e. without slices?

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Theorem 2.7 (OTS 2012)

If K is compact and scattered, then $C(K)^*$ admits a strictly convex dual norm if and only if K has (∗).

(∗) in topological context

A menagerie of compact spaces

Corson ⇒ Valdivia
⇗
Talagrand ⇒ Gul'ko
⇗ ⇘
metrisable ⇒ Eberlein descriptive ⇒ Gruenhage
⇓ ⇘ ⇗ ⇓
Rosenthal σ-discrete ⇒ Namioka-Phelps (∗)
⇓ ⇓ ⇓
scattered ⇒ Radon-Nikodým ⇒ fragmentable

All implications above are strict.

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$$
\begin{array}{ccccccccc}
 & & & & & \nearrow & \text{Corson} \Rightarrow \text{Valdivia} & & \\
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\text{metrisable} \Rightarrow \text{Eberlein} & & & & & \searrow & \text{descriptive} \Rightarrow \text{Gruenhage} & & \\
\Downarrow & & & \searrow & & \nearrow & & & \Downarrow \\
\text{Rosenthal} & & \sigma\text{-discrete} \Rightarrow \text{Namioka-Phelps} & & & & & & (*) \\
 & & \Downarrow \qquad\qquad \Downarrow & & & & & & \Downarrow \\
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\end{array}
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In these lectures, we focus on the implications

$$\text{Gruenhage} \;\Rightarrow\; (*) \;\Rightarrow\; \text{fragmentable}.$$

Gruenhage spaces

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Definition 3.1 (Gruenhage 1987)

A topological space X is called *Gruenhage* if there are families $\mathscr{U}_n$, $n \in \mathbb{N}$, of open subsets of X, and sets R_n, $n \geq 1$, with the property that

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1. if $x, y \in X$, then there is $n_0 \in \mathbb{N}$ and $U \in \mathscr{U}_{n_0}$, such that $\{x, y\} \cap U$ is a singleton, and

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1. if $x, y \in X$, then there is $n_0 \in \mathbb{N}$ and $U \in \mathscr{U}_{n_0}$, such that $\{x, y\} \cap U$ is a singleton, and
2. $V \cap W = R_n$ whenever $V, W \in \mathscr{U}_n$ are distinct.

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Theorem 3.2 (S 2009)

If K is Gruenhage compact then $C(K)^*$ admits a strictly convex dual norm.

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If $K \subseteq (X^*, w^*)$ is Gruenhage compact and satisfies $\overline{\text{span}}^{\|\cdot\|}(K) = X^*$, then X^* admits a strict convex dual norm.

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Proposition 3.4 (OTS 2012)

If X is Gruenhage, then X has $(*)$.

Fragmentable spaces

Definition 3.5 (Jayne, Rogers 1985)

A topological space X is *fragmentable* if there exists a metric d on X (not necessarily related to the topology on X), with the property that whenever $\varepsilon > 0$ and $E \subseteq X$ is non-empty, there exists an open set $U \subseteq X$, such that

$$E \cap U \neq \varnothing \quad \text{and} \quad d\text{-}\operatorname{diam}(E \cap U) < \varepsilon.$$

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Proposition 3.6 (OTS 2012)

If X has ($*$), then X is fragmentable.

Example 3.7 (OTS 2012)

The scattered (hence fragmentable) space ω_1 does not have ($*$).

Countable compactness and countable tightness

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Corollary 3.10 (OTS 2012)

If L is locally compact and has $(*)$, then $L \cup \{\infty\}$ is countably tight and sequentially closed subsets of $L \cup \{\infty\}$ are closed.

Examples of spaces having ($*$)

Example 3.10 (OTS 2012)

1. (CH) Kunen's compact S-space $\mathcal{K}$ is Gruenhage. In particular, $C(\mathcal{K})^*$ admits a strictly convex dual norm.

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Proposition 3.11 (MA) (OTS 2012)

If L is locally compact, locally countable and has (∗), and $\mathrm{card}(L) < \mathfrak{c}$, then L is Gruenhage.

A ZFC example of a non-Gruenhage space having $(*)$

Definition 4.1 (Kurepa)

Let Λ be the tree of injective functions $t : \alpha \longrightarrow \omega$, where α is a (countable) ordinal, and $\omega \setminus \operatorname{ran} t$ is infinite.

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Theorem 4.3 (S 2012)

The space D is locally compact, scattered, non-Gruenhage, and has a G_δ-diagonal.

References

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