1. cvičení

[Ko] Kopáček: Př. z matematiky pro fyziky IV, skripta MFF UK, 1996,
Př. pro UKA, viz kapitola 2 v [Ko]

———— x ————

opakování: počítání v $\mathbb{C}$, odmocniny ...; exponenciála
[Ko, str. 46–48]

———— x ————

Př. $(2+3i)\cdot(1-2i)$ | $\dfrac{2+i^{11}}{1-3i}$ | $\left(\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}\right)^6$; $e^{-i\pi/3}$

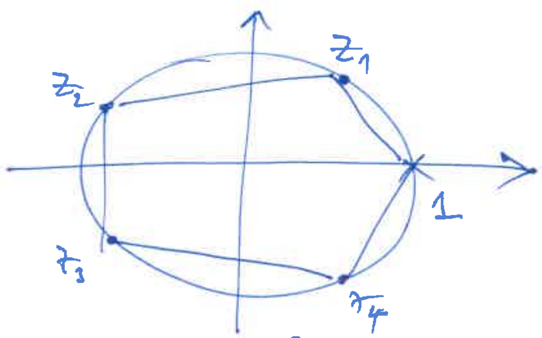
[$8-i$; $\frac{1}{2}(1+i)$; 1]

Př. kořeny v $\mathbb{C}$: (i) $z^4=-1$ [př. 38, 4. odmocniny -1];
(ii) $z^8 = 1$ [40];
(iii) $z^3 = -2+2i$ [43];
(iv) $z^4 - 3iz^2 - 2 = 0$
(v) $z^4+z^3+z^2+z+1=0$

ad (iv): $w = z^2 = i, 2i$, protože

$$\left(w - \frac{3i}{2}\right)^2 + \underbrace{\frac{9}{4} - 2}_{=1/4 = -\left(\frac{1}{2}i\right)^2} = (w-i)\cdot(w-2i) = 0$$

$$z = \pm\frac{1}{\sqrt{2}}(1+i),\ \pm(1+i)$$



ad (v) $\dfrac{z^5-1}{z-1} = 0$, $z \neq 1$

$z^5 = 1$: $e^{i5\alpha} = 1$ $(= e^0)$, $5\alpha = 2\pi k$, $\alpha_k := \frac{2}{5}\pi k$,
$z_k := e^{i\alpha_k}$, $k = 1,2,3,4$

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Př. Nakreslete v $\mathbb{C}$ množinu bodů

(i) $|\mathrm{Re}\, z| < 1$ [57]

(ii) $1 < |z| < 2$ [54]

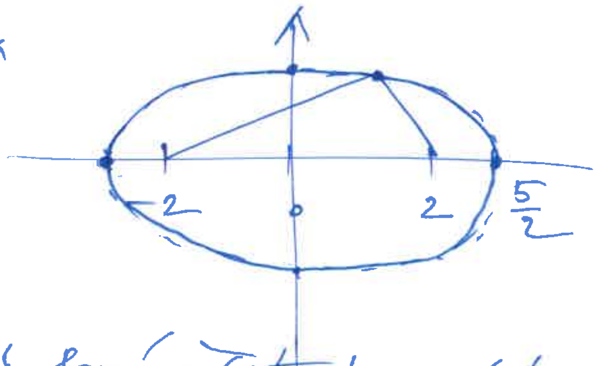
(iii) $|z-2| + |z+2| = 5$ [56]

(iv) $|z-2| - |z+2| > 3$ [57]

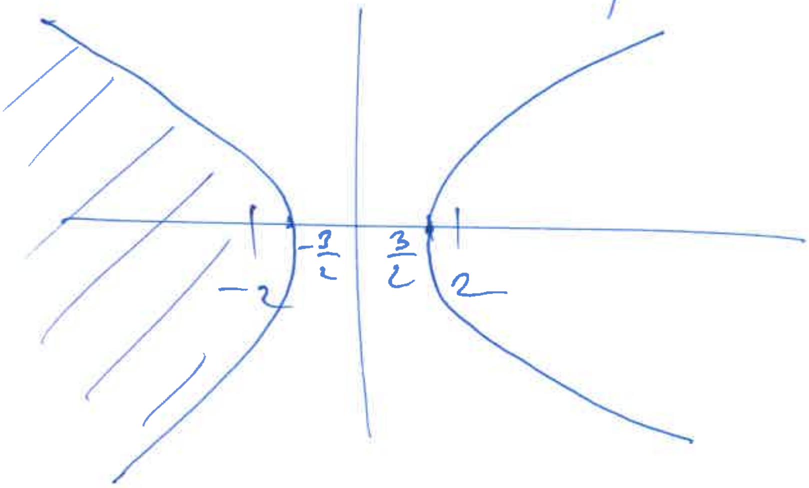
(v) $|\mathrm{Re}\, z| + |\mathrm{Im}\, z| \leq 1$ [61]

(vi) $\mathrm{Re}\, \frac{z-i}{z+i} = 0$ [63] kružnice

ad (iii) elipsa



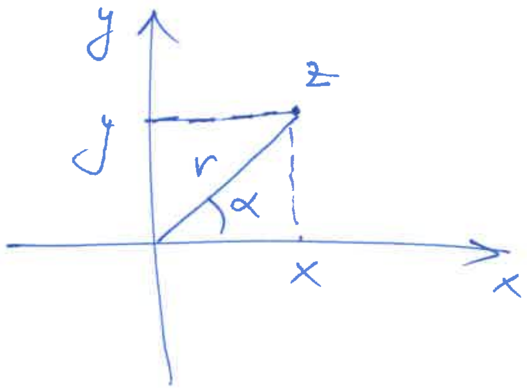
ad (iv) vnitřek levé části hyperboly s ohnisky ± 2 a poloosou $3/2$



Komplexní čísla $\mathbb{C} \simeq \mathbb{R}^2$:

$z = (x,y) = x + iy$

$x = \mathrm{Re}\, z$, $y = \mathrm{Im}\, z$



Polární souřadnice

$x = r \cdot \cos\alpha$

$y = r \cdot \sin\alpha$,

kde $r = \sqrt{x^2+y^2} = |z| \ldots$ modul z (abs. hodn.)

a $\alpha \in \mathbb{R}$ je argument z.

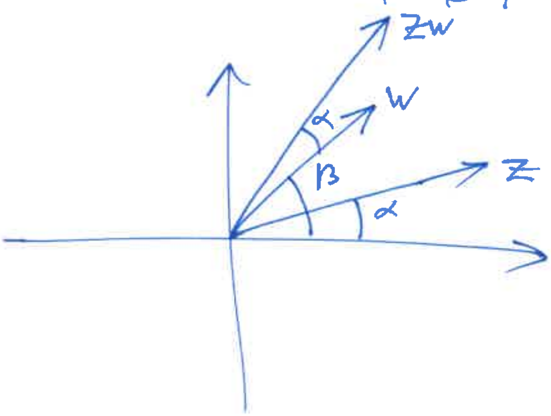
Máme $z = |z|\underbrace{(\cos\alpha + i\sin\alpha)}_{= e^{i\alpha}\ (\text{Euler})} = |z|\, e^{i\alpha}$

Geometrický význam násobení v $\mathbb{C}$:

Platí $z \cdot w = |z| \cdot |w|\, e^{i\alpha} e^{i\beta} \overset{?}{=} |z| \cdot |w|\, e^{i(\alpha+\beta)}$

kde $w = |w| \cdot e^{i\beta}$.

Ověřte!



Důsl. (MOIVRE)

Je-li $\alpha \in \mathbb{R}$ a $n \in \mathbb{Z}$, potom

$$(e^{i\alpha})^n = e^{in\alpha}.$$

Exponenciála v $\mathbb{C}$:

DEF. $\exp(z) := e^x \cdot (\cos y + i \sin y)$, $z = x + iy \in \mathbb{C}$.

Ukažte, že

1. $\exp|_{\mathbb{R}}$ je reálná exponenciála
2. $\exp(z+w) = \exp(z) \cdot \exp(w)$

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(3.) $\exp(\mathbb{C}) = \mathbb{C} \setminus \{0\}$

(4.) $\exp z = \exp w \iff \exists k \in \mathbb{Z}: \; w = z + 2k\pi i$

(5.) $|e^z| = e^{\operatorname{Re} z}$; spec. $|e^{iy}| = 1$, $y \in \mathbb{R}$

(6.) $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $z \in \mathbb{C}$

ad (2.) pro $w = u + iv$ je $e^z e^w = \underbrace{e^x e^u}_{\text{reálná expon.}} \cdot \underbrace{e^{iy} e^{iv}}_{\text{viz (?) výše}} =$

$= e^{x+u} e^{i(y+v)} = e^{z+w}$

ad (4.) $e^z = e^w \iff \begin{matrix} e^x = e^u \\ e^{iy} = e^{iv} \end{matrix} \iff \begin{matrix} x = u \\ v = y + 2k\pi \end{matrix}$

pro nějaké $k \in \mathbb{Z}$,

tj. $w = z + 2k\pi i$

ad (6.) $e^z = e^x e^{iy}$

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
- $e^{iy} = \cos y + i \sin y = \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{(2n+1)!}$ ($(-1)^n = i^{2n}$)

$= \sum_{n=0}^{\infty} \frac{(iy)^n}{n!}$

Tedy $e^z = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{n!}{n!} \frac{x^k}{k!} \frac{(iy)^{n-k}}{(n-k)!} \right) = \sum_{n=0}^{\infty} \frac{1}{n!} (x + iy)^n$.

součin řad

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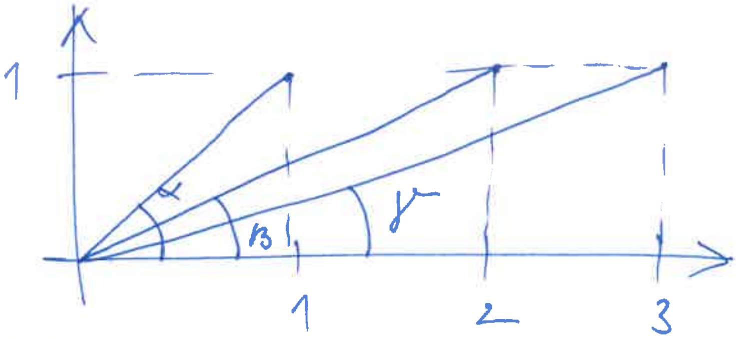
Př. Sečti: $\alpha+\beta+\gamma$, kde



[472]

Př. Sečti: $\sin\theta + \sin(2\theta) + \dots + \sin(n\theta)$, (cos)
kde $\theta \in \mathbb{R}$ a $n \in \mathbb{N}$. [131, 132]

Př. Sečti: $\binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \binom{n}{7} + \dots$

$\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \dots$

pro $n \in \mathbb{N}$

[návod $(1+i)^n$]