Definable cardinals just beyond \mathbb{R}/\mathbb{Q}

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Winter school in abstract analysis Sporthotel Kácov January 13th – 16th, 2013

Benjamin Miller Westfälische Wilhelms-Universität Münster

Part I

Definable cardinality

According to the usual notion of cardinality, one set is smaller than another iff there is an injection of the former into the latter. According to the usual notion of cardinality, one set is smaller than another iff there is an injection of the former into the latter.

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In this first lecture, we will review some of the basic theory behind these developments.

Polish spaces

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Theorem 1

Every uncountable Polish space is Borel isomorphic to \mathbb{R} .

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For simplicity, however, we will focus on Borel sets.

I. Definable cardinality Morphisms

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Definition

A reduction of E to F is a homomorphism from E to F sending E-unrelated points to F-unrelated points.

The definable analog of the continuum hypothesis



Theorem 2 (Silver)

Suppose that X is a Polish space and E is a Borel equivalence relation on X. Then exactly one of the following holds:

- **(**) There is a Borel reduction of E to the equality relation on \mathbb{N} .
- 2 There is a Borel reduction of the equality relation on \mathbb{R} to E.

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Definition

The Vitali equivalence relation is the relation $E_{\mathbb{Q}}$ on \mathbb{R} given by

$$x E_{\mathbb{Q}} y \Leftrightarrow x - y \in \mathbb{Q}.$$

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Theorem 3 (Harrington-Kechris-Louveau)

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- **2** There is a Borel reduction of $E_{\mathbb{O}}$ to E.

I. Definable cardinality Beyond Vitali equivalence

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The orbit equivalence relation associated with a group action $\Gamma \curvearrowright X$ is the relation E_{Γ}^{χ} on X given by $x E_{\Gamma}^{\chi} y \Leftrightarrow \exists \gamma \in \Gamma \ \gamma \cdot x = y$.

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The orbit equivalence relation associated with a group action $\Gamma \curvearrowright X$ is the relation E_{Γ}^{X} on X given by $x E_{\Gamma}^{X} y \Leftrightarrow \exists \gamma \in \Gamma \ \gamma \cdot x = y$.

Definition

A measure μ is Γ -invariant if $\forall B \subseteq X \forall \gamma \in \Gamma \ \mu(B) = \mu(\gamma(B))$.



Theorem 5 (Ornstein-Weiss)

Suppose that X is a Polish space, $\Gamma \curvearrowright X$ is a free Borel action of a countable group, and μ is a Γ -invariant Borel probability measure on X. Then Γ is amenable iff E_{Γ}^X is μ -hyperfinite.

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Question

Suppose that X is a Polish space and E is a countable Borel equivalence relation on X which is not Borel reducible to the Vitali equivalence relation. Is there a Borel probability measure μ on X for which there is no μ -measurable reduction?

I. Definable cardinality Global structure



Theorem 6 (Woodin, Louveau-Velickovic)

There is a family of uncountably many Borel equivalence relations on Polish spaces which are pairwise incomparable under Baire measurable (and therefore Borel) reducibility.



Theorem 7 (Adams-Kechris)

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Theorem 8 (Adams-Kechris, Gao)

There is a Borel reduction of every analytic quasi-order on a Polish space into (codes for) the Borel reducibility quasi-order on the space of countable Borel equivalence relations.

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Theorem 11 (Adams, Hjorth)

There are countable treeable Borel equivalence relations $E \subseteq F$ on a Polish space which are incomparable under Borel reducibility.

Lurking beneath the results for countable Borel equivalence relations are sophisticated rigidity theorems originating in the ergodictheoretic study of actions of linear algebraic groups.

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In the remaining lectures, we will sketch significantly simpler proofs of strengthenings of many of these results.

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Finally, we will use this separability to establish the main results.

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Part I

Local rigidity

I. Local rigidity Basic definitions

Definition

The difference set associated with functions $\varphi \colon X \to Y$ and $\psi \colon X \to Y$ is the set $D(\varphi, \psi)$ given by

$$D(\varphi,\psi) = \{x \in X \mid \varphi(x) \neq \psi(x)\}.$$

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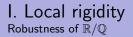
Definition

Given $\Gamma \curvearrowright Y$, we say that a homomorphism $\varphi \colon X \to Y$ from E to E_{Γ}^{Y} is ρ -invariant if $\varphi(x_1) = \rho(x_1, x_2) \cdot \varphi(x_2)$ for all $x_1, x_2 \in X$.

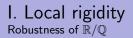
We say that $\Gamma \curvearrowright Y$ is locally rigid if whenever X is a Polish space, E is a countable Borel equivalence relation on X, $\rho: E \to \Gamma$ is a Borel function, and φ, ψ are ρ -invariant countable-to-one Borel homomorphisms from E to E_{Γ}^{Y} , the equivalence relation $E \upharpoonright D(\varphi, \psi)$ is Borel reducible to $E_{\mathbb{Q}}$.

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Today we will prove that $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$ is locally rigid.



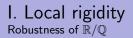
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Definition

A Borel equivalence relation E is hyperfinite if there are finite Borel subequivalence relations $F_0 \subseteq F_1 \subseteq \cdots$ such that $E = \bigcup_{n \in \mathbb{N}} F_n$.



A Borel equivalence relation is smooth if it is Borel reducible to the equality relation on $\mathbb{R}.$

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Definition

A Borel equivalence relation is hypersmooth if there are smooth Borel equivalence relations $F_0 \subseteq F_1 \subseteq \cdots$ such that $E = \bigcup_{n \in \mathbb{N}} F_n$.



Theorem 1 (Dougherty-Jackson-Kechris, Slaman-Steel, Weiss)

Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then the following are equivalent:

- **①** There is a Borel reduction of E to $E_{\mathbb{Q}}$.
- **2** The equivalence relation E is hyperfinite.
- **③** The equivalence relation E is hypersmooth.
- **4** There is a Borel action $\mathbb{Z} \curvearrowright X$ such that $E = E_{\mathbb{Z}}^X$.



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Proposition 2 (Dougherty-Jackson-Kechris)

Suppose that X is a Polish space and E is a Borel equivalence relation on X. Then the family of Borel sets on which E is hyperfinite is closed under countable unions.

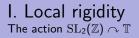


Proposition 2 (Dougherty-Jackson-Kechris)

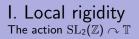
Suppose that X is a Polish space and E is a Borel equivalence relation on X. Then the family of Borel sets on which E is hyperfinite is closed under countable unions.

Proposition 3 (Dougherty-Jackson-Kechris)

Suppose that X and Y are Polish spaces, E and F are countable Borel equivalence relations, F is hyperfinite, and there is a countableto-one Borel homomorphism from E to F. Then E is hyperfinite.



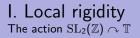
We identify $\mathbb T$ with the set of rays through $\mathbb R^2$ rooted at the origin.



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Definition

Let $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ denote the action induced by $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$.





Proposition 4 (Jackson-Kechris-Louveau)

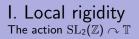
There is a Borel reduction of $E_{SL_2(\mathbb{Z})}^{\mathbb{T}}$ to $E_{\mathbb{Q}}$.





Proposition 5 (Conley-M)

Only countably many points of \mathbb{T} have non-trivial stabilizers under $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$, and they are all infinite cyclic.



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Proof

Suppose that $\theta \in \mathbb{T}$.

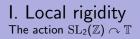
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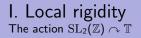
Proof

Suppose that $\theta \in \mathbb{T}$.

There are now two cases, depending on whether $\theta \cap \mathbb{Z}^2 \neq \emptyset$.



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Proof of Proposition 5 (continued)

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Suppose that A is in the stabilizer of θ under $SL_2(\mathbb{Z}) \curvearrowright \mathbb{T}$.

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Then $Av = \lambda v$ for some $\lambda > 0$.

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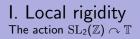
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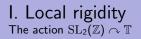
Suppose that A is in the stabilizer of θ under $SL_2(\mathbb{Z}) \curvearrowright \mathbb{T}$.

Then $Av = \lambda v$ for some $\lambda > 0$.

Minimality ensures that $\lambda = 1$, thus the stabilizers of θ and v under $SL_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ and $SL_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$ are one and the same.



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Then $B \in SL_2(\mathbb{Z})$ and Bv = (1, 0).

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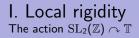
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So conjugation by *B* yields an isomorphism of the stabilizer of *v* under $SL_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ with that of (1, 0).

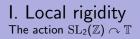


The latter consists of the upper unitriangular matrices in $SL_2(\mathbb{Z})$.

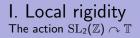


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And this group is trivially infinite cyclic.

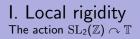


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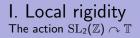
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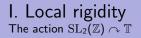


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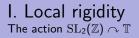
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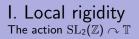
So the stabilizer of v under $SL_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$ is trivial.

Let Λ denote the set of eigenvalues of matrices fixing θ .



Lemma 6

The group Λ is cyclic.

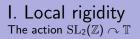


Lemma 6

The group Λ is cyclic.

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We need only show that Λ is not dense.



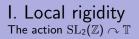
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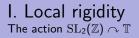
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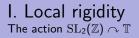
Suppose that v is an eigenvalue of $A \in SL_2(\mathbb{Z})$ with eigenvalue λ .

Let μ denote the other eigenvalue.



Proof of Lemma 6 (continued)

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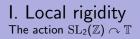
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And the set of such λ cannot be dense.



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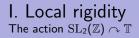
So $A^n B^{-1}$ is the identity matrix, thus $B = A^n$.

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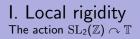
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So $A^n B^{-1}$ is the identity matrix, thus $B = A^n$.

Hence the stabilizer of θ under $SL_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ is infinite cyclic.

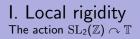


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Note that each non-trivial $A \in SL_2(\mathbb{Z})$ fixes at most two $\theta \in \mathbb{T}$.

As $SL_2(\mathbb{Z})$ is countable, it follows that only countably many $\theta \in \mathbb{T}$ have non-trivial stabilizers.





Proposition 7 (Conley-M)

Suppose that X is a Polish space and $\operatorname{SL}_2(\mathbb{Z}) \curvearrowright X$ is Borel. Then there is a Borel reduction of $E_{\operatorname{SL}_2(\mathbb{Z})}^{\mathbb{T} \times X}$ to $E_{\mathbb{Q}}$.

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Proof

Let \mathbb{T}' denote the $\mathrm{SL}_2(\mathbb{Z})$ -invariant Borel set consisting of all $\theta \in \mathbb{T}$ whose stabilizers under $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ are trivial.

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Any Borel
$$\mathbb{Z}$$
-action generating $E_{\mathrm{SL}_2(\mathbb{Z})}^{\mathbb{T}'}$ induces one for $E_{\mathrm{SL}_2(\mathbb{Z})}^{\mathbb{T}' \times X}$.

Proposition 7 (Conley-M)

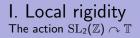
Suppose that X is a Polish space and $\operatorname{SL}_2(\mathbb{Z}) \frown X$ is Borel. Then there is a Borel reduction of $E_{\operatorname{SL}_2(\mathbb{Z})}^{\mathbb{T} \times X}$ to $E_{\mathbb{Q}}$.

Proof

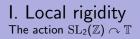
Let \mathbb{T}' denote the $\mathrm{SL}_2(\mathbb{Z})$ -invariant Borel set consisting of all $\theta \in \mathbb{T}$ whose stabilizers under $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ are trivial.

Any Borel
$$\mathbb{Z}$$
-action generating $E_{\mathrm{SL}_2(\mathbb{Z})}^{\mathbb{T}'}$ induces one for $E_{\mathrm{SL}_2(\mathbb{Z})}^{\mathbb{T}' \times X}$.

So there is a Borel reduction of $E_{SL_2(\mathbb{Z})}^{\mathbb{T}' \times X}$ to $E_{\mathbb{Q}}$.

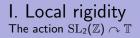


Suppose now that $\theta \in \mathbb{T} \setminus \mathbb{T}'$.



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Let Z denote the stabilizer of θ under $SL_2(\mathbb{Z}) \curvearrowright \mathbb{T}$.



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Then
$$E_{\operatorname{SL}_2(\mathbb{Z})}^{\mathbb{T}\times X} \upharpoonright (\{\theta\} \times X) = E_Z^{\mathbb{T}\times X} \upharpoonright (\{\theta\} \times X).$$

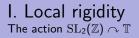
Proof of Proposition 7 (continued)

Suppose now that $\theta \in \mathbb{T} \setminus \mathbb{T}'$.

Let Z denote the stabilizer of θ under $SL_2(\mathbb{Z}) \curvearrowright \mathbb{T}$.

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$$E_{\mathrm{SL}_2(\mathbb{Z})}^{\mathbb{T}\times X} \upharpoonright (\{\theta\} \times X) = E_Z^{\mathbb{T}\times X} \upharpoonright (\{\theta\} \times X).$$

And the latter is Borel reducible to $E_{\mathbb{O}}$.



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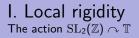
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And the latter is Borel reducible to $E_{\mathbb{Q}}$.

So
$$E_{\mathrm{SL}_2(\mathbb{Z})}^{\mathbb{T} imes X}$$
 is Borel reducible to $E_{\mathbb{Q}}$.

 \boxtimes



Definition

We identify \mathbb{T}^2 with $\mathbb{R}^2/\mathbb{Z}^2$.

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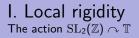
Definition

Let $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}^2$ denote the action induced by $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$.



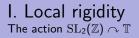
Proposition 8 (Conley-M)

Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, \mathcal{I} is a σ -ideal on X, $\rho \colon E \to \mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ is a Borel function, φ and ψ are ρ -invariant Borel homomorphisms from E to $E_{\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})}^{\mathbb{R}^2}$, and φ is \mathcal{I} -to-one. Then there is an \mathcal{I} -to-one Borel homomorphism from $E \upharpoonright D(\varphi, \psi)$ to $E_{\mathrm{SL}_2(\mathbb{Z})}^{\mathbb{T} \times \mathbb{T}^2}$.



Proof

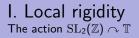
Define $\sigma \colon E \upharpoonright D(\varphi, \psi) \to \operatorname{SL}_2(\mathbb{Z})$ by $\sigma(x, y) = \operatorname{proj}_{\operatorname{SL}_2(\mathbb{Z})} \circ \rho(x, y)$.



Proof

Define $\sigma \colon E \upharpoonright D(\varphi, \psi) \to \operatorname{SL}_2(\mathbb{Z})$ by $\sigma(x, y) = \operatorname{proj}_{\operatorname{SL}_2(\mathbb{Z})} \circ \rho(x, y)$.

Note that $\operatorname{proj}_{\mathbb{T}^2} \circ \varphi(x) \upharpoonright D(\varphi, \psi)$ is σ -invariant.



Proof

Define $\sigma \colon E \upharpoonright D(\varphi, \psi) \to \operatorname{SL}_2(\mathbb{Z})$ by $\sigma(x, y) = \operatorname{proj}_{\operatorname{SL}_2(\mathbb{Z})} \circ \rho(x, y)$.

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Define $\pi \colon D(\varphi, \psi) \to \mathbb{T}$ by $\pi(x) = \operatorname{proj}_{\mathbb{T}}(\varphi(x) - \psi(x))$.

Lemma 9

The function π is σ -invariant.

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Proof

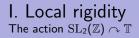
If $x_1, x_2 \in D(arphi, \psi)$ are *E*-related, then

$$\pi(x_1) = \operatorname{proj}_{\mathbb{T}}(\varphi(x_1) - \psi(x_1))$$

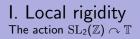
= $\operatorname{proj}_{\mathbb{T}}(\rho(x_1, x_2) \cdot \varphi(x_2) - \rho(x_1, x_2) \cdot \psi(x_2))$
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= $\sigma(x_1, x_2) \cdot \operatorname{proj}_{\mathbb{T}}(\varphi(x_2) - \psi(x_2))$
= $\sigma(x_1, x_2) \cdot \pi(x_2),$

thus π is σ -invariant.

 \square

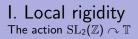


So both $\operatorname{proj}_{\mathbb{T}^2} \circ \varphi \upharpoonright D(\varphi, \psi)$ and π are σ -invariant.



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Thus their product is a homomorphism to $E_{SL_2(\mathbb{Z})}^{\mathbb{T} \times \mathbb{T}^2}$.

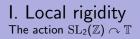


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Thus their product is a homomorphism to
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.

As φ is $\mathcal I\text{-to-one, so too is the product.}$

 \boxtimes





Theorem 10 (Conley-M)

The action $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$ is locally rigid.

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Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\rho: E \to \mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ is Borel, and φ and ψ are countable-to-one Borel homomorphisms from E to $E_{\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})}^{\mathbb{R}^2}$.

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It then follows that there is a countable-to-one Borel homomorphism from $E \upharpoonright D(\varphi, \psi)$ to $E_{\mathbb{Q}}$.

I. Local rigidity The action $\operatorname{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$

Theorem 10 (Conley-M)

The action $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$ is locally rigid.

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It then follows that there is a countable-to-one Borel homomorphism from $E \upharpoonright D(\varphi, \psi)$ to $E_{\mathbb{Q}}$.

So $E \upharpoonright D(\varphi, \psi)$ is Borel reducible to $E_{\mathbb{Q}}$.

Definable cardinals just beyond \mathbb{R}/\mathbb{Q}

4

Winter school in abstract analysis Sporthotel Kácov January 17th, 2013

Benjamin Miller Westfälische Wilhelms-Universität Münster

Part III

Separability

III. Separability A function space

Definition

We use $L(X, \mu, Y)$ to denote the family of all μ -measurable functions $\varphi: D \to Y$ with μ -positive domains $D \subseteq X$.

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Definition

We view $L(X, \mu, Y)$ as a pseudo-metric space, equipped with the pseudo-metric d_{μ} given by $d_{\mu}(\varphi, \psi) = \mu(D(\varphi, \psi))$.

III. Separability A function space

Proposition 1

Suppose that X and Y are Polish spaces, μ is a finite Borel measure on X, and $\mathscr{L} \subseteq L(X, \mu, Y)$. Then \mathscr{L} is separable iff there is a Borel set $R \subseteq X \times Y$, whose vertical sections are all countable, such that

 $\forall \varphi \in \mathscr{L} \ \mu(\{x \in \operatorname{dom}(\varphi) \mid \neg x \ R \ \varphi(x)\}) = 0.$

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$$\forall \varphi \in \mathscr{L} \ \mu(\{x \in \operatorname{dom}(\varphi) \mid \neg x \ R \ \varphi(x)\}) = 0.$$

Proof

Suppose that $R \subseteq X \times Y$ is a Borel set, whose vertical sections are all countable, such that $\forall \varphi \in \mathscr{L} \ \mu(\{x \in \operatorname{dom}(\varphi) \mid \neg x \ R \ \varphi(x)\}) = 0.$



Proof of Proposition 1 (continued)

Fix a countable algebra \mathscr{A} of Borel subsets of X such that for all Borel sets $B \subseteq X$ and $\epsilon > 0$, there exists $A \in \mathscr{A}$ with $\mu(A \bigtriangleup B) \le \epsilon$.

Proof of Proposition 1 (continued)

Fix a countable algebra \mathscr{A} of Borel subsets of X such that for all Borel sets $B \subseteq X$ and $\epsilon > 0$, there exists $A \in \mathscr{A}$ with $\mu(A \bigtriangleup B) \le \epsilon$.

Fix a countable family \mathscr{F} of Borel functions $f: D \to Y$, with Borel domains $D \subseteq X$, such that $R = \bigcup_{f \in \mathscr{F}} \operatorname{graph}(f)$.

Proof of Proposition 1 (continued)

Fix a countable algebra \mathscr{A} of Borel subsets of X such that for all Borel sets $B \subseteq X$ and $\epsilon > 0$, there exists $A \in \mathscr{A}$ with $\mu(A \bigtriangleup B) \le \epsilon$.

Fix a countable family \mathscr{F} of Borel functions $f: D \to Y$, with Borel domains $D \subseteq X$, such that $R = \bigcup_{f \in \mathscr{F}} \operatorname{graph}(f)$.

One obtains a dense set by considering $(f_1 \upharpoonright A_1) \cup \cdots \cup (f_n \upharpoonright A_n)$, where $n \in \mathbb{N}$, $A_1, \ldots, A_n \in \mathscr{A}$, and $f_1, \ldots, f_n \in \mathscr{F}$.

III. Separability

Proposition 2

Suppose that X, Y, and Z are Polish spaces, μ is a finite Borel measure on X, $\mathscr{L}_{XY} \subseteq L(X, \mu, Y)$, $\mathscr{L}_{XZ} \subseteq L(X, \mu, Z)$, there is a countable-to-one Borel function $f: Y \to Z$ with $f \circ \mathscr{L}_{XY} \subseteq \mathscr{L}_{XZ}$, and \mathscr{L}_{XZ} is separable. Then \mathscr{L}_{XY} is separable.

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Proof

Fix a Borel set $S \subseteq Y \times Z$, whose vertical sections are all countable, such that $\forall \varphi \in \mathscr{L}_{XZ} \ \mu(\{x \in \operatorname{dom}(\varphi) \mid \neg x \ S \ \varphi(x)\}) = 0.$

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Proof

Fix a Borel set $S \subseteq Y \times Z$, whose vertical sections are all countable, such that $\forall \varphi \in \mathscr{L}_{XZ} \ \mu(\{x \in \operatorname{dom}(\varphi) \mid \neg x \ S \ \varphi(x)\}) = 0.$

Set
$$R = \{(x, y) \in X \times Y \mid x S f(y)\}.$$

III. Separability Homomorphisms

Definition

Let $\operatorname{Hom}_{\leq\aleph_0\text{-to-1}}(E,\mu,F)$ denote the set of countable-to-one homomorphisms $\varphi \in L(X,\mu,Y)$ from $E \upharpoonright \operatorname{dom}(\varphi)$ to F.

Proposition 3

Suppose that X, Y, and Z are Polish spaces, E, F, and G are countable Borel equivalence relations on X, Y, and Z, μ is a finite Borel measure on X, there is a countable-to-one Borel homomorphism $\varphi: Y \to Z$ from F to G, and $\operatorname{Hom}_{\leq\aleph_0-\text{to-1}}(E,\mu,G)$ is separable. Then $\operatorname{Hom}_{\leq\aleph_0-\text{to-1}}(E,\mu,F)$ is separable.

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Proof

By the previous proposition, it is clearly sufficient to observe that $\varphi \circ \operatorname{Hom}_{\leq \aleph_0-\text{to-1}}(E, \mu, F) \subseteq \operatorname{Hom}_{\leq \aleph_0-\text{to-1}}(E, \mu, G).$

III. Separability

Definition

We say that *E* is μ -nowhere hyperfinite if there is no μ -positive Borel set $B \subseteq X$ with the property that $E \upharpoonright B$ is hyperfinite.

Proposition 4

Suppose that X and Y are Polish spaces, E and F are countable Borel equivalence relations on X and Y, μ is a finite Borel measure on X for which E is μ -nowhere hyperfinite, and F is the orbit equivalence relation of a locally rigid Borel action $\Gamma \curvearrowright Y$ of a countable group. Then $\operatorname{Hom}_{\leq\aleph_0-\text{to-1}}(E,\mu,F)$ is separable.

Proof

Fix real numbers $\epsilon_n > 0$ such that $\sum_{n \in \mathbb{N}} \epsilon_n < \infty$.

III. Separability

Proof

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Fix an increasing sequence of Borel sets $R_n \subseteq X \times X$ such that $E = \bigcup_{n \in \mathbb{N}} R_n$ and each vertical section of R_n has cardinality $\leq n$.

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For each $n \in \mathbb{N}$, set $\nu_n = (\mu \times \mu_c) \upharpoonright R_n$.

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Set
$$\nu = (\mu \times \mu_c) \upharpoonright E$$
.

III. Separability

Proof of Proposition 4 (continued)

Fix countable dense sets $\mathscr{D}_n \subseteq L(R_n, \nu_n, \Gamma)$.

Proof of Proposition 4 (continued)

Fix countable dense sets $\mathscr{D}_n \subseteq L(R_n, \nu_n, \Gamma)$.

Let \mathscr{D}'_n denote the set of $\rho \in \mathscr{D}_n$ for which there exist $\sigma \in L(E, \nu, \Gamma)$ and $\varphi \in \operatorname{Hom}_{\leq \aleph_0 \text{-to-1}}(E, \mu, F)$ such that $\operatorname{dom}(\sigma) = E \upharpoonright \operatorname{dom}(\varphi), \varphi$ is σ -invariant, and $d_{\nu_n}(\sigma \upharpoonright (R_n \upharpoonright \operatorname{dom}(\varphi)), \rho) \leq \epsilon_n$. Proof of Proposition 4 (continued)

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Fix such a $\sigma_{n,\rho}$ and $\varphi_{n,\rho}$ for each $n \in \mathbb{N}$ and $\rho \in \mathscr{D}'_n$.

Proof of Proposition 4 (continued)

Fix countable dense sets $\mathscr{D}_n \subseteq L(R_n, \nu_n, \Gamma)$.

Let \mathscr{D}'_n denote the set of $\rho \in \mathscr{D}_n$ for which there exist $\sigma \in L(E, \nu, \Gamma)$ and $\varphi \in \operatorname{Hom}_{\leq \aleph_0 \text{-to-1}}(E, \mu, F)$ such that $\operatorname{dom}(\sigma) = E \upharpoonright \operatorname{dom}(\varphi), \varphi$ is σ -invariant, and $d_{\nu_n}(\sigma \upharpoonright (R_n \upharpoonright \operatorname{dom}(\varphi)), \rho) \leq \epsilon_n$.

Fix such a $\sigma_{n,\rho}$ and $\varphi_{n,\rho}$ for each $n \in \mathbb{N}$ and $\rho \in \mathscr{D}'_n$.

We will show that the set $\Phi = \{\varphi_{n,\rho} \mid n \in \mathbb{N} \text{ and } \rho \in \mathscr{D}'_n\}$ is dense.

III. Separability

Proof of Proposition 4 (continued)

Suppose that $\epsilon > 0$ and $\varphi \in \operatorname{Hom}_{\leq \aleph_0 \text{-to-1}}(E, \mu, F)$.

Proof of Proposition 4 (continued)

Suppose that $\epsilon > 0$ and $\varphi \in \operatorname{Hom}_{\leq \aleph_0 \text{-to-1}}(E, \mu, F)$.

We can assume that there exists $\sigma \in L(E, \nu, \Gamma)$ with the property that $\operatorname{dom}(\sigma) = E \upharpoonright \operatorname{dom}(\varphi)$ and φ is σ -invariant.

Proof of Proposition 4 (continued)

Suppose that $\epsilon > 0$ and $\varphi \in \operatorname{Hom}_{\leq \aleph_0 \text{-to-1}}(E, \mu, F)$.

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For $n \in \mathbb{N}$, fix $\rho_n \in \mathscr{D}_n$ such that $d_{\nu_n}(\sigma \upharpoonright (R_n \upharpoonright \operatorname{dom}(\varphi)), \rho_n) \leq \epsilon_n$.

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Set $\sigma_n = \sigma_{n,\rho_n}$ and $\varphi_n = \varphi_{n,\rho_n}$.

Proof of Proposition 4 (continued)

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Note that $d_{\nu_n}(\sigma \upharpoonright (R_n \upharpoonright \operatorname{dom}(\varphi)), \sigma_n \upharpoonright (R_n \upharpoonright \operatorname{dom}(\varphi_n))) \leq 2\epsilon_n$.

Proof of Proposition 4 (continued)

For each $n \in \mathbb{N}$, let E_n denote the equivalence relation generated by the set $D_n = \operatorname{dom}(\sigma) \cap R_n \setminus D(\sigma, \sigma_n)$.

Proof of Proposition 4 (continued)

For each $n \in \mathbb{N}$, let E_n denote the equivalence relation generated by the set $D_n = \operatorname{dom}(\sigma) \cap R_n \setminus D(\sigma, \sigma_n)$.

Also for each $n \in \mathbb{N}$, define $F_n = \bigcap_{m \ge n} E_m$ and

 $X_n = \{x \in \operatorname{dom}(\varphi) \mid \exists y \in \operatorname{dom}(\varphi) \cap (R_n)_x \ \sigma(x, y) \neq \sigma_n(x, y)\}.$

Proof of Proposition 4 (continued)

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So $\mu(X_n) \leq d_{\nu_n}(\sigma \upharpoonright (R_n \upharpoonright \operatorname{dom}(\varphi)), \sigma_n \upharpoonright (R_n \upharpoonright \operatorname{dom}(\varphi_n))) \leq 2\epsilon_n$.

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$$X_n = \{x \in \operatorname{dom}(\varphi) \mid \exists y \in \operatorname{dom}(\varphi) \cap (R_n)_x \ \sigma(x, y) \neq \sigma_n(x, y)\}.$$

So $\mu(X_n) \leq d_{\nu_n}(\sigma \upharpoonright (R_n \upharpoonright \operatorname{dom}(\varphi)), \sigma_n \upharpoonright (R_n \upharpoonright \operatorname{dom}(\varphi_n))) \leq 2\epsilon_n.$

Thus the set
$$C = \sim \bigcap_{n \in \mathbb{N}} \bigcup_{m \ge n} X_m$$
 is μ -conull.

III. Separability

Proof of Proposition 4 (continued)

Note that $E \upharpoonright (C \cap \operatorname{dom}(\varphi)) \subseteq \bigcup_{n \in \mathbb{N}} F_n$.

Proof of Proposition 4 (continued)

Note that $E \upharpoonright (C \cap \operatorname{dom}(\varphi)) \subseteq \bigcup_{n \in \mathbb{N}} F_n$.

Fix $n \in \mathbb{N}$ for which there is a Borel set $B \subseteq C \cap \operatorname{dom}(\varphi)$ such that $\mu(\operatorname{dom}(\varphi) \setminus B) \leq \epsilon$ and $F_n \upharpoonright B$ is $(\mu \upharpoonright B)$ -nowhere hyperfinite.

Proof of Proposition 4 (continued)

Note that $E \upharpoonright (C \cap \operatorname{dom}(\varphi)) \subseteq \bigcup_{n \in \mathbb{N}} F_n$.

Fix $n \in \mathbb{N}$ for which there is a Borel set $B \subseteq C \cap \operatorname{dom}(\varphi)$ such that $\mu(\operatorname{dom}(\varphi) \setminus B) \leq \epsilon$ and $F_n \upharpoonright B$ is $(\mu \upharpoonright B)$ -nowhere hyperfinite.

Observe that both φ and φ_n are $(\sigma \upharpoonright (F_n \upharpoonright B))$ -invariant.

Proof of Proposition 4 (continued)

Note that $E \upharpoonright (C \cap \operatorname{dom}(\varphi)) \subseteq \bigcup_{n \in \mathbb{N}} F_n$.

Fix $n \in \mathbb{N}$ for which there is a Borel set $B \subseteq C \cap \operatorname{dom}(\varphi)$ such that $\mu(\operatorname{dom}(\varphi) \setminus B) \leq \epsilon$ and $F_n \upharpoonright B$ is $(\mu \upharpoonright B)$ -nowhere hyperfinite.

Observe that both φ and φ_n are $(\sigma \upharpoonright (F_n \upharpoonright B))$ -invariant.

So local rigidity ensures that $\varphi \upharpoonright B = \varphi_n \upharpoonright B$.

Proof of Proposition 4 (continued)

Note that $E \upharpoonright (C \cap \operatorname{dom}(\varphi)) \subseteq \bigcup_{n \in \mathbb{N}} F_n$.

Fix $n \in \mathbb{N}$ for which there is a Borel set $B \subseteq C \cap \operatorname{dom}(\varphi)$ such that $\mu(\operatorname{dom}(\varphi) \setminus B) \leq \epsilon$ and $F_n \upharpoonright B$ is $(\mu \upharpoonright B)$ -nowhere hyperfinite.

Observe that both φ and φ_n are $(\sigma \upharpoonright (F_n \upharpoonright B))$ -invariant.

So local rigidity ensures that $\varphi \upharpoonright B = \varphi_n \upharpoonright B$.

Thus $d_{\mu}(\varphi, \varphi_n) \leq \epsilon$.

 \boxtimes

Definition

We say that F has separable homomorphisms if whenever X is a Polish space, E is a countable Borel equivalence relation on X, and μ is a finite Borel measure on X for which E is μ -nowhere hyperfinite, the space $\operatorname{Hom}_{\leq\aleph_0-\text{to-1}}(E,\mu,F)$ is separable.



Definition

We say that F has separable homomorphisms if whenever X is a Polish space, E is a countable Borel equivalence relation on X, and μ is a finite Borel measure on X for which E is μ -nowhere hyperfinite, the space $\operatorname{Hom}_{\leq\aleph_0-\text{to-1}}(E,\mu,F)$ is separable.

Theorem 5 (Conley-M)

The family of countable Borel equivalence relations on Polish spaces with separable homomorphisms is closed downward under countableto-one Borel homomorphism, and includes every orbit equivalence relation of a locally rigid Borel action of a countable group.

Part IV

Borel reducibility



The following results are joint with Clinton Conley.

Definition

An equivalence relation is μ -hyperfinite if there is a μ -conull Borel set on which it is hyperfinite.

Definition

An equivalence relation is μ -hyperfinite if there is a μ -conull Borel set on which it is hyperfinite.

Definition

An equivalence relation is measure hyperfinite if it μ -hyperfinite for every finite Borel measure μ on the underlying space.

Theorem 6

Suppose that X is a Polish space, E is a countable Borel equivalence relation on X with separable homomorphisms, and μ is a finite Borel measure on X. Then exactly one of the following holds:

- **①** The equivalence relation E is μ -hyperfinite.
- **2** There is no $(\mu \times m)$ -measurable reduction of $E \times \Delta(\mathbb{R})$ to E.

Theorem 6

Suppose that X is a Polish space, E is a countable Borel equivalence relation on X with separable homomorphisms, and μ is a finite Borel measure on X. Then exactly one of the following holds:

① The equivalence relation E is μ -hyperfinite.

2 There is no $(\mu \times m)$ -measurable reduction of $E \times \Delta(\mathbb{R})$ to E.

Proof

Fix a μ -positive Borel $A \subseteq X$ on which E is μ -nowhere hyperfinite.

Theorem 6

Suppose that X is a Polish space, E is a countable Borel equivalence relation on X with separable homomorphisms, and μ is a finite Borel measure on X. Then exactly one of the following holds:

- **①** The equivalence relation E is μ -hyperfinite.
- **2** There is no $(\mu \times m)$ -measurable reduction of $E \times \Delta(\mathbb{R})$ to E.

Proof

Fix a μ -positive Borel $A \subseteq X$ on which E is μ -nowhere hyperfinite.

Define $\nu(B) = \mu(A \cap B)$.

IV. Borel reducibility Products

Proof of Theorem 6 (continued)

Suppose that $\varphi \colon X \times \mathbb{R} \to X$ is a $(\mu \times m)$ -measurable reduction.

IV. Borel reducibility Products

Proof of Theorem 6 (continued)

Suppose that $\varphi \colon X \times \mathbb{R} \to X$ is a $(\mu \times m)$ -measurable reduction.

For each $r \in \mathbb{R}$, define $\varphi_r \colon A \to X$ by $\varphi_r(x) = \varphi(x, r)$.

IV. Borel reducibility Products

Proof of Theorem 6 (continued)

Suppose that $\varphi \colon X \times \mathbb{R} \to X$ is a $(\mu \times m)$ -measurable reduction.

For each $r \in \mathbb{R}$, define $\varphi_r \colon A \to X$ by $\varphi_r(x) = \varphi(x, r)$.

Fix an uncountable set $R \subseteq \mathbb{R}$ such that $\forall r \in R \varphi_r$ is ν -measurable.

Proof of Theorem 6 (continued)

Suppose that $\varphi \colon X \times \mathbb{R} \to X$ is a $(\mu \times m)$ -measurable reduction.

For each $r \in \mathbb{R}$, define $\varphi_r \colon A \to X$ by $\varphi_r(x) = \varphi(x, r)$.

Fix an uncountable set $R \subseteq \mathbb{R}$ such that $\forall r \in R \varphi_r$ is ν -measurable.

Note that $d_{\nu}(\varphi_r, \varphi_s) = \mu(A)$ for all distinct $r, s \in R$.

Proof of Theorem 6 (continued)

Suppose that $\varphi \colon X \times \mathbb{R} \to X$ is a $(\mu \times m)$ -measurable reduction.

For each $r \in \mathbb{R}$, define $\varphi_r \colon A \to X$ by $\varphi_r(x) = \varphi(x, r)$.

Fix an uncountable set $R \subseteq \mathbb{R}$ such that $\forall r \in R \varphi_r$ is ν -measurable.

Note that $d_{\nu}(\varphi_r, \varphi_s) = \mu(A)$ for all distinct $r, s \in R$.

This contradicts separability of
$$\operatorname{Hom}_{\leq\aleph_0\text{-to-1}}(E,\nu,F)$$
.

Theorem 7

Suppose that X and Y are Polish spaces, E and F are countable Borel equivalence relations on X and Y, μ is a finite Borel measure on X, \mathcal{I} is a σ -ideal on Y, E is μ -nowhere hyperfinite, and F has separable homomorphisms. Then there is an \mathcal{I} -conull set $C \subseteq Y$ such that every $\varphi \in \operatorname{Hom}_{\leq\aleph_0-\text{to}-1}(E,\mu,F \upharpoonright C)$ sends μ -positive sets to \mathcal{I} -positive sets.

Proof

Fix a sequence $(\varphi_{\alpha})_{\alpha < \beta}$ of maximal length consisting of functions in $\operatorname{Hom}_{<\aleph_0-to-1}(E,\mu,F)$ with pairwise disjoint \mathcal{I} -null ranges.

Proof

Fix a sequence $(\varphi_{\alpha})_{\alpha < \beta}$ of maximal length consisting of functions in $\operatorname{Hom}_{\leq\aleph_0-\text{to}-1}(E,\mu,F)$ with pairwise disjoint \mathcal{I} -null ranges.

The separability of $\operatorname{Hom}_{\leq\aleph_0-\text{to}-1}(E,\mu,F)$ ensures that $\beta < \omega_1$.

Proof

Fix a sequence $(\varphi_{\alpha})_{\alpha < \beta}$ of maximal length consisting of functions in $\operatorname{Hom}_{\leq\aleph_0-\text{to}-1}(E,\mu,F)$ with pairwise disjoint \mathcal{I} -null ranges.

The separability of $\operatorname{Hom}_{\leq\aleph_0-\text{to-1}}(E,\mu,F)$ ensures that $\beta < \omega_1$.

Set
$$C = Y \setminus \bigcup_{\alpha < \beta} \operatorname{rng}(\varphi_{\alpha}).$$

IV. Borel reducibility Small products

Theorem 8

Suppose that X is a Polish space and E is a countable Borel equivalence relation on X with separable homomorphisms. Then exactly one of the following holds:

- **(**) The equivalence relation E is measure hyperfinite.
- ② There is a Borel set B ⊆ X for which there is a finite Borel measure µ on B with the property that there is no (µ × 2)-measurable reduction of (E ↾ B × Δ(2)) to E ↾ B.

Theorem 8

Suppose that X is a Polish space and E is a countable Borel equivalence relation on X with separable homomorphisms. Then exactly one of the following holds:

- **(**) The equivalence relation E is measure hyperfinite.
- ② There is a Borel set B ⊆ X for which there is a finite Borel measure µ on B with the property that there is no (µ × 2)-measurable reduction of (E ↾ B × Δ(2)) to E ↾ B.

Proof

We can assume there is a finite Borel measure ν on X with the property that E is not ν -hyperfinite.

IV. Borel reducibility Small products

Proof of Theorem 8 (continued)

We can assume that every *E*-invariant Borel set is ν -null or ν -conull.

Proof of Theorem 8 (continued)

We can assume that every *E*-invariant Borel set is ν -null or ν -conull.

Then there exists a ν -conull Borel set $B \subseteq X$ with the property that every $\varphi \in \operatorname{Hom}_{\leq \aleph_0 \text{-to-1}}((E \upharpoonright B) \times \Delta(2), \nu, E \upharpoonright B)$ sends sets of positive measure to sets of positive measure.

Proof of Theorem 8 (continued)

We can assume that every *E*-invariant Borel set is ν -null or ν -conull.

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But clearly there can be no such φ .

 \boxtimes

Theorem 9

Suppose that X is a Polish space, E is a countable treeable Borel equivalence relation on X with separable homomorphisms, and μ is a finite Borel measure on X such that E is not μ -hyperfinite. Then there is a μ -positive Borel set $B \subseteq X$ for which there is an increasing sequence $(F_r)_{r \in \mathbb{R}}$ of Borel subequivalence relations of E such that for no distinct $r, s \in \mathbb{R}$ is there a μ -positive set $A \subseteq B$ on which there is a μ -measurable reduction of $F_r \upharpoonright A$ to F_s .

IV. Borel reducibility Increasing sequences

Proof (Sketch)

Fix a μ -positive Borel $C \subseteq X$ on which E is μ -nowhere hyperfinite.

IV. Borel reducibility Increasing sequences

Proof (Sketch)

Fix a μ -positive Borel $C \subseteq X$ on which E is μ -nowhere hyperfinite.

Fix an acyclic Borel graph G generating $E \upharpoonright C$.

Proof (Sketch)

Fix a μ -positive Borel $C \subseteq X$ on which E is μ -nowhere hyperfinite.

Fix an acyclic Borel graph G generating $E \upharpoonright C$.

Construct a Borel subgraph $H \subseteq G$ whose induced equivalence relation F is hyperfinite but μ -nowhere smooth.

Proof of Theorem 9 (continued)

Construct an increasing sequence $(H_r)_{r\in\mathbb{R}}$ of Borel graphs lying between H and G with the property that for all real numbers r < s, the projection of $H_s \setminus H_r$ contains points of μ -almost every E-class.

Proof of Theorem 9 (continued)

Construct an increasing sequence $(H_r)_{r\in\mathbb{R}}$ of Borel graphs lying between H and G with the property that for all real numbers r < s, the projection of $H_s \setminus H_r$ contains points of μ -almost every E-class.

Then for no distinct $r, s \in \mathbb{R}$ is there a μ -positive set on which the equivalence relations F_r and F_s induced by H_r and H_s agree.

IV. Borel reducibility Increasing sequences

Proof of Theorem 9 (continued)

We can assume there is a μ -positive Borel set $B \subseteq C$ on which each F_r is μ -nowhere hyperfinite.

Proof of Theorem 9 (continued)

We can assume there is a μ -positive Borel set $B \subseteq C$ on which each F_r is μ -nowhere hyperfinite.

Then for each $r \in \mathbb{R}$, there are only countably many $s \in \mathbb{R}$ for which there is a μ -positive set $A \subseteq B$ with the property that there is a $(\mu \upharpoonright A)$ -measurable reduction of $F_r \upharpoonright A$ to F_s .

Proof of Theorem 9 (continued)

We can assume there is a μ -positive Borel set $B \subseteq C$ on which each F_r is μ -nowhere hyperfinite.

Then for each $r \in \mathbb{R}$, there are only countably many $s \in \mathbb{R}$ for which there is a μ -positive set $A \subseteq B$ with the property that there is a $(\mu \upharpoonright A)$ -measurable reduction of $F_r \upharpoonright A$ to F_s .

So we can assume that for no distinct $r, s \in \mathbb{R}$ is there a μ -positive set $A \subseteq B$ on which there is a μ -measurable reduction.

IV. Borel reducibility Cardinality of bases

Theorem 10

Suppose that X is a Polish space and E is a countable treeable Borel equivalence relation on X which has separable homomorphisms but is not measure hyperfinite. Then there is no basis of cardinality strictly less than add(null) for the family of non-measure hyperfinite countable Borel equivalence relations which admit countable-to-one Borel homomorphisms to E.

IV. Borel reducibility Cardinality of bases

Proof

Fix a finite Borel measure μ for which *E* is not μ -hyperfinite.

Fix a finite Borel measure μ for which E is not μ -hyperfinite.

Fix a μ -positive Borel set $B \subseteq X$ and a sequence $(F_r)_{r \in \mathbb{R}}$ of Borel subequivalence relations of E such that for no distinct $r, s \in \mathbb{R}$ is there a μ -positive set $A \subseteq B$ on which there is a μ -measurable reduction of $F_r \upharpoonright A$ to F_s .

IV. Borel reducibility Cardinality of bases

Proof of Theorem 10 (continued)

Suppose that \mathscr{B} is a basis of cardinality strictly less than add(null).

Proof of Theorem 10 (continued)

Suppose that \mathscr{B} is a basis of cardinality strictly less than add(null).

For each $E \in \mathscr{B}$, fix a finite Borel measure μ_E such that E is μ_E -nowhere hyperfinite.

Proof of Theorem 10 (continued)

Suppose that \mathscr{B} is a basis of cardinality strictly less than add(null).

For each $E \in \mathscr{B}$, fix a finite Borel measure μ_E such that E is μ_E -nowhere hyperfinite.

We can assume that every μ_E -measurable reduction of E to any F_r sends μ_E -positive sets to μ -positive sets.

IV. Borel reducibility Cardinality of bases

Proof of Theorem 10 (continued)

Fix $E_r \in \mathscr{B}$ and μ -measurable reductions φ_r of E_r to F_r .

Proof of Theorem 10 (continued)

Fix $E_r \in \mathscr{B}$ and μ -measurable reductions φ_r of E_r to F_r .

Fix distinct $r, s \in \mathbb{R}$ such that $E_r = E_s$ and φ_r and φ_s agree on a μ_{E_r} -positive set.

Proof of Theorem 10 (continued)

Fix $E_r \in \mathscr{B}$ and μ -measurable reductions φ_r of E_r to F_r .

Fix distinct $r, s \in \mathbb{R}$ such that $E_r = E_s$ and φ_r and φ_s agree on a μ_{E_r} -positive set.

Then F_r and F_s agree on a μ -positive set.

 \boxtimes

Theorem 11

Suppose that X is a Polish space and E is a countable treeable Borel equivalence relation on X which has separable homomorphisms but is not measure hyperfinite. Then the initial segment of the Borel reducibility hierarchy consisting of relations with countable-to-one Borel homomorphisms to $E \times \Delta(2^{\mathbb{N}})$ contains copies of all Borel quasi-orders on Polish spaces.

IV. Borel reducibility

Proof

Fix $(F_r)_{r \in \mathbb{R}}$ as before.

Fix $(F_r)_{r\in\mathbb{R}}$ as before.

We can assume there are finite Borel measures μ_r such that every F_r -invariant Borel set is μ_r -null or μ_r -conull, and for no distinct $r, s \in \mathbb{R}$ is there a μ_r -measurable reduction of F_r to F_s .

Fix $(F_r)_{r\in\mathbb{R}}$ as before.

We can assume there are finite Borel measures μ_r such that every F_r -invariant Borel set is μ_r -null or μ_r -conull, and for no distinct $r, s \in \mathbb{R}$ is there a μ_r -measurable reduction of F_r to F_s .

Associate with each Borel set $B \subseteq \mathbb{R}$ the disjoint union of the equivalence relations F_r , for $r \in B$.

Fix $(F_r)_{r\in\mathbb{R}}$ as before.

We can assume there are finite Borel measures μ_r such that every F_r -invariant Borel set is μ_r -null or μ_r -conull, and for no distinct $r, s \in \mathbb{R}$ is there a μ_r -measurable reduction of F_r to F_s .

Associate with each Borel set $B \subseteq \mathbb{R}$ the disjoint union of the equivalence relations F_r , for $r \in B$.

This reduces \subseteq on Borel sets to Borel reducibility.

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Definable cardinals just beyond \mathbb{R}/\mathbb{Q}

Winter school in abstract analysis Sporthotel Kácov January 18th, 2013

Benjamin Miller Westfälische Wilhelms-Universität Münster

Part IV

Borel reducibility



The following results are joint with Clinton Conley.

Definition

An equivalence relation is μ -hyperfinite if there is a μ -conull Borel set on which it is hyperfinite.

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An equivalence relation is μ -hyperfinite if there is a μ -conull Borel set on which it is hyperfinite.

Definition

An equivalence relation is measure hyperfinite if it μ -hyperfinite for every finite Borel measure μ on the underlying space.

Theorem 1

Suppose that X is a Polish space, E is a countable Borel equivalence relation on X with separable homomorphisms, and μ is a finite Borel measure on X. Then exactly one of the following holds:

① The equivalence relation E is μ -hyperfinite.

2 There is no $(\mu \times m)$ -measurable reduction of $E \times \Delta(\mathbb{R})$ to E.

Theorem 1

Suppose that X is a Polish space, E is a countable Borel equivalence relation on X with separable homomorphisms, and μ is a finite Borel measure on X. Then exactly one of the following holds:

① The equivalence relation E is μ -hyperfinite.

2 There is no $(\mu \times m)$ -measurable reduction of $E \times \Delta(\mathbb{R})$ to E.

Proof

Fix a μ -positive Borel $A \subseteq X$ on which E is μ -nowhere hyperfinite.

IV. Borel reducibility Products

Proof of Theorem 1 (continued)

Suppose that $\varphi \colon X \times \mathbb{R} \to X$ is a $(\mu \times m)$ -measurable reduction.

IV. Borel reducibility Products

Proof of Theorem 1 (continued)

Suppose that $\varphi \colon X \times \mathbb{R} \to X$ is a $(\mu \times m)$ -measurable reduction.

For each $r \in \mathbb{R}$, define $\varphi_r \colon A \to X$ by $\varphi_r(x) = \varphi(x, r)$.

IV. Borel reducibility Products

Proof of Theorem 1 (continued)

Suppose that $\varphi \colon X \times \mathbb{R} \to X$ is a $(\mu \times m)$ -measurable reduction.

For each $r \in \mathbb{R}$, define $\varphi_r \colon A \to X$ by $\varphi_r(x) = \varphi(x, r)$.

Fix an uncountable set $R \subseteq \mathbb{R}$ such that $\forall r \in R \varphi_r$ is μ -measurable.

Proof of Theorem 1 (continued)

Suppose that $\varphi \colon X \times \mathbb{R} \to X$ is a $(\mu \times m)$ -measurable reduction.

For each $r \in \mathbb{R}$, define $\varphi_r \colon A \to X$ by $\varphi_r(x) = \varphi(x, r)$.

Fix an uncountable set $R \subseteq \mathbb{R}$ such that $\forall r \in R \varphi_r$ is μ -measurable.

Note that $d_{\nu}(\varphi_r, \varphi_s) = \mu(A)$ for all distinct $r, s \in R$.

Proof of Theorem 1 (continued)

Suppose that $\varphi \colon X \times \mathbb{R} \to X$ is a $(\mu \times m)$ -measurable reduction.

For each $r \in \mathbb{R}$, define $\varphi_r \colon A \to X$ by $\varphi_r(x) = \varphi(x, r)$.

Fix an uncountable set $R \subseteq \mathbb{R}$ such that $\forall r \in R \varphi_r$ is μ -measurable.

Note that $d_{\nu}(\varphi_r, \varphi_s) = \mu(A)$ for all distinct $r, s \in R$.

This contradicts separability of
$$\operatorname{Hom}_{\leq\aleph_0\text{-to-1}}(E,\nu,F)$$
.

Theorem 2

Suppose that X and Y are Polish spaces, E and F are countable Borel equivalence relations on X and Y, μ is a finite Borel measure on X, \mathcal{I} is a σ -ideal on Y, E is μ -nowhere hyperfinite, and F has separable homomorphisms. Then there is an \mathcal{I} -conull set $C \subseteq Y$ such that $\operatorname{rng}(\varphi) \notin \mathcal{I}$ for all $\varphi \in \operatorname{Hom}_{\leq\aleph_0-\operatorname{to-1}}(E, \mu, F \upharpoonright C)$.

Proof

Fix a sequence $(\varphi_{\alpha})_{\alpha < \beta}$ of maximal length consisting of functions in $\operatorname{Hom}_{<\aleph_0-to-1}(E,\mu,F)$ with pairwise disjoint \mathcal{I} -null ranges.

Proof

Fix a sequence $(\varphi_{\alpha})_{\alpha < \beta}$ of maximal length consisting of functions in $\operatorname{Hom}_{\leq\aleph_0-\text{to}-1}(E,\mu,F)$ with pairwise disjoint \mathcal{I} -null ranges.

The separability of $\operatorname{Hom}_{\leq\aleph_0-\text{to}-1}(E,\mu,F)$ ensures that $\beta < \omega_1$.

Proof

Fix a sequence $(\varphi_{\alpha})_{\alpha < \beta}$ of maximal length consisting of functions in $\operatorname{Hom}_{\leq\aleph_0-\text{to}-1}(E,\mu,F)$ with pairwise disjoint \mathcal{I} -null ranges.

The separability of $\operatorname{Hom}_{\leq\aleph_0-\text{to-1}}(E,\mu,F)$ ensures that $\beta < \omega_1$.

Set
$$C = Y \setminus \bigcup_{\alpha < \beta} \operatorname{rng}(\varphi_{\alpha}).$$

IV. Borel reducibility Small products

Theorem 3

Suppose that X is a Polish space and E is a countable Borel equivalence relation on X with separable homomorphisms. Then exactly one of the following holds:

- **(**) The equivalence relation E is measure hyperfinite.
- ② There is a Borel set B ⊆ X for which there is a finite Borel measure µ on B with the property that there is no (µ × 2)-measurable reduction of (E ↾ B × Δ(2)) to E ↾ B.

IV. Borel reducibility Small products

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Proof

We can assume there is a finite Borel measure μ on X with the property that E is not μ -hyperfinite.

IV. Borel reducibility Small products

Proof of Theorem 3 (continued)

We can assume every *E*-invariant Borel set is μ -null or μ -conull.

Proof of Theorem 3 (continued)

We can assume every *E*-invariant Borel set is μ -null or μ -conull.

Then there exists a μ -conull Borel set $B \subseteq X$ with the property that every $\varphi \in \operatorname{Hom}_{\leq \aleph_0 \text{-to-1}}((E \upharpoonright B) \times \Delta(2), \mu, E \upharpoonright B)$ sends sets of positive measure to sets of positive measure.

Proof of Theorem 3 (continued)

We can assume every *E*-invariant Borel set is μ -null or μ -conull.

Then there exists a μ -conull Borel set $B \subseteq X$ with the property that every $\varphi \in \operatorname{Hom}_{\leq \aleph_0 \text{-to-1}}((E \upharpoonright B) \times \Delta(2), \mu, E \upharpoonright B)$ sends sets of positive measure to sets of positive measure.

But clearly there can be no such φ .

 \boxtimes

Increasing sequences

Definition

We say that *E* is μ -nowhere smooth if there is no μ -positive Borel set on which *E* is smooth.

Theorem 4

Suppose that X is a Polish space, E is a countable treeable Borel equivalence relation on X with separable homomorphisms, and μ is a finite Borel measure on X such that E is μ -nowhere hyperfinite. Then there is an increasing sequence $(E_r)_{r \in \mathbb{R}}$ of μ -nowhere hyperfinite Borel subequivalence relations of E such that for no distinct $r, s \in \mathbb{R}$ is there a μ -positive set on which E_r and E_s agree.

Proof (Sketch)

We can assume that the μ -null sets are closed under *E*-saturation.

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We can assume that the μ -null sets are closed under *E*-saturation.

Fix an acyclic Borel graph G generating E.

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We can assume that the μ -null sets are closed under *E*-saturation.

Fix an acyclic Borel graph G generating E.

Construct a Borel subgraph $H \subseteq G$ whose induced equivalence relation F is hyperfinite but μ -nowhere smooth.

Proof of Theorem 4 (continued)

Construct an increasing sequence $(H_r)_{r\in\mathbb{R}}$ of Borel graphs lying between H and G with the property that for all real numbers r < s, the projection of $H_s \setminus H_r$ intersects μ -almost every $(E \upharpoonright B)$ -class.

Proof of Theorem 4 (continued)

Construct an increasing sequence $(H_r)_{r\in\mathbb{R}}$ of Borel graphs lying between H and G with the property that for all real numbers r < s, the projection of $H_s \setminus H_r$ intersects μ -almost every $(E \upharpoonright B)$ -class.

Then for no distinct $r, s \in \mathbb{R}$ is there a μ -positive $A \subseteq B$ on which the relations F_r and F_s induced by H_r and H_s agree.

Proof of Theorem 4 (continued)

Fix a Borel contraction $f: X \to B$ such that $\forall x \in X \times E f(x)$.

Proof of Theorem 4 (continued)

Fix a Borel contraction $f: X \to B$ such that $\forall x \in X \times E f(x)$.

Define $x E_r y \Leftrightarrow f(x) F_r f(y)$.

Proof of Theorem 4 (continued)

Fix a Borel contraction $f: X \to B$ such that $\forall x \in X \times E f(x)$.

Define $x E_r y \Leftrightarrow f(x) F_r f(y)$.

Then for no distinct $r, s \in \mathbb{R}$ is there a μ -positive set on which E_r and E_s agree.

Theorem 5

Suppose that X is a Polish space, E is a countable treeable Borel equivalence relation on X with separable homomorphisms, and μ is a finite Borel measure on X such that E is μ -nowhere hyperfinite. Then there is an increasing sequence $(E_r)_{r\in\mathbb{R}}$ of Borel subequivalence relations of E such that for no distinct $r, s \in \mathbb{R}$ is there a μ -measurable reduction of E_r to E_s on a μ -positive set.

Proof

Fix an increasing sequence $(E_r)_{r\in\mathbb{R}}$ of μ -nowhere hyperfinite Borel subequivalence relations of E such that for no distinct $r, s \in \mathbb{R}$ is there a μ -positive set on which E_r and E_s agree.

Proof

Fix an increasing sequence $(E_r)_{r \in \mathbb{R}}$ of μ -nowhere hyperfinite Borel subequivalence relations of E such that for no distinct $r, s \in \mathbb{R}$ is there a μ -positive set on which E_r and E_s agree.

We can assume that $\bigcap_{r \in \mathbb{R}} E_r$ is μ -nowhere hyperfinite on B.

Proof of Theorem 5 (continued)

Then for each $r \in \mathbb{R}$, there are only countably many $s \in \mathbb{R}$ for which there is a μ -measurable reduction of E_s to E_r on a μ -positive set.

Proof of Theorem 5 (continued)

Then for each $r \in \mathbb{R}$, there are only countably many $s \in \mathbb{R}$ for which there is a μ -measurable reduction of E_s to E_r on a μ -positive set.

So we can assume that for no distinct $r, s \in \mathbb{R}$ is there a μ -measurable reduction of E_r to E_s on a μ -positive set.

Theorem 6

Suppose that X is a Polish space and E is a countable treeable Borel equivalence relation on X which has separable homomorphisms but is not measure hyperfinite. Then there is no basis of cardinality strictly less than add(null) for the family of non-measure hyperfinite countable Borel equivalence relations which admit countable-to-one Borel homomorphisms to E.

Proof

Fix a finite Borel measure μ for which *E* is μ -nowhere hyperfinite.

Proof

Fix a finite Borel measure μ for which *E* is μ -nowhere hyperfinite.

Fix a sequence $(E_r)_{r\in\mathbb{R}}$ of Borel subequivalence relations of E such that for no distinct $r, s \in \mathbb{R}$ is there a μ -measurable reduction of E_r to E_s on a μ -positive set.

Proof of Theorem 6 (continued)

Suppose that \mathscr{F} is a basis of cardinality $< \operatorname{add}(\operatorname{null})$.

Proof of Theorem 6 (continued)

Suppose that \mathscr{F} is a basis of cardinality $< \operatorname{add}(\operatorname{null})$.

For each $F \in \mathscr{F}$, fix a finite Borel measure μ_F such that F is μ_F -nowhere hyperfinite.

Proof of Theorem 6 (continued)

Suppose that \mathscr{F} is a basis of cardinality $< \operatorname{add}(\operatorname{null})$.

For each $F \in \mathscr{F}$, fix a finite Borel measure μ_F such that F is μ_F -nowhere hyperfinite.

We can assume that every μ_F -measurable reduction of F to E_r sends μ_F -positive sets to μ -positive sets.

Proof of Theorem 6 (continued)

Fix $F_r \in \mathscr{F}$ and μ -measurable reductions φ_r of F_r to E_r .

Proof of Theorem 6 (continued)

Fix $F_r \in \mathscr{F}$ and μ -measurable reductions φ_r of F_r to E_r .

Fix distinct $r, s \in \mathbb{R}$ such that $F_r = F_s$ and φ_r and φ_s agree on a μ_{F_r} -positive set.

Proof of Theorem 6 (continued)

Fix $F_r \in \mathscr{F}$ and μ -measurable reductions φ_r of F_r to E_r .

Fix distinct $r, s \in \mathbb{R}$ such that $F_r = F_s$ and φ_r and φ_s agree on a μ_{F_r} -positive set.

Then E_r and E_s agree on a μ -positive set.

 \boxtimes

Theorem 7

Suppose that X is a Polish space and E is a countable treeable Borel equivalence relation on X which has separable homomorphisms but is not measure hyperfinite. Then the initial segment of the Borel reducibility hierarchy consisting of relations with countable-to-one Borel homomorphisms to $E \times \Delta(\mathbb{R})$ contains copies of all Borel quasi-orders on Polish spaces.

IV. Borel reducibility Complexity

Proof

Fix a finite Borel measure μ for which *E* is μ -nowhere hyperfinite.

IV. Borel reducibility Complexity

Proof

Fix a finite Borel measure μ for which *E* is μ -nowhere hyperfinite.

Fix a sequence $(E_r)_{r\in\mathbb{R}}$ of Borel subequivalence relations of E such that for no distinct $r, s \in \mathbb{R}$ is there a μ -measurable reduction of E_r to E_s on a μ -positive set.

Proof of Theorem 7 (continued)

We can assume there are finite Borel measures μ_r such that every E_{r} -invariant Borel set is μ_r -null or μ_r -conull, and for no distinct $r, s \in \mathbb{R}$ is there a μ_r -measurable reduction of E_r to E_s on a μ_r -positive set.

Proof of Theorem 7 (continued)

We can assume there are finite Borel measures μ_r such that every E_r -invariant Borel set is μ_r -null or μ_r -conull, and for no distinct $r, s \in \mathbb{R}$ is there a μ_r -measurable reduction of E_r to E_s on a μ_r -positive set.

Associate with each Borel set $B \subseteq \mathbb{R}$ the disjoint union of the equivalence relations E_r , for $r \in B$.

Proof of Theorem 7 (continued)

We can assume there are finite Borel measures μ_r such that every E_{r} -invariant Borel set is μ_r -null or μ_r -conull, and for no distinct $r, s \in \mathbb{R}$ is there a μ_r -measurable reduction of E_r to E_s on a μ_r -positive set.

Associate with each Borel set $B \subseteq \mathbb{R}$ the disjoint union of the equivalence relations E_r , for $r \in B$.

This reduces \subseteq on Borel sets to Borel reducibility.