Definable cardinals just beyond $\mathbb{R}/\mathbb{Q}$

Winter school in abstract analysis
Sporthotel Kácov
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Benjamin Miller
Westfälische Wilhelms-Universität Münster

Part I

Definable cardinality

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I. Definable cardinality
Introduction

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In this first lecture, we will review some of the basic theory behind these developments.

Definition

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Theorem 1

Every uncountable Polish space is Borel isomorphic to $\mathbb{R}$.

Under determinacy, many properties of Borel sets in Polish spaces generalize to broader families of definable sets.

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For simplicity, however, we will focus on Borel sets.

Definition

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Definition

A reduction of E to F is a homomorphism from E to F sending E-unrelated points to F-unrelated points.

Theorem 2 (Silver)

Suppose that X is a Polish space and E is a Borel equivalence relation on X. Then exactly one of the following holds:

1. There is a Borel reduction of E to the equality relation on $\mathbb{N}$.
2. There is a Borel reduction of the equality relation on $\mathbb{R}$ to E.

Definition

The Vitali equivalence relation is the relation $E_{\mathbb{Q}}$ on $\mathbb{R}$ given by

$$x \; E_{\mathbb{Q}} \; y \Leftrightarrow x - y \in \mathbb{Q}.$$

I. Definable cardinality

The definable analog of the next continuum hypothesis

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Theorem 3 (Harrington-Kechris-Louveau)

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1. There is a Borel reduction of E to the equality relation on $\mathbb{R}$.
2. There is a Borel reduction of $E_{\mathbb{Q}}$ to E.

Definition

An equivalence relation is countable if its classes are all countable.

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Theorem 4 (Sullivan-Weiss-Wright, Woodin, Kechris-Hjorth)

Every countable Borel equivalence relation on a Polish space admits a Baire measurable reduction to the Vitali equivalence relation.

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Definition

The orbit equivalence relation associated with a group action $\Gamma \curvearrowright X$ is the relation E_Γ^X on X given by $x \; E_\Gamma^X \; y \Leftrightarrow \exists \gamma \in \Gamma \; \gamma \cdot x = y$.

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Definition

A measure μ is Γ-invariant if $\forall B \subseteq X \forall \gamma \in \Gamma \; \mu(B) = \mu(\gamma(B))$.

Theorem 5 (Ornstein-Weiss)

Suppose that X is a Polish space, $\Gamma \curvearrowright X$ is a free Borel action of a countable group, and μ is a Γ-invariant Borel probability measure on X. Then Γ is amenable iff E_Γ^X is μ-hyperfinite.

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Question

Suppose that X is a Polish space and E is a countable Borel equivalence relation on X which is not Borel reducible to the Vitali equivalence relation. Is there a Borel probability measure μ on X for which there is no μ-measurable reduction?

Theorem 6 (Woodin, Louveau-Velickovic)

There is a family of uncountably many Borel equivalence relations on Polish spaces which are pairwise incomparable under Baire measurable (and therefore Borel) reducibility.

Theorem 7 (Adams-Kechris)

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Theorem 8 (Adams-Kechris, Gao)

There is a Borel reduction of every analytic quasi-order on a Polish space into (codes for) the Borel reducibility quasi-order on the space of countable Borel equivalence relations.

Definition

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Theorem 9 (Hjorth)

There is a family of continuum-many countable Borel equivalence relations on Polish spaces which are pairwise incomparable under Borel reducibility.

Theorem 10 (Thomas)

There is a countable Borel equivalence relation E on a Polish space with the property that the disjoint union of two copies of E is not Borel reducible to E.

I. Definable cardinality
Pathology

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Theorem 10 (Thomas)

There is a countable Borel equivalence relation E on a Polish space with the property that the disjoint union of two copies of E is not Borel reducible to E.

Theorem 11 (Adams, Hjorth)

There are countable treeable Borel equivalence relations $E \subseteq F$ on a Polish space which are incomparable under Borel reducibility.

Lurking beneath the results for countable Borel equivalence relations are sophisticated rigidity theorems originating in the ergodic-theoretic study of actions of linear algebraic groups.

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In the remaining lectures, we will sketch significantly simpler proofs of strengthenings of many of these results.

We will first establish a simple purely Borel rigidity theorem.

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From this, we will obtain separability of certain spaces of measurable homomorphisms connected with equivalence relations.

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From this, we will obtain separability of certain spaces of measurable homomorphisms connected with equivalence relations.

Finally, we will use this separability to establish the main results.

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Part I

Local rigidity

Definition

The difference set associated with functions $\varphi\colon X \to Y$ and $\psi\colon X \to Y$ is the set $D(\varphi, \psi)$ given by

$$D(\varphi, \psi) = \{x \in X \mid \varphi(x) \neq \psi(x)\}.$$

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$$D(\varphi, \psi) = \{x \in X \mid \varphi(x) \neq \psi(x)\}.$$

Definition

Given $\Gamma \curvearrowright Y$, we say that a homomorphism $\varphi\colon X \to Y$ from E to E_Γ^Y is ρ-invariant if $\varphi(x_1) = \rho(x_1, x_2) \cdot \varphi(x_2)$ for all $x_1, x_2 \in X$.

Definition

We say that $\Gamma \curvearrowright Y$ is locally rigid if whenever X is a Polish space, E is a countable Borel equivalence relation on X, $\rho\colon E \to \Gamma$ is a Borel function, and φ, ψ are ρ-invariant countable-to-one Borel homomorphisms from E to E_Γ^Y, the equivalence relation $E \restriction D(\varphi, \psi)$ is Borel reducible to $E_\mathbb{Q}$.

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Today we will prove that $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$ is locally rigid.

Definition

An equivalence relation is finite if its classes are all finite.

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Definition

A Borel equivalence relation E is hyperfinite if there are finite Borel subequivalence relations $F_0 \subseteq F_1 \subseteq \cdots$ such that $E = \bigcup_{n \in \mathbb{N}} F_n$.

Definition

A Borel equivalence relation is smooth if it is Borel reducible to the equality relation on $\mathbb{R}$.

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A Borel equivalence relation is hypersmooth if there are smooth Borel equivalence relations $F_0 \subseteq F_1 \subseteq \cdots$ such that $E = \bigcup_{n \in \mathbb{N}} F_n$.

Theorem 1 (Dougherty-Jackson-Kechris, Slaman-Steel, Weiss)

Suppose that X is a Polish space and E is a countable Borel equivalence relation on X. Then the following are equivalent:

1. There is a Borel reduction of E to $E_{\mathbb{Q}}$.
2. The equivalence relation E is hyperfinite.
3. The equivalence relation E is hypersmooth.
4. There is a Borel action $\mathbb{Z} \curvearrowright X$ such that $E = E_{\mathbb{Z}}^{X}$.

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Proposition 2 (Dougherty-Jackson-Kechris)

Suppose that X is a Polish space and E is a Borel equivalence relation on X. Then the family of Borel sets on which E is hyperfinite is closed under countable unions.

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Proposition 3 (Dougherty-Jackson-Kechris)

Suppose that X and Y are Polish spaces, E and F are countable Borel equivalence relations, F is hyperfinite, and there is a countable-to-one Borel homomorphism from E to F. Then E is hyperfinite.

Definition

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Definition

Let $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ denote the action induced by $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$.

Proposition 4 (Jackson-Kechris-Louveau)

There is a Borel reduction of $E^{\mathbb{T}}_{\mathrm{SL}_2(\mathbb{Z})}$ to $E_{\mathbb{Q}}$.

I. Local rigidity

The action $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$

Proposition 5 (Conley-M)

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Proof

Suppose that $\theta \in \mathbb{T}$.

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Only countably many points of $\mathbb{T}$ have non-trivial stabilizers under $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$, and they are all infinite cyclic.

Proof

Suppose that $\theta \in \mathbb{T}$.

There are now two cases, depending on whether $\theta \cap \mathbb{Z}^2 \neq \emptyset$.

Proof of Proposition 5 (continued)

Suppose first that the set $\theta \cap \mathbb{Z}^2$ is non-empty.

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Let $v = (v_1, v_2)$ be the element of this set of minimal magnitude.

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Let $v = (v_1, v_2)$ be the element of this set of minimal magnitude.

Suppose that A is in the stabilizer of θ under $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$.

Proof of Proposition 5 (continued)

Suppose first that the set $\theta \cap \mathbb{Z}^2$ is non-empty.

Let $v = (v_1, v_2)$ be the element of this set of minimal magnitude.

Suppose that A is in the stabilizer of θ under $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$.

Then $Av = \lambda v$ for some $\lambda > 0$.

Proof of Proposition 5 (continued)

Suppose first that the set $\theta \cap \mathbb{Z}^2$ is non-empty.

Let $v = (v_1, v_2)$ be the element of this set of minimal magnitude.

Suppose that A is in the stabilizer of θ under $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$.

Then $Av = \lambda v$ for some $\lambda > 0$.

Minimality ensures that $\lambda = 1$, thus the stabilizers of θ and v under $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ and $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$ are one and the same.

Proof of Proposition 5 (continued)

Minimality also ensures that v_1 and v_2 are relatively prime.

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Set $B = \begin{bmatrix} a_1 & a_2 \\ -v_1 & v_2 \end{bmatrix}$.

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So there exists $a \in \mathbb{Z}^2$ such that $a \cdot v = 1$.

Set $B = \begin{bmatrix} a_1 & a_2 \\ -v_1 & v_2 \end{bmatrix}$.

Then $B \in \mathrm{SL}_2(\mathbb{Z})$ and $Bv = (1, 0)$.

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So there exists $a \in \mathbb{Z}^2$ such that $a \cdot v = 1$.

Set $B = \begin{bmatrix} a_1 & a_2 \\ -v_1 & v_2 \end{bmatrix}$.

Then $B \in \mathrm{SL}_2(\mathbb{Z})$ and $Bv = (1, 0)$.

So conjugation by B yields an isomorphism of the stabilizer of v under $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ with that of $(1, 0)$.

Proof of Proposition 5 (continued)

The latter consists of the upper unitriangular matrices in $\mathrm{SL}_2(\mathbb{Z})$.

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And this group is trivially infinite cyclic.

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So the stabilizer of v under $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$ is trivial.

Proof of Proposition 5 (continued)

We now handle the case that $\theta \cap \mathbb{Z}^2$ is empty.

Fix any $v \in \theta$.

Then v_1 and v_2 are independent over $\mathbb{Q}$.

So the stabilizer of v under $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$ is trivial.

Let Λ denote the set of eigenvalues of matrices fixing θ.

Lemma 6

The group Λ is cyclic.

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Proof

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Suppose that v is an eigenvalue of $A \in \mathrm{SL}_2(\mathbb{Z})$ with eigenvalue λ.

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Proof

We need only show that Λ is not dense.

Suppose that v is an eigenvalue of $A \in \mathrm{SL}_2(\mathbb{Z})$ with eigenvalue λ.

Let μ denote the other eigenvalue.

Proof of Lemma 6 (continued)

Then $\lambda\mu = \det(A) = 1$.

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Proof of Lemma 6 (continued)

Then $\lambda\mu = \det(A) = 1$.

So $\mathrm{trace}(A) = \lambda + \mu = \lambda + 1/\lambda$.

Thus $\lambda + 1/\lambda \in \mathbb{Z}$.

And the set of such λ cannot be dense. ⊠

Proof of Proposition 5 (continued)

Fix A in the stabilizer of θ under $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ whose corresponding eigenvalue λ generates Λ.

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If B is also in the stabilizer of θ under $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$, then the corresponding eigenvalue is λ^n, for some $n \in \mathbb{Z}$.

Proof of Proposition 5 (continued)

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So $A^n B^{-1}$ is the identity matrix, thus $B = A^n$.

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If B is also in the stabilizer of θ under $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$, then the corresponding eigenvalue is λ^n, for some $n \in \mathbb{Z}$.

So $A^n B^{-1}$ is the identity matrix, thus $B = A^n$.

Hence the stabilizer of θ under $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ is infinite cyclic.

Proof of Proposition 5 (continued)

Note that each non-trivial $A \in \mathrm{SL}_2(\mathbb{Z})$ fixes at most two $\theta \in \mathbb{T}$.

Proof of Proposition 5 (continued)

Note that each non-trivial $A \in \mathrm{SL}_2(\mathbb{Z})$ fixes at most two $\theta \in \mathbb{T}$.

As $\mathrm{SL}_2(\mathbb{Z})$ is countable, it follows that only countably many $\theta \in \mathbb{T}$ have non-trivial stabilizers. ⊠

Proposition 7 (Conley-M)

Suppose that X is a Polish space and $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright X$ is Borel. Then there is a Borel reduction of $E_{\mathrm{SL}_2(\mathbb{Z})}^{\mathbb{T} \times X}$ to $E_{\mathbb{Q}}$.

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Proof

Let $\mathbb{T}'$ denote the $\mathrm{SL}_2(\mathbb{Z})$-invariant Borel set consisting of all $\theta \in \mathbb{T}$ whose stabilizers under $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ are trivial.

Proposition 7 (Conley-M)

Suppose that X is a Polish space and $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright X$ is Borel. Then there is a Borel reduction of $E^{\mathbb{T}\times X}_{\mathrm{SL}_2(\mathbb{Z})}$ to $E_{\mathbb{Q}}$.

Proof

Let $\mathbb{T}'$ denote the $\mathrm{SL}_2(\mathbb{Z})$-invariant Borel set consisting of all $\theta \in \mathbb{T}$ whose stabilizers under $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ are trivial.

Any Borel $\mathbb{Z}$-action generating $E^{\mathbb{T}'}_{\mathrm{SL}_2(\mathbb{Z})}$ induces one for $E^{\mathbb{T}'\times X}_{\mathrm{SL}_2(\mathbb{Z})}$.

Proposition 7 (Conley-M)

Suppose that X is a Polish space and $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright X$ is Borel. Then there is a Borel reduction of $E_{\mathrm{SL}_2(\mathbb{Z})}^{\mathbb{T} \times X}$ to $E_{\mathbb{Q}}$.

Proof

Let $\mathbb{T}'$ denote the $\mathrm{SL}_2(\mathbb{Z})$-invariant Borel set consisting of all $\theta \in \mathbb{T}$ whose stabilizers under $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$ are trivial.

Any Borel $\mathbb{Z}$-action generating $E_{\mathrm{SL}_2(\mathbb{Z})}^{\mathbb{T}'}$ induces one for $E_{\mathrm{SL}_2(\mathbb{Z})}^{\mathbb{T}' \times X}$.

So there is a Borel reduction of $E_{\mathrm{SL}_2(\mathbb{Z})}^{\mathbb{T}' \times X}$ to $E_{\mathbb{Q}}$.

Proof of Proposition 7 (continued)

Suppose now that $\theta \in \mathbb{T} \setminus \mathbb{T}'$.

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Proof of Proposition 7 (continued)

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Let Z denote the stabilizer of θ under $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$.

Then $E_{\mathrm{SL}_2(\mathbb{Z})}^{\mathbb{T} \times X} \upharpoonright (\{\theta\} \times X) = E_Z^{\mathbb{T} \times X} \upharpoonright (\{\theta\} \times X)$.

Proof of Proposition 7 (continued)

Suppose now that $\theta \in \mathbb{T} \setminus \mathbb{T}'$.

Let Z denote the stabilizer of θ under $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$.

Then $E_{\mathrm{SL}_2(\mathbb{Z})}^{\mathbb{T} \times X} \upharpoonright (\{\theta\} \times X) = E_Z^{\mathbb{T} \times X} \upharpoonright (\{\theta\} \times X)$.

And the latter is Borel reducible to $E_{\mathbb{Q}}$.

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Suppose now that $\theta \in \mathbb{T} \setminus \mathbb{T}'$.

Let Z denote the stabilizer of θ under $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}$.

Then $E_{\mathrm{SL}_2(\mathbb{Z})}^{\mathbb{T}\times X} \upharpoonright (\{\theta\} \times X) = E_Z^{\mathbb{T}\times X} \upharpoonright (\{\theta\} \times X)$.

And the latter is Borel reducible to $E_{\mathbb{Q}}$.

So $E_{\mathrm{SL}_2(\mathbb{Z})}^{\mathbb{T}\times X}$ is Borel reducible to $E_{\mathbb{Q}}$. ⊠

Definition

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Definition

Let $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}^2$ denote the action induced by $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$.

Proposition 8 (Conley-M)

Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\mathcal{I}$ is a σ-ideal on X, $\rho\colon E \to \mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ is a Borel function, φ and ψ are ρ-invariant Borel homomorphisms from E to $E_{\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})}^{\mathbb{R}^2}$, and φ is $\mathcal{I}$-to-one. Then there is an $\mathcal{I}$-to-one Borel homomorphism from $E \restriction D(\varphi, \psi)$ to $E_{\mathrm{SL}_2(\mathbb{Z})}^{\mathbb{T} \times \mathbb{T}^2}$.

Proof

Define $\sigma\colon E \restriction D(\varphi, \psi) \to \mathrm{SL}_2(\mathbb{Z})$ by $\sigma(x, y) = \mathrm{proj}_{\mathrm{SL}_2(\mathbb{Z})} \circ \rho(x, y)$.

Proof

Define $\sigma\colon E \restriction D(\varphi,\psi) \to \mathrm{SL}_2(\mathbb{Z})$ by $\sigma(x,y) = \mathrm{proj}_{\mathrm{SL}_2(\mathbb{Z})} \circ \rho(x,y)$.

Note that $\mathrm{proj}_{\mathbb{T}^2} \circ \varphi(x) \restriction D(\varphi,\psi)$ is σ-invariant.

Proof

Define $\sigma \colon E \restriction D(\varphi, \psi) \to \mathrm{SL}_2(\mathbb{Z})$ by $\sigma(x, y) = \mathrm{proj}_{\mathrm{SL}_2(\mathbb{Z})} \circ \rho(x, y)$.

Note that $\mathrm{proj}_{\mathbb{T}^2} \circ \varphi(x) \restriction D(\varphi, \psi)$ is σ-invariant.

Define $\pi \colon D(\varphi, \psi) \to \mathbb{T}$ by $\pi(x) = \mathrm{proj}_{\mathbb{T}}(\varphi(x) - \psi(x))$.

Lemma 9

The function π is σ-invariant.

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Proof

If $x_1, x_2 \in D(\varphi, \psi)$ are E-related, then

$$\begin{aligned}
\pi(x_1) &= \mathrm{proj}_{\mathbb{T}}(\varphi(x_1) - \psi(x_1)) \\
&= \mathrm{proj}_{\mathbb{T}}(\rho(x_1, x_2) \cdot \varphi(x_2) - \rho(x_1, x_2) \cdot \psi(x_2)) \\
&= \mathrm{proj}_{\mathbb{T}}(\sigma(x_1, x_2) \cdot \varphi(x_2) - \sigma(x_1, x_2) \cdot \psi(x_2)) \\
&= \mathrm{proj}_{\mathbb{T}}(\sigma(x_1, x_2) \cdot (\varphi(x_2) - \psi(x_2))) \\
&= \sigma(x_1, x_2) \cdot \mathrm{proj}_{\mathbb{T}}(\varphi(x_2) - \psi(x_2)) \\
&= \sigma(x_1, x_2) \cdot \pi(x_2),
\end{aligned}$$

thus π is σ-invariant. ⊠

Proof of Proposition 8 (continued)

So both $\mathrm{proj}_{\mathbb{T}^2} \circ \varphi \restriction D(\varphi, \psi)$ and π are σ-invariant.

Proof of Proposition 8 (continued)

So both $\mathrm{proj}_{\mathbb{T}^2} \circ \varphi \restriction D(\varphi, \psi)$ and π are σ-invariant.

Thus their product is a homomorphism to $E^{\mathbb{T}\times\mathbb{T}^2}_{\mathrm{SL}_2(\mathbb{Z})}$.

Proof of Proposition 8 (continued)

So both $\mathrm{proj}_{\mathbb{T}^2} \circ \varphi \upharpoonright D(\varphi, \psi)$ and π are σ-invariant.

Thus their product is a homomorphism to $E_{\mathrm{SL}_2(\mathbb{Z})}^{\mathbb{T} \times \mathbb{T}^2}$.

As φ is $\mathcal{I}$-to-one, so too is the product. ⊠

Theorem 10 (Conley-M)

The action $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2$ is locally rigid.

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Proof

Suppose that X is a Polish space, E is a countable Borel equivalence relation on X, $\rho\colon E \to \mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ is Borel, and φ and ψ are countable-to-one Borel homomorphisms from E to $E^{\mathbb{R}^2}_{\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})}$.

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It then follows that there is a countable-to-one Borel homomorphism from $E \restriction D(\varphi, \psi)$ to $E_{\mathbb{Q}}$.

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It then follows that there is a countable-to-one Borel homomorphism from $E \restriction D(\varphi, \psi)$ to $E_{\mathbb{Q}}$.

So $E \restriction D(\varphi, \psi)$ is Borel reducible to $E_{\mathbb{Q}}$. ⊠

Definable cardinals just beyond $\mathbb{R}/\mathbb{Q}$

Winter school in abstract analysis
Sporthotel Kácov
January 17th, 2013

Benjamin Miller
Westfälische Wilhelms-Universität Münster

Part III

Separability

Definition

We use $L(X, \mu, Y)$ to denote the family of all μ-measurable functions $\varphi\colon D \to Y$ with μ-positive domains $D \subseteq X$.

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Definition

We view $L(X, \mu, Y)$ as a pseudo-metric space, equipped with the pseudo-metric d_μ given by $d_\mu(\varphi, \psi) = \mu(D(\varphi, \psi))$.

Proposition 1

Suppose that X and Y are Polish spaces, μ is a finite Borel measure on X, and $\mathscr{L} \subseteq L(X, \mu, Y)$. Then $\mathscr{L}$ is separable iff there is a Borel set $R \subseteq X \times Y$, whose vertical sections are all countable, such that

$$\forall \varphi \in \mathscr{L} \;\; \mu(\{x \in \mathrm{dom}(\varphi) \mid \neg x \, R \, \varphi(x)\}) = 0.$$

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$$\forall \varphi \in \mathscr{L} \ \mu(\{x \in \mathrm{dom}(\varphi) \mid \neg x \, R \, \varphi(x)\}) = 0.$$

Proof

Suppose that $R \subseteq X \times Y$ is a Borel set, whose vertical sections are all countable, such that $\forall \varphi \in \mathscr{L} \ \mu(\{x \in \mathrm{dom}(\varphi) \mid \neg x \, R \, \varphi(x)\}) = 0$.

Proof of Proposition 1 (continued)

Fix a countable algebra $\mathscr{A}$ of Borel subsets of X such that for all Borel sets $B \subseteq X$ and $\epsilon > 0$, there exists $A \in \mathscr{A}$ with $\mu(A \triangle B) \leq \epsilon$.

Proof of Proposition 1 (continued)

Fix a countable algebra $\mathscr{A}$ of Borel subsets of X such that for all Borel sets $B \subseteq X$ and $\epsilon > 0$, there exists $A \in \mathscr{A}$ with $\mu(A \triangle B) \leq \epsilon$.

Fix a countable family $\mathscr{F}$ of Borel functions $f\colon D \to Y$, with Borel domains $D \subseteq X$, such that $R = \bigcup_{f \in \mathscr{F}} \text{graph}(f)$.

Proof of Proposition 1 (continued)

Fix a countable algebra $\mathscr{A}$ of Borel subsets of X such that for all Borel sets $B \subseteq X$ and $\epsilon > 0$, there exists $A \in \mathscr{A}$ with $\mu(A \triangle B) \leq \epsilon$.

Fix a countable family $\mathscr{F}$ of Borel functions $f \colon D \to Y$, with Borel domains $D \subseteq X$, such that $R = \bigcup_{f \in \mathscr{F}} \text{graph}(f)$.

One obtains a dense set by considering $(f_1 \restriction A_1) \cup \cdots \cup (f_n \restriction A_n)$, where $n \in \mathbb{N}$, $A_1, \ldots, A_n \in \mathscr{A}$, and $f_1, \ldots, f_n \in \mathscr{F}$. ⊠

Proposition 2

Suppose that X, Y, and Z are Polish spaces, μ is a finite Borel measure on X, $\mathscr{L}_{XY} \subseteq L(X, \mu, Y)$, $\mathscr{L}_{XZ} \subseteq L(X, \mu, Z)$, there is a countable-to-one Borel function $f \colon Y \to Z$ with $f \circ \mathscr{L}_{XY} \subseteq \mathscr{L}_{XZ}$, and $\mathscr{L}_{XZ}$ is separable. Then $\mathscr{L}_{XY}$ is separable.

Proposition 2

Suppose that X, Y, and Z are Polish spaces, μ is a finite Borel measure on X, $\mathscr{L}_{XY} \subseteq L(X, \mu, Y)$, $\mathscr{L}_{XZ} \subseteq L(X, \mu, Z)$, there is a countable-to-one Borel function $f \colon Y \to Z$ with $f \circ \mathscr{L}_{XY} \subseteq \mathscr{L}_{XZ}$, and $\mathscr{L}_{XZ}$ is separable. Then $\mathscr{L}_{XY}$ is separable.

Proof

Fix a Borel set $S \subseteq Y \times Z$, whose vertical sections are all countable, such that $\forall \varphi \in \mathscr{L}_{XZ}\ \mu(\{x \in \mathrm{dom}(\varphi) \mid \neg x \mathrel{S} \varphi(x)\}) = 0$.

Proposition 2

Suppose that X, Y, and Z are Polish spaces, μ is a finite Borel measure on X, $\mathscr{L}_{XY} \subseteq L(X, \mu, Y)$, $\mathscr{L}_{XZ} \subseteq L(X, \mu, Z)$, there is a countable-to-one Borel function $f \colon Y \to Z$ with $f \circ \mathscr{L}_{XY} \subseteq \mathscr{L}_{XZ}$, and $\mathscr{L}_{XZ}$ is separable. Then $\mathscr{L}_{XY}$ is separable.

Proof

Fix a Borel set $S \subseteq Y \times Z$, whose vertical sections are all countable, such that $\forall \varphi \in \mathscr{L}_{XZ}\ \mu(\{x \in \mathrm{dom}(\varphi) \mid \neg x\ S\ \varphi(x)\}) = 0$.

Set $R = \{(x, y) \in X \times Y \mid x\ S\ f(y)\}$. ⊠

Definition

Let $\mathrm{Hom}_{\leq\aleph_0\text{-to-1}}(E, \mu, F)$ denote the set of countable-to-one homomorphisms $\varphi \in L(X, \mu, Y)$ from $E \restriction \mathrm{dom}(\varphi)$ to F.

Proposition 3

Suppose that X, Y, and Z are Polish spaces, E, F, and G are countable Borel equivalence relations on X, Y, and Z, μ is a finite Borel measure on X, there is a countable-to-one Borel homomorphism $\varphi \colon Y \to Z$ from F to G, and $\mathrm{Hom}_{\leq \aleph_0\text{-to-}1}(E, \mu, G)$ is separable. Then $\mathrm{Hom}_{\leq \aleph_0\text{-to-}1}(E, \mu, F)$ is separable.

Proposition 3

Suppose that X, Y, and Z are Polish spaces, E, F, and G are countable Borel equivalence relations on X, Y, and Z, μ is a finite Borel measure on X, there is a countable-to-one Borel homomorphism $\varphi \colon Y \to Z$ from F to G, and $\mathrm{Hom}_{\leq\aleph_0\text{-to-}1}(E, \mu, G)$ is separable. Then $\mathrm{Hom}_{\leq\aleph_0\text{-to-}1}(E, \mu, F)$ is separable.

Proof

By the previous proposition, it is clearly sufficient to observe that $\varphi \circ \mathrm{Hom}_{\leq\aleph_0\text{-to-}1}(E, \mu, F) \subseteq \mathrm{Hom}_{\leq\aleph_0\text{-to-}1}(E, \mu, G)$. ⊠

Definition

We say that E is μ-nowhere hyperfinite if there is no μ-positive Borel set $B \subseteq X$ with the property that $E \upharpoonright B$ is hyperfinite.

Proposition 4

Suppose that X and Y are Polish spaces, E and F are countable Borel equivalence relations on X and Y, μ is a finite Borel measure on X for which E is μ-nowhere hyperfinite, and F is the orbit equivalence relation of a locally rigid Borel action $\Gamma \curvearrowright Y$ of a countable group. Then $\mathrm{Hom}_{\leq \aleph_0\text{-to-}1}(E, \mu, F)$ is separable.

Proof

Fix real numbers $\epsilon_n > 0$ such that $\sum_{n\in\mathbb{N}} \epsilon_n < \infty$.

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Let μ_c denote the counting measure on X.

Fix an increasing sequence of Borel sets $R_n \subseteq X \times X$ such that $E = \bigcup_{n \in \mathbb{N}} R_n$ and each vertical section of R_n has cardinality $\leq n$.

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For each $n \in \mathbb{N}$, set $\nu_n = (\mu \times \mu_c) \restriction R_n$.

Proof

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Proof of Proposition 4 (continued)

Fix countable dense sets $\mathscr{D}_n \subseteq L(R_n, \nu_n, \Gamma)$.

Proof of Proposition 4 (continued)

Fix countable dense sets $\mathscr{D}_n \subseteq L(R_n, \nu_n, \Gamma)$.

Let $\mathscr{D}'_n$ denote the set of $\rho \in \mathscr{D}_n$ for which there exist $\sigma \in L(E, \nu, \Gamma)$ and $\varphi \in \mathrm{Hom}_{\leq \aleph_0\text{-to-}1}(E, \mu, F)$ such that $\mathrm{dom}(\sigma) = E \restriction \mathrm{dom}(\varphi)$, φ is σ-invariant, and $d_{\nu_n}(\sigma \restriction (R_n \restriction \mathrm{dom}(\varphi)), \rho) \leq \epsilon_n$.

Proof of Proposition 4 (continued)

Fix countable dense sets $\mathscr{D}_n \subseteq L(R_n, \nu_n, \Gamma)$.

Let $\mathscr{D}'_n$ denote the set of $\rho \in \mathscr{D}_n$ for which there exist $\sigma \in L(E, \nu, \Gamma)$ and $\varphi \in \mathrm{Hom}_{\leq \aleph_0\text{-to-}1}(E, \mu, F)$ such that $\mathrm{dom}(\sigma) = E \upharpoonright \mathrm{dom}(\varphi)$, φ is σ-invariant, and $d_{\nu_n}(\sigma \upharpoonright (R_n \upharpoonright \mathrm{dom}(\varphi)), \rho) \leq \epsilon_n$.

Fix such a $\sigma_{n,\rho}$ and $\varphi_{n,\rho}$ for each $n \in \mathbb{N}$ and $\rho \in \mathscr{D}'_n$.

Proof of Proposition 4 (continued)

Fix countable dense sets $\mathscr{D}_n \subseteq L(R_n, \nu_n, \Gamma)$.

Let $\mathscr{D}'_n$ denote the set of $\rho \in \mathscr{D}_n$ for which there exist $\sigma \in L(E, \nu, \Gamma)$ and $\varphi \in \mathrm{Hom}_{\leq\aleph_0\text{-to-}1}(E, \mu, F)$ such that $\mathrm{dom}(\sigma) = E \restriction \mathrm{dom}(\varphi)$, φ is σ-invariant, and $d_{\nu_n}(\sigma \restriction (R_n \restriction \mathrm{dom}(\varphi)), \rho) \leq \epsilon_n$.

Fix such a $\sigma_{n,\rho}$ and $\varphi_{n,\rho}$ for each $n \in \mathbb{N}$ and $\rho \in \mathscr{D}'_n$.

We will show that the set $\Phi = \{\varphi_{n,\rho} \mid n \in \mathbb{N} \text{ and } \rho \in \mathscr{D}'_n\}$ is dense.

Proof of Proposition 4 (continued)

Suppose that $\epsilon > 0$ and $\varphi \in \mathrm{Hom}_{\leq \aleph_0\text{-to-1}}(E, \mu, F)$.

Proof of Proposition 4 (continued)

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We can assume that there exists $\sigma \in L(E, \nu, \Gamma)$ with the property that $\mathrm{dom}(\sigma) = E \restriction \mathrm{dom}(\varphi)$ and φ is σ-invariant.

Proof of Proposition 4 (continued)

Suppose that $\epsilon > 0$ and $\varphi \in \mathrm{Hom}_{\leq\aleph_0\text{-to-}1}(E, \mu, F)$.

We can assume that there exists $\sigma \in L(E, \nu, \Gamma)$ with the property that $\mathrm{dom}(\sigma) = E \restriction \mathrm{dom}(\varphi)$ and φ is σ-invariant.

For $n \in \mathbb{N}$, fix $\rho_n \in \mathscr{D}_n$ such that $d_{\nu_n}(\sigma \restriction (R_n \restriction \mathrm{dom}(\varphi)), \rho_n) \leq \epsilon_n$.

Proof of Proposition 4 (continued)

Suppose that $\epsilon > 0$ and $\varphi \in \mathrm{Hom}_{\leq \aleph_0\text{-to-}1}(E, \mu, F)$.

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Set $\sigma_n = \sigma_{n,\rho_n}$ and $\varphi_n = \varphi_{n,\rho_n}$.

Proof of Proposition 4 (continued)

Suppose that $\epsilon > 0$ and $\varphi \in \mathrm{Hom}_{\leq\aleph_0\text{-to-}1}(E, \mu, F)$.

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For $n \in \mathbb{N}$, fix $\rho_n \in \mathscr{D}_n$ such that $d_{\nu_n}(\sigma \restriction (R_n \restriction \mathrm{dom}(\varphi)), \rho_n) \leq \epsilon_n$.

Set $\sigma_n = \sigma_{n,\rho_n}$ and $\varphi_n = \varphi_{n,\rho_n}$.

Note that $d_{\nu_n}(\sigma \restriction (R_n \restriction \mathrm{dom}(\varphi)), \sigma_n \restriction (R_n \restriction \mathrm{dom}(\varphi_n))) \leq 2\epsilon_n$.

Proof of Proposition 4 (continued)

For each $n \in \mathbb{N}$, let E_n denote the equivalence relation generated by the set $D_n = \mathrm{dom}(\sigma) \cap R_n \setminus D(\sigma, \sigma_n)$.

Proof of Proposition 4 (continued)

For each $n \in \mathbb{N}$, let E_n denote the equivalence relation generated by the set $D_n = \mathrm{dom}(\sigma) \cap R_n \setminus D(\sigma, \sigma_n)$.

Also for each $n \in \mathbb{N}$, define $F_n = \bigcap_{m \geq n} E_m$ and

$$X_n = \{x \in \mathrm{dom}(\varphi) \mid \exists y \in \mathrm{dom}(\varphi) \cap (R_n)_x \ \sigma(x, y) \neq \sigma_n(x, y)\}.$$

Proof of Proposition 4 (continued)

For each $n \in \mathbb{N}$, let E_n denote the equivalence relation generated by the set $D_n = \mathrm{dom}(\sigma) \cap R_n \setminus D(\sigma, \sigma_n)$.

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$$X_n = \{x \in \mathrm{dom}(\varphi) \mid \exists y \in \mathrm{dom}(\varphi) \cap (R_n)_x \; \sigma(x, y) \neq \sigma_n(x, y)\}.$$

So $\mu(X_n) \leq d_{\nu_n}(\sigma \restriction (R_n \restriction \mathrm{dom}(\varphi)), \sigma_n \restriction (R_n \restriction \mathrm{dom}(\varphi_n))) \leq 2\epsilon_n$.

Proof of Proposition 4 (continued)

For each $n \in \mathbb{N}$, let E_n denote the equivalence relation generated by the set $D_n = \mathrm{dom}(\sigma) \cap R_n \setminus D(\sigma, \sigma_n)$.

Also for each $n \in \mathbb{N}$, define $F_n = \bigcap_{m \geq n} E_m$ and

$$X_n = \{x \in \mathrm{dom}(\varphi) \mid \exists y \in \mathrm{dom}(\varphi) \cap (R_n)_x \ \sigma(x, y) \neq \sigma_n(x, y)\}.$$

So $\mu(X_n) \leq d_{\nu_n}(\sigma \restriction (R_n \restriction \mathrm{dom}(\varphi)), \sigma_n \restriction (R_n \restriction \mathrm{dom}(\varphi_n))) \leq 2\epsilon_n$.

Thus the set $C = \sim \bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} X_m$ is μ-conull.

Proof of Proposition 4 (continued)

Note that $E \restriction (C \cap \mathrm{dom}(\varphi)) \subseteq \bigcup_{n \in \mathbb{N}} F_n$.

Proof of Proposition 4 (continued)

Note that $E \restriction (C \cap \mathrm{dom}(\varphi)) \subseteq \bigcup_{n \in \mathbb{N}} F_n$.

Fix $n \in \mathbb{N}$ for which there is a Borel set $B \subseteq C \cap \mathrm{dom}(\varphi)$ such that $\mu(\mathrm{dom}(\varphi) \setminus B) \leq \epsilon$ and $F_n \restriction B$ is $(\mu \restriction B)$-nowhere hyperfinite.

Proof of Proposition 4 (continued)

Note that $E \restriction (C \cap \mathrm{dom}(\varphi)) \subseteq \bigcup_{n \in \mathbb{N}} F_n$.

Fix $n \in \mathbb{N}$ for which there is a Borel set $B \subseteq C \cap \mathrm{dom}(\varphi)$ such that $\mu(\mathrm{dom}(\varphi) \setminus B) \leq \epsilon$ and $F_n \restriction B$ is $(\mu \restriction B)$-nowhere hyperfinite.

Observe that both φ and φ_n are $(\sigma \restriction (F_n \restriction B))$-invariant.

Proof of Proposition 4 (continued)

Note that $E \restriction (C \cap \mathrm{dom}(\varphi)) \subseteq \bigcup_{n \in \mathbb{N}} F_n$.

Fix $n \in \mathbb{N}$ for which there is a Borel set $B \subseteq C \cap \mathrm{dom}(\varphi)$ such that $\mu(\mathrm{dom}(\varphi) \setminus B) \leq \epsilon$ and $F_n \restriction B$ is $(\mu \restriction B)$-nowhere hyperfinite.

Observe that both φ and φ_n are $(\sigma \restriction (F_n \restriction B))$-invariant.

So local rigidity ensures that $\varphi \restriction B = \varphi_n \restriction B$.

Proof of Proposition 4 (continued)

Note that $E \restriction (C \cap \mathrm{dom}(\varphi)) \subseteq \bigcup_{n \in \mathbb{N}} F_n$.

Fix $n \in \mathbb{N}$ for which there is a Borel set $B \subseteq C \cap \mathrm{dom}(\varphi)$ such that $\mu(\mathrm{dom}(\varphi) \setminus B) \leq \epsilon$ and $F_n \restriction B$ is $(\mu \restriction B)$-nowhere hyperfinite.

Observe that both φ and φ_n are $(\sigma \restriction (F_n \restriction B))$-invariant.

So local rigidity ensures that $\varphi \restriction B = \varphi_n \restriction B$.

Thus $d_\mu(\varphi, \varphi_n) \leq \epsilon$. ⊠

Definition

We say that F has separable homomorphisms if whenever X is a Polish space, E is a countable Borel equivalence relation on X, and μ is a finite Borel measure on X for which E is μ-nowhere hyperfinite, the space $\mathrm{Hom}_{\leq \aleph_0\text{-to-}1}(E, \mu, F)$ is separable.

Definition

We say that F has separable homomorphisms if whenever X is a Polish space, E is a countable Borel equivalence relation on X, and μ is a finite Borel measure on X for which E is μ-nowhere hyperfinite, the space $\mathrm{Hom}_{\leq\aleph_0\text{-to-}1}(E, \mu, F)$ is separable.

Theorem 5 (Conley-M)

The family of countable Borel equivalence relations on Polish spaces with separable homomorphisms is closed downward under countable-to-one Borel homomorphism, and includes every orbit equivalence relation of a locally rigid Borel action of a countable group.

Part IV

Borel reducibility

The following results are joint with Clinton Conley.

Definition

An equivalence relation is μ-hyperfinite if there is a μ-conull Borel set on which it is hyperfinite.

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Definition

An equivalence relation is measure hyperfinite if it μ-hyperfinite for every finite Borel measure μ on the underlying space.

Theorem 6

Suppose that X is a Polish space, E is a countable Borel equivalence relation on X with separable homomorphisms, and μ is a finite Borel measure on X. Then exactly one of the following holds:

1. The equivalence relation E is μ-hyperfinite.
2. There is no $(\mu \times m)$-measurable reduction of $E \times \Delta(\mathbb{R})$ to E.

Theorem 6

Suppose that X is a Polish space, E is a countable Borel equivalence relation on X with separable homomorphisms, and μ is a finite Borel measure on X. Then exactly one of the following holds:

1. The equivalence relation E is μ-hyperfinite.
2. There is no $(\mu \times m)$-measurable reduction of $E \times \Delta(\mathbb{R})$ to E.

Proof

Fix a μ-positive Borel $A \subseteq X$ on which E is μ-nowhere hyperfinite.

Theorem 6

Suppose that X is a Polish space, E is a countable Borel equivalence relation on X with separable homomorphisms, and μ is a finite Borel measure on X. Then exactly one of the following holds:

1. The equivalence relation E is μ-hyperfinite.
2. There is no $(\mu \times m)$-measurable reduction of $E \times \Delta(\mathbb{R})$ to E.

Proof

Fix a μ-positive Borel $A \subseteq X$ on which E is μ-nowhere hyperfinite.

Define $\nu(B) = \mu(A \cap B)$.

Proof of Theorem 6 (continued)

Suppose that $\varphi\colon X \times \mathbb{R} \to X$ is a $(\mu \times m)$-measurable reduction.

Proof of Theorem 6 (continued)

Suppose that $\varphi\colon X \times \mathbb{R} \to X$ is a $(\mu \times m)$-measurable reduction.

For each $r \in \mathbb{R}$, define $\varphi_r\colon A \to X$ by $\varphi_r(x) = \varphi(x, r)$.

Proof of Theorem 6 (continued)

Suppose that $\varphi\colon X \times \mathbb{R} \to X$ is a $(\mu \times m)$-measurable reduction.

For each $r \in \mathbb{R}$, define $\varphi_r\colon A \to X$ by $\varphi_r(x) = \varphi(x, r)$.

Fix an uncountable set $R \subseteq \mathbb{R}$ such that $\forall r \in R$ φ_r is ν-measurable.

Proof of Theorem 6 (continued)

Suppose that $\varphi\colon X \times \mathbb{R} \to X$ is a $(\mu \times m)$-measurable reduction.

For each $r \in \mathbb{R}$, define $\varphi_r\colon A \to X$ by $\varphi_r(x) = \varphi(x, r)$.

Fix an uncountable set $R \subseteq \mathbb{R}$ such that $\forall r \in R$ φ_r is ν-measurable.

Note that $d_\nu(\varphi_r, \varphi_s) = \mu(A)$ for all distinct $r, s \in R$.

Proof of Theorem 6 (continued)

Suppose that $\varphi\colon X \times \mathbb{R} \to X$ is a $(\mu \times m)$-measurable reduction.

For each $r \in \mathbb{R}$, define $\varphi_r\colon A \to X$ by $\varphi_r(x) = \varphi(x, r)$.

Fix an uncountable set $R \subseteq \mathbb{R}$ such that $\forall r \in R$ φ_r is ν-measurable.

Note that $d_\nu(\varphi_r, \varphi_s) = \mu(A)$ for all distinct $r, s \in R$.

This contradicts separability of $\mathrm{Hom}_{\leq \aleph_0\text{-to-}1}(E, \nu, F)$. ☒

Theorem 7

Suppose that X and Y are Polish spaces, E and F are countable Borel equivalence relations on X and Y, μ is a finite Borel measure on X, $\mathcal{I}$ is a σ-ideal on Y, E is μ-nowhere hyperfinite, and F has separable homomorphisms. Then there is an $\mathcal{I}$-conull set $C \subseteq Y$ such that every $\varphi \in \mathrm{Hom}_{\leq \aleph_0\text{-to-}1}(E, \mu, F \upharpoonright C)$ sends μ-positive sets to $\mathcal{I}$-positive sets.

Proof

Fix a sequence $(\varphi_\alpha)_{\alpha<\beta}$ of maximal length consisting of functions in $\mathrm{Hom}_{\leq\aleph_0\text{-to-}1}(E, \mu, F)$ with pairwise disjoint $\mathcal{I}$-null ranges.

Proof

Fix a sequence $(\varphi_\alpha)_{\alpha<\beta}$ of maximal length consisting of functions in $\mathrm{Hom}_{\leq\aleph_0\text{-to-}1}(E, \mu, F)$ with pairwise disjoint $\mathcal{I}$-null ranges.

The separability of $\mathrm{Hom}_{\leq\aleph_0\text{-to-}1}(E, \mu, F)$ ensures that $\beta < \omega_1$.

Proof

Fix a sequence $(\varphi_\alpha)_{\alpha<\beta}$ of maximal length consisting of functions in $\mathrm{Hom}_{\leq\aleph_0\text{-to-}1}(E, \mu, F)$ with pairwise disjoint $\mathcal{I}$-null ranges.

The separability of $\mathrm{Hom}_{\leq\aleph_0\text{-to-}1}(E, \mu, F)$ ensures that $\beta < \omega_1$.

Set $C = Y \setminus \bigcup_{\alpha<\beta} \mathrm{rng}(\varphi_\alpha)$. ⊠

Theorem 8

Suppose that X is a Polish space and E is a countable Borel equivalence relation on X with separable homomorphisms. Then exactly one of the following holds:

1. The equivalence relation E is measure hyperfinite.
2. There is a Borel set $B \subseteq X$ for which there is a finite Borel measure μ on B with the property that there is no $(\mu \times 2)$-measurable reduction of $(E \upharpoonright B \times \Delta(2))$ to $E \upharpoonright B$.

Theorem 8

Suppose that X is a Polish space and E is a countable Borel equivalence relation on X with separable homomorphisms. Then exactly one of the following holds:

1. The equivalence relation E is measure hyperfinite.
2. There is a Borel set $B \subseteq X$ for which there is a finite Borel measure μ on B with the property that there is no $(\mu \times 2)$-measurable reduction of $(E \restriction B \times \Delta(2))$ to $E \restriction B$.

Proof

We can assume there is a finite Borel measure ν on X with the property that E is not ν-hyperfinite.

Proof of Theorem 8 (continued)

We can assume that every E-invariant Borel set is ν-null or ν-conull.

Proof of Theorem 8 (continued)

We can assume that every E-invariant Borel set is ν-null or ν-conull.

Then there exists a ν-conull Borel set $B \subseteq X$ with the property that every $\varphi \in \mathrm{Hom}_{\leq\aleph_0\text{-to-}1}((E \upharpoonright B) \times \Delta(2), \nu, E \upharpoonright B)$ sends sets of positive measure to sets of positive measure.

Proof of Theorem 8 (continued)

We can assume that every E-invariant Borel set is ν-null or ν-conull.

Then there exists a ν-conull Borel set $B \subseteq X$ with the property that every $\varphi \in \mathrm{Hom}_{\leq \aleph_0\text{-to-}1}((E \restriction B) \times \Delta(2), \nu, E \restriction B)$ sends sets of positive measure to sets of positive measure.

But clearly there can be no such φ. ☒

Theorem 9

Suppose that X is a Polish space, E is a countable treeable Borel equivalence relation on X with separable homomorphisms, and μ is a finite Borel measure on X such that E is not μ-hyperfinite. Then there is a μ-positive Borel set $B \subseteq X$ for which there is an increasing sequence $(F_r)_{r \in \mathbb{R}}$ of Borel subequivalence relations of E such that for no distinct $r, s \in \mathbb{R}$ is there a μ-positive set $A \subseteq B$ on which there is a μ-measurable reduction of $F_r \upharpoonright A$ to F_s.

Proof (Sketch)

Fix a μ-positive Borel $C \subseteq X$ on which E is μ-nowhere hyperfinite.

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Fix a μ-positive Borel $C \subseteq X$ on which E is μ-nowhere hyperfinite.

Fix an acyclic Borel graph G generating $E \restriction C$.

Proof (Sketch)

Fix a μ-positive Borel $C \subseteq X$ on which E is μ-nowhere hyperfinite.

Fix an acyclic Borel graph G generating $E \upharpoonright C$.

Construct a Borel subgraph $H \subseteq G$ whose induced equivalence relation F is hyperfinite but μ-nowhere smooth.

Proof of Theorem 9 (continued)

Construct an increasing sequence $(H_r)_{r \in \mathbb{R}}$ of Borel graphs lying between H and G with the property that for all real numbers $r < s$, the projection of $H_s \setminus H_r$ contains points of μ-almost every E-class.

Proof of Theorem 9 (continued)

Construct an increasing sequence $(H_r)_{r \in \mathbb{R}}$ of Borel graphs lying between H and G with the property that for all real numbers $r < s$, the projection of $H_s \setminus H_r$ contains points of μ-almost every E-class.

Then for no distinct $r, s \in \mathbb{R}$ is there a μ-positive set on which the equivalence relations F_r and F_s induced by H_r and H_s agree.

Proof of Theorem 9 (continued)

We can assume there is a μ-positive Borel set $B \subseteq C$ on which each F_r is μ-nowhere hyperfinite.

Proof of Theorem 9 (continued)

We can assume there is a μ-positive Borel set $B \subseteq C$ on which each F_r is μ-nowhere hyperfinite.

Then for each $r \in \mathbb{R}$, there are only countably many $s \in \mathbb{R}$ for which there is a μ-positive set $A \subseteq B$ with the property that there is a $(\mu \restriction A)$-measurable reduction of $F_r \restriction A$ to F_s.

Proof of Theorem 9 (continued)

We can assume there is a μ-positive Borel set $B \subseteq C$ on which each F_r is μ-nowhere hyperfinite.

Then for each $r \in \mathbb{R}$, there are only countably many $s \in \mathbb{R}$ for which there is a μ-positive set $A \subseteq B$ with the property that there is a $(\mu \restriction A)$-measurable reduction of $F_r \restriction A$ to F_s.

So we can assume that for no distinct $r, s \in \mathbb{R}$ is there a μ-positive set $A \subseteq B$ on which there is a μ-measurable reduction. ⊠

Theorem 10

Suppose that X is a Polish space and E is a countable treeable Borel equivalence relation on X which has separable homomorphisms but is not measure hyperfinite. Then there is no basis of cardinality strictly less than $\mathtt{add}(\mathtt{null})$ for the family of non-measure hyperfinite countable Borel equivalence relations which admit countable-to-one Borel homomorphisms to E.

Proof

Fix a finite Borel measure μ for which E is not μ-hyperfinite.

Proof

Fix a finite Borel measure μ for which E is not μ-hyperfinite.

Fix a μ-positive Borel set $B \subseteq X$ and a sequence $(F_r)_{r \in \mathbb{R}}$ of Borel subequivalence relations of E such that for no distinct $r, s \in \mathbb{R}$ is there a μ-positive set $A \subseteq B$ on which there is a μ-measurable reduction of $F_r \upharpoonright A$ to F_s.

Proof of Theorem 10 (continued)

Suppose that $\mathscr{B}$ is a basis of cardinality strictly less than $\mathtt{add}(\mathtt{null})$.

Proof of Theorem 10 (continued)

Suppose that $\mathscr{B}$ is a basis of cardinality strictly less than $\texttt{add}(\texttt{null})$.

For each $E \in \mathscr{B}$, fix a finite Borel measure μ_E such that E is μ_E-nowhere hyperfinite.

Proof of Theorem 10 (continued)

Suppose that $\mathscr{B}$ is a basis of cardinality strictly less than $\texttt{add}(\texttt{null})$.

For each $E \in \mathscr{B}$, fix a finite Borel measure μ_E such that E is μ_E-nowhere hyperfinite.

We can assume that every μ_E-measurable reduction of E to any F_r sends μ_E-positive sets to μ-positive sets.

Proof of Theorem 10 (continued)

Fix $E_r \in \mathscr{B}$ and μ-measurable reductions φ_r of E_r to F_r.

Proof of Theorem 10 (continued)

Fix $E_r \in \mathscr{B}$ and μ-measurable reductions φ_r of E_r to F_r.

Fix distinct $r, s \in \mathbb{R}$ such that $E_r = E_s$ and φ_r and φ_s agree on a μ_{E_r}-positive set.

Proof of Theorem 10 (continued)

Fix $E_r \in \mathscr{B}$ and μ-measurable reductions φ_r of E_r to F_r.

Fix distinct $r, s \in \mathbb{R}$ such that $E_r = E_s$ and φ_r and φ_s agree on a μ_{E_r}-positive set.

Then F_r and F_s agree on a μ-positive set. ⊠

Theorem 11

Suppose that X is a Polish space and E is a countable treeable Borel equivalence relation on X which has separable homomorphisms but is not measure hyperfinite. Then the initial segment of the Borel reducibility hierarchy consisting of relations with countable-to-one Borel homomorphisms to $E \times \Delta(2^{\mathbb{N}})$ contains copies of all Borel quasi-orders on Polish spaces.

Proof

Fix $(F_r)_{r\in\mathbb{R}}$ as before.

Proof

Fix $(F_r)_{r \in \mathbb{R}}$ as before.

We can assume there are finite Borel measures μ_r such that every F_r-invariant Borel set is μ_r-null or μ_r-conull, and for no distinct $r, s \in \mathbb{R}$ is there a μ_r-measurable reduction of F_r to F_s.

Proof

Fix $(F_r)_{r\in\mathbb{R}}$ as before.

We can assume there are finite Borel measures μ_r such that every F_r-invariant Borel set is μ_r-null or μ_r-conull, and for no distinct $r, s \in \mathbb{R}$ is there a μ_r-measurable reduction of F_r to F_s.

Associate with each Borel set $B \subseteq \mathbb{R}$ the disjoint union of the equivalence relations F_r, for $r \in B$.

Proof

Fix $(F_r)_{r\in\mathbb{R}}$ as before.

We can assume there are finite Borel measures μ_r such that every F_r-invariant Borel set is μ_r-null or μ_r-conull, and for no distinct $r, s \in \mathbb{R}$ is there a μ_r-measurable reduction of F_r to F_s.

Associate with each Borel set $B \subseteq \mathbb{R}$ the disjoint union of the equivalence relations F_r, for $r \in B$.

This reduces $\subseteq$ on Borel sets to Borel reducibility. ⊠

Definable cardinals just beyond $\mathbb{R}/\mathbb{Q}$

Winter school in abstract analysis
Sporthotel Kácov
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Benjamin Miller
Westfälische Wilhelms-Universität Münster

Part IV

Borel reducibility

IV. Borel reducibility
Products

The following results are joint with Clinton Conley.

Definition

An equivalence relation is μ-hyperfinite if there is a μ-conull Borel set on which it is hyperfinite.

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An equivalence relation is μ-hyperfinite if there is a μ-conull Borel set on which it is hyperfinite.

Definition

An equivalence relation is measure hyperfinite if it μ-hyperfinite for every finite Borel measure μ on the underlying space.

Theorem 1

Suppose that X is a Polish space, E is a countable Borel equivalence relation on X with separable homomorphisms, and μ is a finite Borel measure on X. Then exactly one of the following holds:

1. The equivalence relation E is μ-hyperfinite.
2. There is no $(\mu \times m)$-measurable reduction of $E \times \Delta(\mathbb{R})$ to E.

Theorem 1

Suppose that X is a Polish space, E is a countable Borel equivalence relation on X with separable homomorphisms, and μ is a finite Borel measure on X. Then exactly one of the following holds:

1. The equivalence relation E is μ-hyperfinite.
2. There is no $(\mu \times m)$-measurable reduction of $E \times \Delta(\mathbb{R})$ to E.

Proof

Fix a μ-positive Borel $A \subseteq X$ on which E is μ-nowhere hyperfinite.

Proof of Theorem 1 (continued)

Suppose that $\varphi\colon X \times \mathbb{R} \to X$ is a $(\mu \times m)$-measurable reduction.

Proof of Theorem 1 (continued)

Suppose that $\varphi\colon X \times \mathbb{R} \to X$ is a $(\mu \times m)$-measurable reduction.

For each $r \in \mathbb{R}$, define $\varphi_r\colon A \to X$ by $\varphi_r(x) = \varphi(x, r)$.

Proof of Theorem 1 (continued)

Suppose that $\varphi\colon X \times \mathbb{R} \to X$ is a $(\mu \times m)$-measurable reduction.

For each $r \in \mathbb{R}$, define $\varphi_r\colon A \to X$ by $\varphi_r(x) = \varphi(x, r)$.

Fix an uncountable set $R \subseteq \mathbb{R}$ such that $\forall r \in R$ φ_r is μ-measurable.

Proof of Theorem 1 (continued)

Suppose that $\varphi\colon X \times \mathbb{R} \to X$ is a $(\mu \times m)$-measurable reduction.

For each $r \in \mathbb{R}$, define $\varphi_r\colon A \to X$ by $\varphi_r(x) = \varphi(x, r)$.

Fix an uncountable set $R \subseteq \mathbb{R}$ such that $\forall r \in R\ \varphi_r$ is μ-measurable.

Note that $d_\nu(\varphi_r, \varphi_s) = \mu(A)$ for all distinct $r, s \in R$.

Proof of Theorem 1 (continued)

Suppose that $\varphi\colon X \times \mathbb{R} \to X$ is a $(\mu \times m)$-measurable reduction.

For each $r \in \mathbb{R}$, define $\varphi_r\colon A \to X$ by $\varphi_r(x) = \varphi(x, r)$.

Fix an uncountable set $R \subseteq \mathbb{R}$ such that $\forall r \in R$ φ_r is μ-measurable.

Note that $d_\nu(\varphi_r, \varphi_s) = \mu(A)$ for all distinct $r, s \in R$.

This contradicts separability of $\mathrm{Hom}_{\leq \aleph_0\text{-to-}1}(E, \nu, F)$. ☒

Theorem 2

Suppose that X and Y are Polish spaces, E and F are countable Borel equivalence relations on X and Y, μ is a finite Borel measure on X, $\mathcal{I}$ is a σ-ideal on Y, E is μ-nowhere hyperfinite, and F has separable homomorphisms. Then there is an $\mathcal{I}$-conull set $C \subseteq Y$ such that $\mathrm{rng}(\varphi) \notin \mathcal{I}$ for all $\varphi \in \mathrm{Hom}_{\leq \aleph_0\text{-to-}1}(E, \mu, F \upharpoonright C)$.

Proof

Fix a sequence $(\varphi_\alpha)_{\alpha<\beta}$ of maximal length consisting of functions in $\mathrm{Hom}_{\leq\aleph_0\text{-to-}1}(E, \mu, F)$ with pairwise disjoint $\mathcal{I}$-null ranges.

Proof

Fix a sequence $(\varphi_\alpha)_{\alpha<\beta}$ of maximal length consisting of functions in $\mathrm{Hom}_{\leq\aleph_0\text{-to-}1}(E, \mu, F)$ with pairwise disjoint $\mathcal{I}$-null ranges.

The separability of $\mathrm{Hom}_{\leq\aleph_0\text{-to-}1}(E, \mu, F)$ ensures that $\beta < \omega_1$.

Proof

Fix a sequence $(\varphi_\alpha)_{\alpha<\beta}$ of maximal length consisting of functions in $\mathrm{Hom}_{\leq\aleph_0\text{-to-}1}(E, \mu, F)$ with pairwise disjoint $\mathcal{I}$-null ranges.

The separability of $\mathrm{Hom}_{\leq\aleph_0\text{-to-}1}(E, \mu, F)$ ensures that $\beta < \omega_1$.

Set $C = Y \setminus \bigcup_{\alpha<\beta} \mathrm{rng}(\varphi_\alpha)$. ⊠

Theorem 3

Suppose that X is a Polish space and E is a countable Borel equivalence relation on X with separable homomorphisms. Then exactly one of the following holds:

1. The equivalence relation E is measure hyperfinite.
2. There is a Borel set $B \subseteq X$ for which there is a finite Borel measure μ on B with the property that there is no $(\mu \times 2)$-measurable reduction of $(E \restriction B \times \Delta(2))$ to $E \restriction B$.

Theorem 3

Suppose that X is a Polish space and E is a countable Borel equivalence relation on X with separable homomorphisms. Then exactly one of the following holds:

1. The equivalence relation E is measure hyperfinite.
2. There is a Borel set $B \subseteq X$ for which there is a finite Borel measure μ on B with the property that there is no $(\mu \times 2)$-measurable reduction of $(E \restriction B \times \Delta(2))$ to $E \restriction B$.

Proof

We can assume there is a finite Borel measure μ on X with the property that E is not μ-hyperfinite.

Proof of Theorem 3 (continued)

We can assume every E-invariant Borel set is μ-null or μ-conull.

Proof of Theorem 3 (continued)

We can assume every E-invariant Borel set is μ-null or μ-conull.

Then there exists a μ-conull Borel set $B \subseteq X$ with the property that every $\varphi \in \mathrm{Hom}_{\leq\aleph_0\text{-to-}1}((E \restriction B) \times \Delta(2), \mu, E \restriction B)$ sends sets of positive measure to sets of positive measure.

Proof of Theorem 3 (continued)

We can assume every E-invariant Borel set is μ-null or μ-conull.

Then there exists a μ-conull Borel set $B \subseteq X$ with the property that every $\varphi \in \mathrm{Hom}_{\leq\aleph_0\text{-to-}1}((E \restriction B) \times \Delta(2), \mu, E \restriction B)$ sends sets of positive measure to sets of positive measure.

But clearly there can be no such φ. $\boxtimes$

Definition

We say that E is μ-nowhere smooth if there is no μ-positive Borel set on which E is smooth.

Theorem 4

Suppose that X is a Polish space, E is a countable treeable Borel equivalence relation on X with separable homomorphisms, and μ is a finite Borel measure on X such that E is μ-nowhere hyperfinite. Then there is an increasing sequence $(E_r)_{r \in \mathbb{R}}$ of μ-nowhere hyperfinite Borel subequivalence relations of E such that for no distinct $r, s \in \mathbb{R}$ is there a μ-positive set on which E_r and E_s agree.

Proof (Sketch)

We can assume that the μ-null sets are closed under E-saturation.

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Fix an acyclic Borel graph G generating E.

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We can assume that the μ-null sets are closed under E-saturation.

Fix an acyclic Borel graph G generating E.

Construct a Borel subgraph $H \subseteq G$ whose induced equivalence relation F is hyperfinite but μ-nowhere smooth.

Proof of Theorem 4 (continued)

Construct an increasing sequence $(H_r)_{r \in \mathbb{R}}$ of Borel graphs lying between H and G with the property that for all real numbers $r < s$, the projection of $H_s \setminus H_r$ intersects μ-almost every $(E \restriction B)$-class.

Proof of Theorem 4 (continued)

Construct an increasing sequence $(H_r)_{r \in \mathbb{R}}$ of Borel graphs lying between H and G with the property that for all real numbers $r < s$, the projection of $H_s \setminus H_r$ intersects μ-almost every $(E \restriction B)$-class.

Then for no distinct $r, s \in \mathbb{R}$ is there a μ-positive $A \subseteq B$ on which the relations F_r and F_s induced by H_r and H_s agree.

Proof of Theorem 4 (continued)

Fix a Borel contraction $f\colon X \to B$ such that $\forall x \in X \; x \, E \, f(x)$.

Proof of Theorem 4 (continued)

Fix a Borel contraction $f\colon X \to B$ such that $\forall x \in X\ x\, E\, f(x)$.

Define $x\ E_r\ y \Leftrightarrow f(x)\ F_r\ f(y)$.

Proof of Theorem 4 (continued)

Fix a Borel contraction $f \colon X \to B$ such that $\forall x \in X \; x \, E \, f(x)$.

Define $x \; E_r \; y \Leftrightarrow f(x) \; F_r \; f(y)$.

Then for no distinct $r, s \in \mathbb{R}$ is there a μ-positive set on which E_r and E_s agree. $\boxtimes$

Theorem 5

Suppose that X is a Polish space, E is a countable treeable Borel equivalence relation on X with separable homomorphisms, and μ is a finite Borel measure on X such that E is μ-nowhere hyperfinite. Then there is an increasing sequence $(E_r)_{r \in \mathbb{R}}$ of Borel subequivalence relations of E such that for no distinct $r, s \in \mathbb{R}$ is there a μ-measurable reduction of E_r to E_s on a μ-positive set.

Proof

Fix an increasing sequence $(E_r)_{r\in\mathbb{R}}$ of μ-nowhere hyperfinite Borel subequivalence relations of E such that for no distinct $r, s \in \mathbb{R}$ is there a μ-positive set on which E_r and E_s agree.

Proof

Fix an increasing sequence $(E_r)_{r \in \mathbb{R}}$ of μ-nowhere hyperfinite Borel subequivalence relations of E such that for no distinct $r, s \in \mathbb{R}$ is there a μ-positive set on which E_r and E_s agree.

We can assume that $\bigcap_{r \in \mathbb{R}} E_r$ is μ-nowhere hyperfinite on B.

Proof of Theorem 5 (continued)

Then for each $r \in \mathbb{R}$, there are only countably many $s \in \mathbb{R}$ for which there is a μ-measurable reduction of E_s to E_r on a μ-positive set.

Proof of Theorem 5 (continued)

Then for each $r \in \mathbb{R}$, there are only countably many $s \in \mathbb{R}$ for which there is a μ-measurable reduction of E_s to E_r on a μ-positive set.

So we can assume that for no distinct $r, s \in \mathbb{R}$ is there a μ-measurable reduction of E_r to E_s on a μ-positive set. ⊠

Theorem 6

Suppose that X is a Polish space and E is a countable treeable Borel equivalence relation on X which has separable homomorphisms but is not measure hyperfinite. Then there is no basis of cardinality strictly less than $\mathtt{add}(\mathtt{null})$ for the family of non-measure hyperfinite countable Borel equivalence relations which admit countable-to-one Borel homomorphisms to E.

Proof

Fix a finite Borel measure μ for which E is μ-nowhere hyperfinite.

Proof

Fix a finite Borel measure μ for which E is μ-nowhere hyperfinite.

Fix a sequence $(E_r)_{r\in\mathbb{R}}$ of Borel subequivalence relations of E such that for no distinct $r, s \in \mathbb{R}$ is there a μ-measurable reduction of E_r to E_s on a μ-positive set.

Proof of Theorem 6 (continued)

Suppose that $\mathscr{F}$ is a basis of cardinality $< \mathtt{add}(\mathtt{null})$.

Proof of Theorem 6 (continued)

Suppose that $\mathscr{F}$ is a basis of cardinality $< \texttt{add}(\texttt{null})$.

For each $F \in \mathscr{F}$, fix a finite Borel measure μ_F such that F is μ_F-nowhere hyperfinite.

Proof of Theorem 6 (continued)

Suppose that $\mathscr{F}$ is a basis of cardinality $< \texttt{add}(\texttt{null})$.

For each $F \in \mathscr{F}$, fix a finite Borel measure μ_F such that F is μ_F-nowhere hyperfinite.

We can assume that every μ_F-measurable reduction of F to E_r sends μ_F-positive sets to μ-positive sets.

Proof of Theorem 6 (continued)

Fix $F_r \in \mathscr{F}$ and μ-measurable reductions φ_r of F_r to E_r.

Proof of Theorem 6 (continued)

Fix $F_r \in \mathscr{F}$ and μ-measurable reductions φ_r of F_r to E_r.

Fix distinct $r, s \in \mathbb{R}$ such that $F_r = F_s$ and φ_r and φ_s agree on a μ_{F_r}-positive set.

Proof of Theorem 6 (continued)

Fix $F_r \in \mathscr{F}$ and μ-measurable reductions φ_r of F_r to E_r.

Fix distinct $r, s \in \mathbb{R}$ such that $F_r = F_s$ and φ_r and φ_s agree on a μ_{F_r}-positive set.

Then E_r and E_s agree on a μ-positive set. ⊠

Theorem 7

Suppose that X is a Polish space and E is a countable treeable Borel equivalence relation on X which has separable homomorphisms but is not measure hyperfinite. Then the initial segment of the Borel reducibility hierarchy consisting of relations with countable-to-one Borel homomorphisms to $E \times \Delta(\mathbb{R})$ contains copies of all Borel quasi-orders on Polish spaces.

Proof

Fix a finite Borel measure μ for which E is μ-nowhere hyperfinite.

Proof

Fix a finite Borel measure μ for which E is μ-nowhere hyperfinite.

Fix a sequence $(E_r)_{r \in \mathbb{R}}$ of Borel subequivalence relations of E such that for no distinct $r, s \in \mathbb{R}$ is there a μ-measurable reduction of E_r to E_s on a μ-positive set.

Proof of Theorem 7 (continued)

We can assume there are finite Borel measures μ_r such that every E_r-invariant Borel set is μ_r-null or μ_r-conull, and for no distinct $r, s \in \mathbb{R}$ is there a μ_r-measurable reduction of E_r to E_s on a μ_r-positive set.

Proof of Theorem 7 (continued)

We can assume there are finite Borel measures μ_r such that every E_r-invariant Borel set is μ_r-null or μ_r-conull, and for no distinct $r, s \in \mathbb{R}$ is there a μ_r-measurable reduction of E_r to E_s on a μ_r-positive set.

Associate with each Borel set $B \subseteq \mathbb{R}$ the disjoint union of the equivalence relations E_r, for $r \in B$.

Proof of Theorem 7 (continued)

We can assume there are finite Borel measures μ_r such that every E_r-invariant Borel set is μ_r-null or μ_r-conull, and for no distinct $r, s \in \mathbb{R}$ is there a μ_r-measurable reduction of E_r to E_s on a μ_r-positive set.

Associate with each Borel set $B \subseteq \mathbb{R}$ the disjoint union of the equivalence relations E_r, for $r \in B$.

This reduces $\subseteq$ on Borel sets to Borel reducibility. ⊠