

Strictly convex norms and topology

41st Winter School in Abstract Analysis, Kácov

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Strictly convex norms

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Problem 1.3 (Lindenstrauss 1975/6)

Characterize those Banach spaces X which admit an equivalent strictly convex norm.

Strictly convex *dual* norms

Problem 1.4

When does a dual Banach space admit a strictly convex *dual* norm?

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Theorem 1.5 (Šmulyan 1940)

If $\|\cdot\|$ is a strictly convex *dual* on X^* , then the predual norm on X is Gâteaux smooth.

Definition of (*)

Definition 2.1 (Orihuela, Troyanski, S 2012)

A topological space X has (*) if there are families \mathcal{U}_n , $n \in \mathbb{N}$, of open sets, such that for any $x, y \in X$, there is $n \in \mathbb{N}$ satisfying

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Example 2.2

- 1 $X = \mathbb{R}$, $\mathcal{U}_n = \{\text{open intervals of length } n^{-1}\}$.
- 2 Spaces having G_δ -diagonals have (*).
- 3 There are many compact non-metrizable spaces having (*).

Motivation for (*)

Proposition 2.3 (OTS 2012)

If a dual Banach space X^* admits a strictly convex *dual* norm, then (B_{X^*}, w^*) has (*).

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Theorem 2.4 (OTS 2012)

The space X^* admits a strictly convex dual norm if and only if (B_{X^*}, w^*) has (*) *with slices*.

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Given *compact* (B_{X^*}, w^*) , to what extent can we do without the geometry, i.e. without slices?

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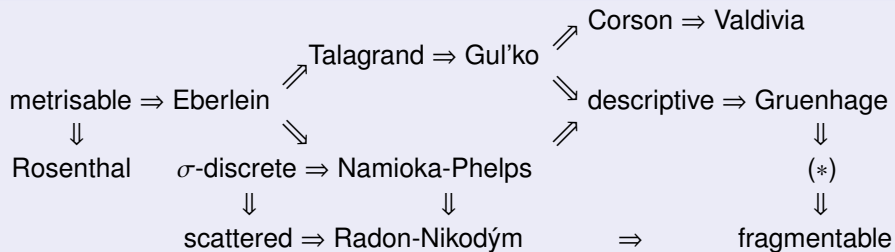
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Theorem 2.7 (OTS 2012)

If K is compact and scattered, then $C(K)^*$ admits a strictly convex dual norm if and only if K has (*).

(*) in topological context

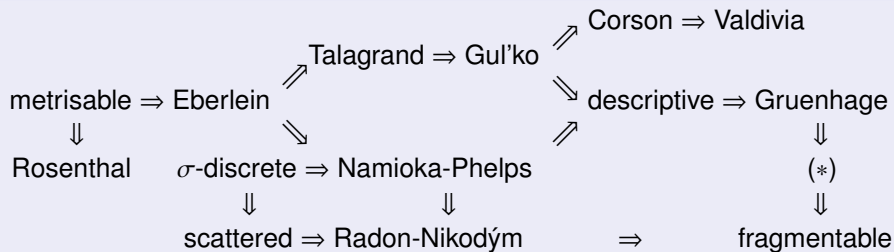
A menagerie of compact spaces



All implications above are strict.

(*) in topological context

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In these lectures, we focus on the implications

$$\text{Gruenhage} \Rightarrow (*) \Rightarrow \text{fragmentable}.$$

Gruenhagen spaces

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Definition 3.1 (Gruenhage 1987)

A topological space X is called *Gruenhage* if there are families \mathcal{U}_n , $n \in \mathbb{N}$, of open subsets of X , and sets R_n , $n \geq 1$, with the property that

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- 1 if $x, y \in X$, then there is $n_0 \in \mathbb{N}$ and $U \in \mathcal{U}_{n_0}$, such that $\{x, y\} \cap U$ is a singleton, and
- 2 $V \cap W = R_n$ whenever $V, W \in \mathcal{U}_n$ are distinct.

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Theorem 3.2 (S 2009)

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Corollary 3.3 (S 2009)

If $K \subseteq (X^*, w^*)$ is Gruenhagen compact and satisfies $\overline{\text{span}}^{\|\cdot\|}(K) = X^*$, then X^* admits a strict convex dual norm.

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Proposition 3.4 (OTS 2012)

If X is Gruenhagen, then X has (*).

Fragmentable spaces

Definition 3.5 (Jayne, Rogers 1985)

A topological space X is *fragmentable* if there exists a metric d on X (not necessarily related to the topology on X), with the property that whenever $\varepsilon > 0$ and $E \subseteq X$ is non-empty, there exists an open set $U \subseteq X$, such that

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Proposition 3.6 (OTS 2012)

If X has (*), then X is fragmentable.

Example 3.7 (OTS 2012)

The scattered (hence fragmentable) space ω_1 does not have (*).

Countable compactness and countable tightness

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Corollary 3.10 (OTS 2012)

If L is locally compact and has (*), then $L \cup \{\infty\}$ is countably tight and sequentially closed subsets of $L \cup \{\infty\}$ are closed.

Examples of spaces having (*)

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- 1 (CH) Kunen's compact S-space \mathcal{K} is Gruenhage. In particular, $C(\mathcal{K})^*$ admits a strictly convex dual norm.

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Proposition 3.11 (MA) (OTS 2012)

If L is locally compact, locally countable and has (*), and $\text{card}(L) < \mathfrak{c}$, then L is Gruenhage.

A ZFC example of a non-Gruenhagen space having (*)

Definition 4.1 (Kurepa)

Let Λ be the tree of injective functions $t : \alpha \rightarrow \omega$, where α is a (countable) ordinal, and $\omega \setminus \text{ran } t$ is infinite.

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Theorem 4.3 (S 2012)

The space D is locally compact, scattered, non-Gruenhagen, and has a G_δ -diagonal.

References

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