Measurable equidecompositions

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based on joint work with Lukasz Grabowski and Oleg Pikhurko

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Dissecting polygons and polyhedra

Wallace–Bolyai–Gerwien theorem

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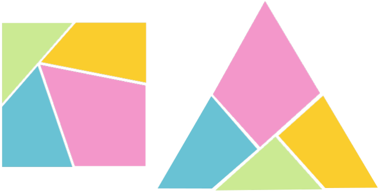

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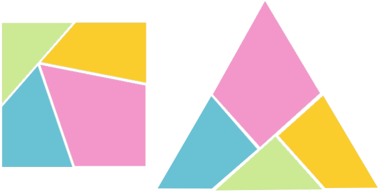

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Theorem (Dehn)

No.
Dehn invariant. For example, cube and regular tetrahedron.

Banach–Tarski paradox (1924)

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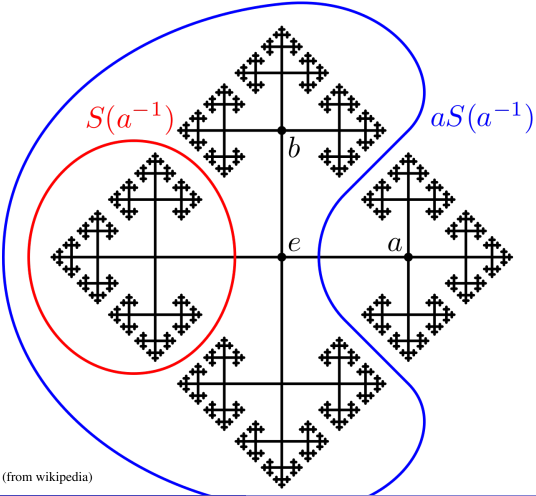

(from wikipedia)

There are two rotations in $SO(3)$ generating the free group $\mathbb{F}_2$.

$\mathbb{F}_2 = \{e\} \cup S(a) \cup S(b) \cup S(a^{-1}) \cup S(b^{-1})$

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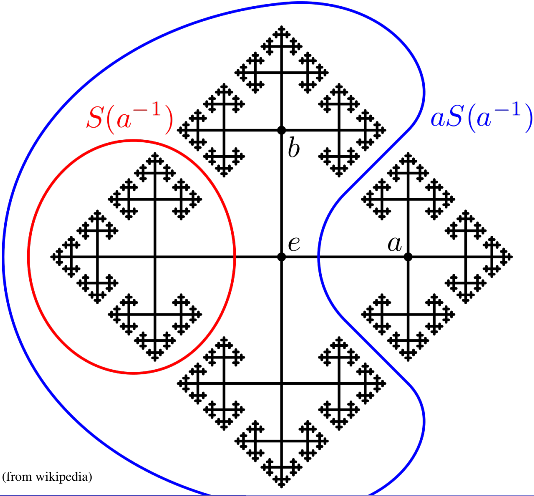

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von Neumann $\rightarrow$ *amenable* groups

Equidecompositions in the plane

Banach–Tarski paradox does not hold because...

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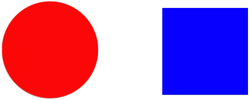
If two measurable sets $A, B \subset \mathbb{R}^2$ are equidecomposable (with non-measurable pieces) then A and B have the same Lebesgue measure.

Tarski's circle squaring problem (1920s)

Question

Is it possible to cut a disc into finitely many pieces and rearrange them to obtain a square of the same area?
(Is the disc equidecomposable to a square?)



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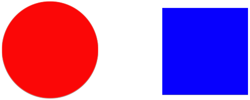

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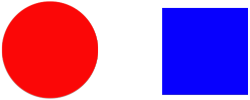

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Let $A, B \in \mathbb{R}^d$, $d \geq 1$, be bounded measurable sets with $\lambda(A) = \lambda(B) > 0$ and $\dim_B(\partial A) < d$, $\dim_B(\partial B) < d$.
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Corollary

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Theorem (Grabowski–M–Pikhurko 2014)

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Corollary (Grabowski–M–Pikhurko)

The cube and the tetrahedron are equidecomposable using measurable pieces.

Measurable/Borel circle squaring

Theorem (Grabowski–M–Pikhurko 2015)

Let $A, B \subset \mathbb{R}^d$, $d \geq 1$, be measurable sets with the same positive measure. Let $\dim_B(\partial A) < d,\ \dim_B(\partial B) < d$.
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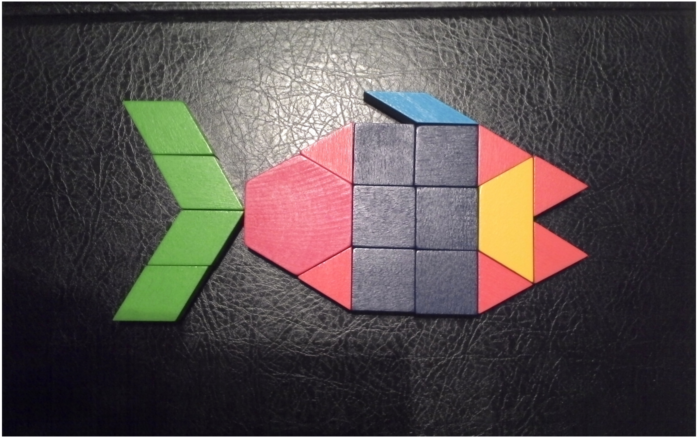
No picture

Laczkovich needs about 10^{40} pieces to equidecompose the disc to a square.
We need a bit more.

How not to look for equidecompositions

Dividing one set into pieces and then trying to reassemble to yield the other usually does not work.



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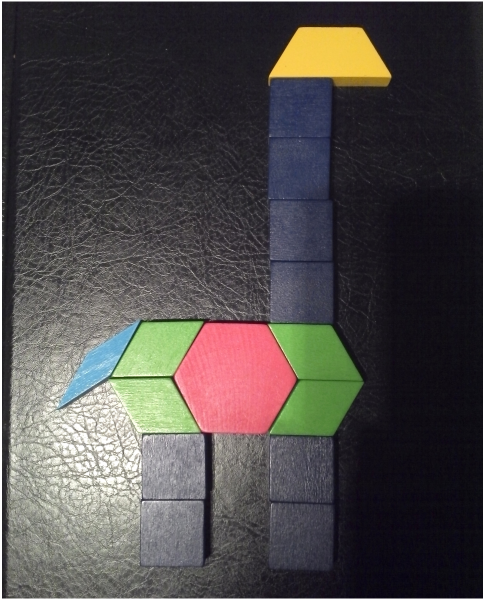

The right way to find equidecompositions

Take a lot of isometries / translations, then take even more, and then try to find the partitions that work.

“Take even more” usually means to take compositions of the isometries we already have.

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Fix isometries $\gamma_1, \ldots, \gamma_n$.
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- For Banach–Tarski paradox: we need isometries generating a free group.
- For this theorem: an analytic/quantitative analogue.

Spectral gap of averaging operators

Theorem (Margulis, Sullivan $d \geq 5$, Drinfeld $d \geq 3$)

There exist rotations $\gamma_1, \ldots, \gamma_k \in SO(d)$ for which we have a spectral gap for the operator

$$T : L^2(S^{d-1}) \to L^2(S^{d-1})$$

$$Tf(x) = \frac{f(\gamma_1(x)) + \ldots + f(\gamma_k(x))}{k}.$$

That is,
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Corollary (expansion property)

For every $\delta > 0$ there exists a finite set of rotations Γ such that

$$\lambda\big(\cup_{\gamma \in \Gamma} \gamma(X)\big) \geq \min\Big(1 - \delta,\ \lambda(X)/\delta\Big) \qquad \text{for every } X \subset S^{d-1}.$$

Here λ is the probability Lebesgue measure on S^{d-1}.

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$A, B \subset S^{d-1}$ disjoint measurable sets with non-empty interiors.
We would like to have an equidecomposition between A and B using rotations in Γ.

Bi-partite graph $G = G_\Gamma$

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Lemma (expansion in G)

By adding more isometries (increasing Γ),

$$\lambda\Big(\underbrace{\cup_{\gamma\in\Gamma}\gamma(X) \cap B}_{N(X)}\Big) \geq \min\Big(\tfrac{2}{3}\lambda(B),\ 2\lambda(X)\Big) \qquad \text{for every } X \subset A.$$

That is, for every set the set of neighbours is large.

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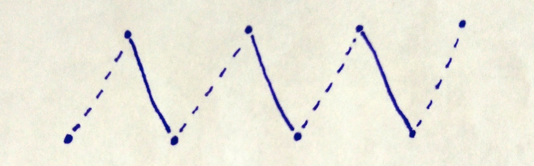
Theorem (Lyons–Nazarov)

Borel graphs with this expansion property have a Borel perfect matching up to a nullset.

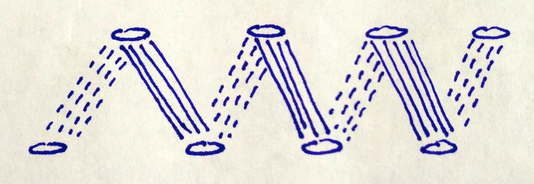
Finding maximum matchings in finite bi-partite graphs



Maximum matching algorithm

- Start with any matching.
- Find an augmenting path.
- Increase the size of the matching using the augmenting path.
- Iterate.
- The algorithm finishes in finite time.

Finding measurable maximum matchings in infinite bi-partite graphs?



- Start with any matching.
- Find a large family of disjoint augmenting paths.
- Increase the size of the matching using these augmenting paths.
- Iterate.
- The algorithm does not finish in finite time. The matchings might or might not converge.

We need short augmenting paths to have convergence.

Putting together the proof

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3. M_{k+1} is obtained from M_k by changing it on a set of vertices of measure $\leq c' k(1 + \varepsilon)^{-k}$.
4. Since $\sum_i k(1 + \varepsilon)^{-k} < \infty$, Borel–Cantelli implies that $\lim_k M_k$ exists (almost everywhere). This is a Borel perfect matching up to a nullset.

Previously...

Definition

We say that two sets $A, B \subset \mathbb{R}^d$ are equidecomposable if there exist finite partitions
$A = A_1 \cup^* \ldots \cup^* A_n$
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Grabowski–M–Pikhurko

The ball is equidecomposable to a cube using measurable pieces.

In $\mathbb{R}^d$, $d \geq 3$, any two bounded measurable sets with non-empty interior of the same measure are equidecomposable using measurable pieces.

Baire equidecompositions

Theorem (Dougherty–Foreman 1992)

Banach–Tarski paradox works with Baire pieces.
(Any two bounded sets in $\mathbb{R}^d$, $d \geq 3$, with the Baire property and having non-empty interiors are equidecomposable using Baire pieces.)

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The real statement

There are disjoint open sets $V_1, \dots, V_n \subset \mathbb{R}^3$ and isometries γ_i such that

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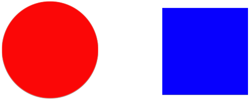
Question (Tarski, 1920s)

Is the disc equidecomposable to a square?



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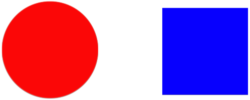

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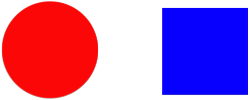

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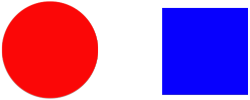

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Theorem (Marks–Unger, 2016)

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Negative results

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The square and the disc are not “scissor-congruent”.
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Theorem (Gardner, 1985)

The square and the disc are not equidecomposable if the pieces are moved by a locally discrete group of isometries.

A and B are equidecomposable using translations if and only if there is a bijection $\varphi : A \to B$ such that $\{\, \varphi(x) - x \,:\, x \in A \}$ is finite.

Proof.

A and B are equidecomposable using translations if and only if there is a bijection $\varphi : A \to B$ such that $\{\, \varphi(x) - x \,:\, x \in A\}$ is finite.

Proof.
If $A = A_1 \cup^* \ldots \cup^* A_n$, $B = B_1 \cup^* \ldots \cup^* B_n$, $B_i = A_i + t_i$, then let $\varphi(x) = x + a_i \ (x \in A_i)$.

If $\{\, \varphi(x) - x \,:\, x \in A\} = \{t_1, \ldots, t_n\}$, then let $A_i = \{x \in A \,:\, \varphi(x) - x = t_i\}$. □

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Look at the associated $\mathbb{Z}^d$ action. Look at the orbits / cosets.

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$\Longleftrightarrow$

Existence of bijections $f_x : A_x^* \to B_x^*$ for every x such that $\forall n \in \mathbb{Z}^d \ \|f_x(n) - n\|_\infty \le C$.

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Axiom of choice.
Aim:

- $\forall x$ the density of A_x^* and B_x^* is $\lambda(A)$ (which is $= \lambda(B)$)
- these sets are "uniformly spread" in $\mathbb{Z}^d$.

$$A_x^* = \Big\{ (n_1, \ldots, n_d) \in \mathbb{Z}^d \,:\, x + \sum_{i=1}^{d} n_i v_i \in A \Big\}.$$

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Step 1

If A is a rectangle, then A_x^* is known to be ‘uniformly spread’:
$|A_x^* \cap Q| = \lambda(A)|Q| \pm c \log^c N$ for every cube $Q \subset \mathbb{Z}^d$ of side length N.
(Application of the Erdős–Turán–Koksma inequality.)

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Erdős–Turán inequality

(Quantitative result implying Weyl's ciretrion for equidistribution.)
For every probability measure μ on the unit circle,

$$\sup_A |\mu(A) - \lambda(A)| \leq C \left(\frac{1}{n} + \sum_{k=1}^{n} \frac{\hat{\mu}(k)}{k} \right)$$

supremum taken over arcs $A \subset [0, 1) = \mathbb{R}/\mathbb{Z}$.

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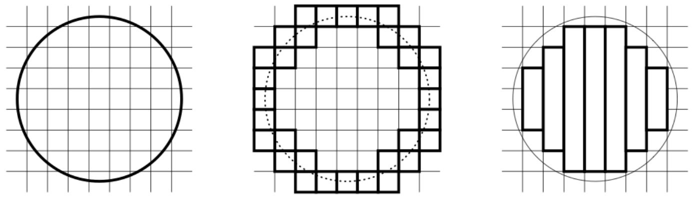

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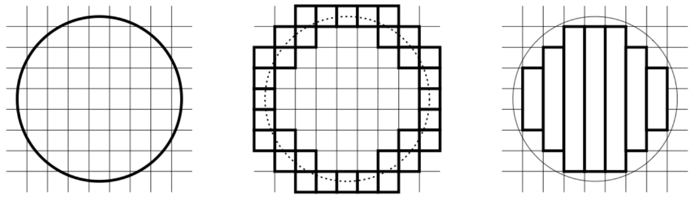

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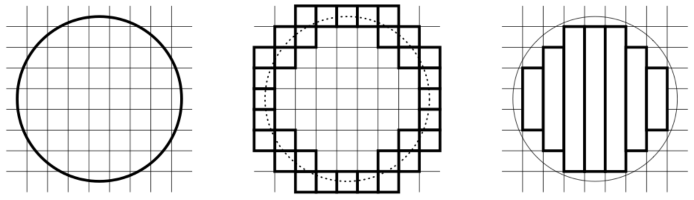

Let $\dim_B \partial A < (1-\varepsilon) \cdot 2$.
Then $|A_x^* \cap Q| = \lambda(A)|Q| \pm cN^{d(1-\varepsilon)}$ for every cube $Q \subset \mathbb{Z}^d$ of side length N.
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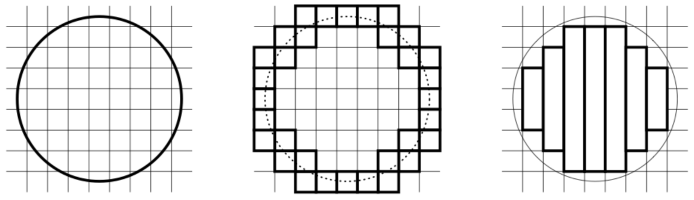

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Crucial: for large cubes, error term $cN^{d(1-\varepsilon)}$ is less than the size of boundary of the cube if we choose d to be large enough.

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Step 3 – most difficult part in Laczkovich's proof

Assume that $A^* \subset \mathbb{Z}^d$, $B^* \subset \mathbb{Z}^d$ satisfy

$$\big||A^* \cap Q| - \alpha|Q|\big| \leq cN^{d-1-\delta}$$

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Then there is a bijection $f : A^* \to B^*$ for which $\forall n \ \|f(n) - n\|_\infty \le C(c,d,\delta)$.

The existence of the bijection is obtained by checking Hall's condition:

$$|N(X)| \ge |X| \quad \text{for every set of vertices } X.$$

That is,

$$|\underbrace{X_C}_{C\text{-width neighbourhood of } X} \cap B^*| \ge |X| \quad (X \subset A^*).$$

□

To obtain a measurable circle squaring

Cosets: $\{x + \sum_{i=1}^{d} n_i v_i \in \mathbb{T} : (n_1, \ldots, n_d) \in \mathbb{Z}^d\}$

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Solution

- We use augmenting paths to build up a sequence of matchings.

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- We have to find perfect matchings in (almost) all cosets of $\mathbb{Z}^d$ in a Borel way.
- There is no distinguished origin in these cosets. (There is no Borel set E which intersects every coset in exactly 1 point.)

Solution

- We use augmenting paths to build up a sequence of matchings.
- We show that short augmenting paths exist.

To obtain a measurable circle squaring

Cosets: $\{x + \sum_{i=1}^{d} n_i v_i \in \mathbb{T} \,:\, (n_1, \ldots, n_d) \in \mathbb{Z}^d\}$

$$A_x^* = \left\{(n_1, \ldots, n_d) \in \mathbb{Z}^d \,:\, x + \sum_{i=1}^{d} n_i v_i \in A\right\}.$$

B_x^* $\qquad\qquad\qquad\qquad\qquad$ B

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- We show that short augmenting paths exist.
- We find Borel sets E_i which intersect cosets in sparse sets and use these as "local origins".
- We use a local algorithm to build up the matchings, ensuring that everything is Borel.

Short augmenting paths

For large enough C,

$$|N(X)| \geq |X| + |\partial X| \geq |X| + |X|^{(d-1)/d} \quad \text{for every set of vertices } X.$$

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Lemma

Let $Q \subset \mathbb{Z}^d$ be a cube and assume we have a matching between $A^* \cap Q$ and $B^* \cap Q$. If there are unmatched vertices in A^* and B^* of distance t, then there is an augmenting path of length ct.

Algorithm to find perfect matchings

Pretend that there is a Borel set $E \subset \mathbb{T}$ intersecting every coset in exactly 1 point.

1. Take A^* and B^*.
2. Take a sequence $N_i \to \infty$. $N_i | N_{i+1}$.
   Divide $\mathbb{Z}^d$ into the family $\mathcal{Q}_i$ of grid cubes of side N_i.
3. We will define matchings M_i (bijection of a subset of A^* into B^*, every point is moved by at most C)
   such that all the edges are inside one of the grid cubes of $\mathcal{Q}_i$.
4. M_i is a maximal matching in each of the grid cubes.
5. The density of unmatched vertices is $\leq N_i^{-\varepsilon d}$.
6. To obtain the matching M_{i+1} from M_i
   - For each grid cube in $\mathcal{Q}_{i+1}$ take M_i and increase it using the shortest possible augmenting paths to a maximal matching.
   - The density where the matching is changed is small.
7. Borel–Cantelli can be used, the limit of the matchings M_i exists (almost everywhere).
8. This gives a Borel algorithm to find a Borel a.e. equidecomposition of A and B provided that E exists (but it does not).

Measurable equidecompositions (continued)

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46th Winter School in Abstract Analysis, Svratka
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Tarski's circle squaring with Borel pieces

Theorem (Marks–Unger 2016)

Let $A, B \subset \mathbb{R}^d$, $d \geq 1$, be measurable sets with the same positive measure. Let $\dim_M(\partial A) < d$, $\dim_M(\partial B) < d$.
Then A and B are equidecomposable with Borel pieces, using translations only.

$A = A_1 \cup^* \ldots \cup^* A_n$, $B = B_1 \cup^* \ldots \cup^* B_n$, $B_i = A_i + t_i$

Borel circle squaring (Marks–Unger)

Matchings and augmenting paths are replaced by flows.

Let (V, E) be a graph and $f : V \to \mathbb{R}$. An f-flow is a function φ on the edges with

$$\varphi(x, y) = -\varphi(y, x) \quad (xy \in E)$$

such that

$$f(x) = \sum_{y \in N(x)} \varphi(x, y) \quad (x \in V).$$

(f replaces the usual source and sink)

Connection to matchings

Let $E \subset A \times B$ be a bi-partite graph, $M \subset E$ a matching. Then

$$\varphi(x, y) = \begin{cases} 1 & \text{if } (x, y) \in M, \\ -1 & \text{if } (y, x) \in M, \\ 0 & \text{otherwise.} \end{cases}$$

is an f-flow for $f = 1_A - 1_B$.

$V = \mathbb{T}$

$E = \{(x, y) \ : \ y - x = n_1 v_1 + \ldots + n_d v_d, \ n_i = -1, 0, 1\}.$

Marks–Unger, Step 1

Under Laczkovich's conditions, there exists a bounded Borel f-flow on E with $f = 1_A - 1_B$.

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There exists an integer valued bounded Borel f-flow on E with $f = 1_A - 1_B$.

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Marks–Unger, Step 2

There exists an integer valued bounded Borel f-flow on E with $f = 1_A - 1_B$.

Marks–Unger, Step 3

There exists a Borel equidecomposition of A to B.

Hyperfinite Borel equivalence relations

X standard Borel space, $E \subset X \times X$ a Borel equivalence relation.

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Theorem (Weiss, 1981)

If $G = \mathbb{Z}^d$ and $G \curvearrowright X$ is a Borel action, then
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Theorem (Gao–Jackson–Krohne–Seward)

If $G = \mathbb{Z}^d$ and $G \curvearrowright X$ is a free Borel action, then X is the union of a Borel family of finite sets whose $\mathbb{Z}^d$-boundary are disjoint and far away from each other (say, the n-neighbourhood of the boundaries are disjoint too).

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Lemma (Marks–Unger, 2016)

Suppose $G = \mathbb{Z}^d$, $d \geq 2$, $G \curvearrowright X$ is a free Borel action.
If $f : X \to \mathbb{Z}$ is Borel, φ is a Borel f-flow, then there is an integer valued Borel f-flow ψ such that $|\varphi - \psi| \leq 3^d$.

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Corollary: Step 1 $\Longrightarrow$ Step 2.

Theorem (Gao–Jackson, 2015)

Suppose $G = \mathbb{Z}^d$, and $G \curvearrowright X$ is a free Borel action. Then for every $n \geq 1$ there is a Borel partion of X into sets of the form

$$\{g_{n_1 \dots n_d}(x) : 0 \leq n_i < n \text{ or } n+1\}.$$

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This theorem is used to obtain the Borel equidecomposition from the integer valued Borel flow.

Open questions

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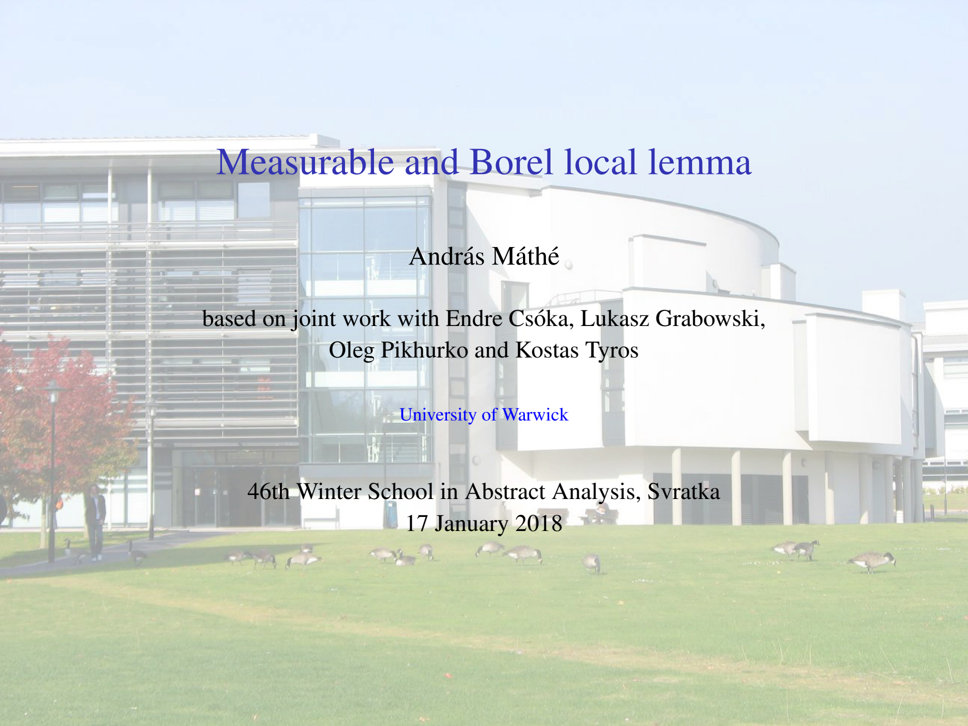
Question (Mycielski, Wagon)

Is it possible to divide the sphere into three congruent measurable sets?

$$S^2 = A \cup^* B \cup^* C, \qquad A \sim B \sim C$$



Measurable and Borel local lemma

András Máthé

based on joint work with Endre Csóka, Lukasz Grabowski, Oleg Pikhurko and Kostas Tyros

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46th Winter School in Abstract Analysis, Svratka
17 January 2018

Let $P \subset \mathbb{R}$ be a non-empty perfect set (closed set without isolated points). Is there a (closed) set of Lebesgue measure zero $E \subset \mathbb{R}$ such that $P + E = \mathbb{R}$?

$P + E = \{p + e \,:\, p \in P,\ e \in E\}$

For any set $S \subset \mathbb{Z}$ with $|S| \geq 100$, there are eight disjoint sets $A_1, \ldots, A_8 \subset \mathbb{Z}$ such that every translate $S + m$ intersects all the eight sets A_i.

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In general,

For every k there is n such that if $S \subset \mathbb{R}^d$, $|S| \geq n$, then there are k disjoint sets $A_i \in \mathbb{R}^d$ ($i = 1, \dots, k$) such that every translate of S intersects every set A_i.

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The Borel version of this problem is actually not much harder than the non-Borel one partly because we are not interested here about sharp statements.

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Local lemma

These statements are all corollaries of the Lovász Local Lemma (Erdős–Lovász 1975).

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Those involving Borel sets also rely on our infinite/measurable/Borel version of this local lemma (Csóka–Grabowski–M–Pikhurko–Tyros).

The probabilistic method

Erdős 1947

$R(k,k) > \lfloor 2^{k/2} \rfloor$ (Ramsey number)

That is, the edges of the complete graph on $n = \lfloor 2^{k/2} \rfloor$ vertices can be coloured red and blue such that every complete subgraph on k vertices contains both red and blue edges.

Proof.

Colour the edges independently randomly red or blue with equal probability.

For any complete subgraph on k vertices, the probability that it is *monochromatic* (all its edges are red or all are blue) is

$$2^{1-\binom{k}{2}}.$$

There are $\binom{n}{k}$ ways to choose k vertices.

The probability that one of the subgraphs on k vertices is monochromatic is at most

$$\binom{n}{k} 2^{1-\binom{k}{2}} < 1.$$

Hence, with positive probability, all complete subgraphs on k vertices contain both red and blue edges. □

Local lemma example

Assume $e(d+1) \leq 2^{k-1}$.
Consider a finite set X and finitely many subsets $A_i \subset X$ containing at least k elements.
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Then the elements of X can be coloured red and blue such that each set A_i contains a red element and a blue element.

Colour the points of X randomly red or blue.
The probability that A_i is monocoloured ("bad event") is 2^{1-k}. The "good event" is if A_i is multicoloured.

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Lovász Local Lemma: If $ep(d+1) \leq 1$, then with positive probability, all events are good.

Lovász Local Lemma (Erdős–Lovász 1975)

Let $A_1, \ldots, A_m$ be events in an arbitrary probability space.
Suppose that each event A_i is mutually independent of a set of all the other events A_j but at most d, and that $\Pr(A_i) \le p$ for all i. If

$$ep(d+1) \le 1$$

then $\Pr(\wedge_i \overline{A_i}) > 0$.

Multicoloured translates

Multicoloured translates 1

For every k there is $n = LLL(k)$ such that if $S \subset \mathbb{Z}$, $|S| \geq n$, then $\mathbb{Z}$ can be coloured by k colours such that every translate of S contains all k colours.

Proof.

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Multicoloured translates 3 (Csóka–Grabowski–M–Pikhurko–Tyros)

For every k there is $n = LLL(k)$ such that if $S \subset \mathbb{R}$, $|S| \geq n$, then $\mathbb{R}$ can be coloured by k colours in a Borel way such that every translate of S contains all k colours.

Proof.

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Proof. Lovász Local Lemma for finitely many translates $S + m$ $(m = -M, \dots, M)$. Then diagonal argument. □

Multicoloured translates 2

For every k there is $n = LLL(k)$ such that if $S \subset \mathbb{R}$, $|S| \geq n$, then $\mathbb{R}$ can be coloured by k colours such that every translate of S contains all k colours.

Proof. Lovász Local Lemma for finitely many translates.
Then Tikhonov. □

Multicoloured translates 3 (Csóka–Grabowski–M–Pikhurko–Tyros)

For every k there is $n = LLL(k)$ such that if $S \subset \mathbb{R}$, $|S| \geq n$, then $\mathbb{R}$ can be coloured by k colours in a Borel way such that every translate of S contains all k colours.

Proof. Borel version of the Lovász Local Lemma. □

Proof of Local Lemma

Original proof: Induction.

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Algorithmic versions. Beck 1991, Alon 1991, Molloy–Reed 1998, Czumaj-Scheideler 2000, Srinivasan 2009, Moser 2008, Moser 2009, ...

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Assume $e(d+1) \leq 2^{k-1}$.
Consider a finite set X and finitely many (m) subsets $A_i \subset X$ containing at least k elements.
Assume that each A_i is disjoint from all but at most d other sets.
Then the elements of X can be coloured red and blue such that each set A_i contains a red element and a blue element.

Algorithm.

Colour elements of X randomly (independently).

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Algorithm.

Colour elements of X randomly (independently).
If there are sets A_i that are monocoloured, choose one arbitrarily, and colour its elements randomly.
Repeat.
(It is possible that a set was multicoloured but becomes monocoloured in this process.)
Claim: this algorithm finishes in finite time (almost surely).
In fact, the expected running time is at most

$$\frac{m}{d-1}.$$

Borel Local Lemma (through multicoloured translates)

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Modified Moser–Tardos algorithm with limited randomness (GLMPT)

Assume a subexponentiality condition.
There is $K > 0$ such that it is enough to assume that random bits of "distance" at most K are independent. The algorithm still finishes almost surely. The output is a good Borel colouring.

For multicoloured translates of S, this "distance" of $x, y \in \mathbb{R}$ is actually the minimal d such that $x - y \in \underbrace{(S - S) + (S - S) + \ldots + (S - S)}_{d \text{ times}}$.

The size of this set is polynomial (thus subexponential) in d.

Related results

Gábor Kun 2013+
Infinite countable graph
Bernoulli shift $\Gamma \curvearrowright (\{0,1\}^{\mathbb{N}})^{\Gamma}$

Anton Bernshteyn 2016+

Open questions

Borel or measurable local lemma in the general (not subexponential) case.

Question

$(X, \mathcal{B}, \mu)$ standard Borel probability space.
Let n be large compared to k.
Let $T_i : X \to X$ $(i = 1, \ldots, n)$ be measure preserving Borel bijections.
Is there a measurable colouring of X with k colours such that for almost every $x \in X$,

$$\{T_i(x) \;:\; i = 1, \ldots, n\}$$

is multicoloured (includes all k colours)?

If the transformations are commuting, we have polynomial (subexponential) growth rate, so the answer is positive.
Is it still true if an amenable group acts preserving the measure μ? $(\to$ hyperfinite$)$

On the other hand, if there is no measure:

Marks 2016

There is an action of $\mathbb{F}_{2n}$ and a colouring problem for which LLL inequality holds but there is no Borel colouring.