1. $\log(i)$, $\mathcal{L}og(i)$, i^i, $\mathcal{M}_i(i)$

$$\log z = \log|z| + i \cdot \arg z, \quad z \neq 0$$

$$\mathcal{L}og\, z = \{ \log z + 2k\pi i \mid k \in \mathbb{Z} \}$$

———— x ————

$$\log i = i\frac{\pi}{2}, \quad \mathcal{L}og(i) = \{ i\frac{\pi}{2} + 2k\pi i \mid k \in \mathbb{Z} \}$$

$$i^i = e^{i \log i} = e^{-\frac{\pi}{2}}$$

$$\mathcal{M}_i(i) = \{ e^{-\frac{\pi}{2} + 2k\pi} \mid k \in \mathbb{Z} \}$$

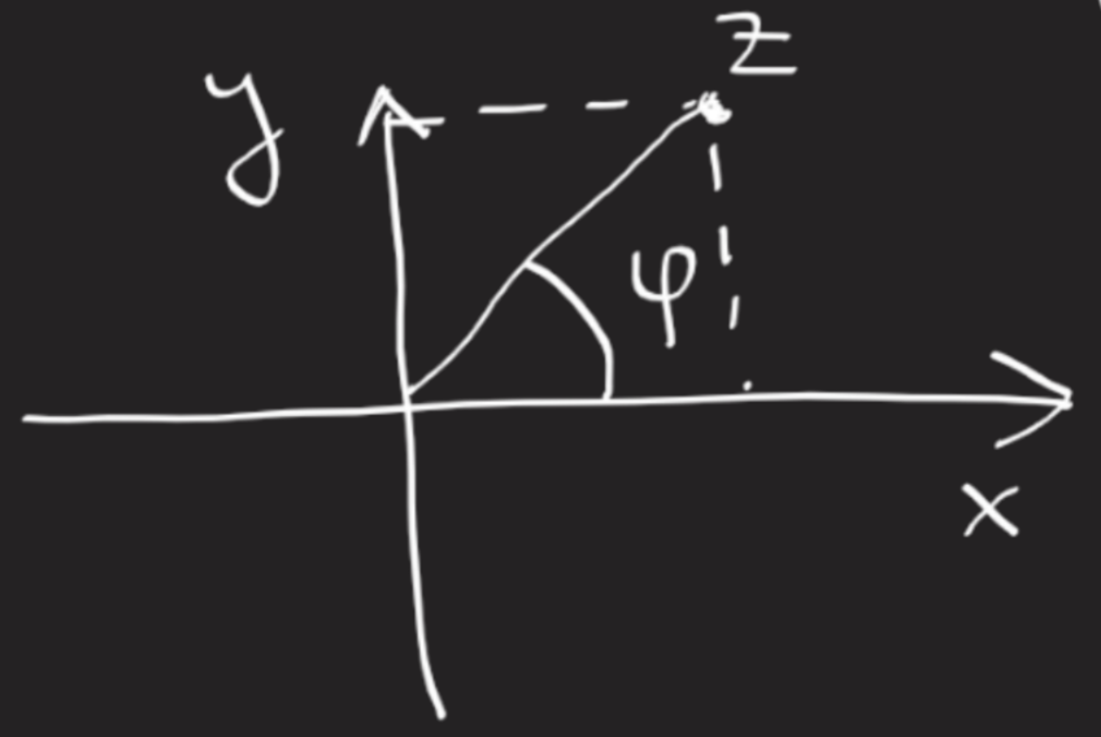
———— x ————

2. $\log' z = \frac{1}{z}, \quad z \in \mathbb{C} \setminus (-\infty, 0]$

$$f(z) = \log z = \log|z| + i \cdot \arg z, \quad z = x + iy \in \mathbb{C}$$

$$f_1(x,y) = \log(x^2+y^2)^{1/2} = \frac{1}{2}\log(x^2+y^2)$$

$$f_2(x,y) = \operatorname{arctg}\left(\frac{y}{x}\right), \quad x > 0$$



$$\operatorname{tg}\varphi = \frac{y}{x}$$

$$\varphi = \operatorname{arctg}\left(\frac{y}{x}\right), \quad \varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\frac{\partial f_1}{\partial x} = \frac{1}{2}\,\frac{2x}{x^2+y^2} = \frac{x}{x^2+y^2} \qquad \frac{\partial f_1}{\partial y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial f_2}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial f_2}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$$

Máme $f \in \mathcal{C}^\infty(\{\operatorname{Re} z > 0\})$ a splňuje tam (CR).

Tedy $f'(z) = \frac{\partial f}{\partial x} = \frac{x - iy}{x^2+y^2} = \frac{\bar z}{|z|^2} = \frac{1}{z}$, $\operatorname{Re} z > 0$.

(Pozn.) Tímto najít f' i na ostatních.

———— x ————

(3.) $\log(z_1 \cdot z_2) \overset{?}{=} \log z_1 + \log z_2$

———— x ————

(i) Nechť $z_1, z_2 \in \mathbb{C} \setminus \{0\}$. Potom $z_j = |z_j| e^{i \arg z_j}$

a $z_1 \cdot z_2 = |z_1| \cdot |z_2| e^{i(\arg z_1 + \arg z_2)}$, neboli

$\theta := \arg z_1 + \arg z_2 \in \operatorname{Arg}(z_1 \cdot z_2)$. Máme ale

$$\arg(z_1 \cdot z_2) = \theta, \text{ je-li } \theta \in (-\pi, \pi]$$
$$= \theta - 2\pi, \text{ je-li } \theta \in (\pi, 2\pi]$$
$$= \theta + 2\pi, \text{ je-li } \theta \in (-2\pi, -\pi]$$

(ii) $\log|z_1 \cdot z_2| = \log|z_1| + \log|z_2|$

Z (i) a (ii) dostaneme

$$\log(z_1 \cdot z_2) = \log z_1 + \log z_2 + 2k\pi i, \text{ kde}$$

$k = 0$, je-li $\theta \in (-\pi, \pi]$

$k = -1$, $(\pi, 2\pi]$

$k = +1$, $(-2\pi, -\pi]$

(4.) $z = e^{\log z}, \; z \neq 0$

$$z = |z| e^{i \arg z} = e^{\log|z| + i \arg z}$$

———— × ————

(Dcv) $\log(e^z) \neq z$

(5.) $\sin(\mathbb{C}) = \mathbb{C}$, spec. sin není v $\mathbb{C}$ omezený

Pro daná $w \in \mathbb{C}$ řešme rovnici v $\mathbb{C}$

$$\sin z = w$$

$$\frac{e^{iz} - e^{-iz}}{2i} = w$$

$$e^{iz} - e^{-iz} = 2iw$$

$$(e^{iz})^2 - 2iw e^{iz} - 1 = 0$$

$$(e^{iz} - iw)^2 = 1 - w^2$$

$$e^{iz} - iw = \pm\sqrt{1-w^2}, \quad \text{kde } \sqrt{w} := e^{\frac{1}{2}\log w}, \; w \neq 0$$
$$:= 0, \; w = 0$$

$$e^{iz} = iw \pm \sqrt{1-w^2} \neq 0$$

⌈protože $\left(iw + \sqrt{1-w^2}\right) \cdot \left(iw - \sqrt{1-w^2}\right) = -w^2 - (1-w^2) = -1$⌋

$$e^{iz} = A \overset{\neq 0}{=} e^{\log A}$$

$$iz = \log A + 2k\pi i, \quad k \in \mathbb{Z}$$

$$z_{k,\pm} = \frac{1}{i}\log\left(i\cdot w \pm \sqrt{1-w^2}\right) + 2k\pi, \quad k\in\mathbb{Z}$$

Pozn: (i) $\sin(z) = 0 \iff z = k\pi,\ k\in\mathbb{Z}$

[Pro $w=0$ je $z_{k,+} = 2k\pi$, $z_{k,-} = \pi + 2k\pi$]

$$\log(-1) = i\pi$$

(ii) Je-li $w\in[-1,1]$, potom

$$\arcsin(w) = \frac{1}{i}\log\left(iw+\sqrt{1-w^2}\right).$$

Komplexní funkce reálné proměnné

$f:\mathbb{R}\to\mathbb{C}=\mathbb{R}^2 \qquad f(t) = (f_1(t), f_2(t)) = f_1(t)+i f_2(t)$

- vše po složkách:

lim, derivace, prim. fce, integrál, posl., řady

$f(t)$

např. $\int f(t)\,dt = \int f_1(t)\,dt + i\int f_2(t)\,dt$, pokud neurč. integrály vpravo ex.

Př. $f(x) := e^{(3+i)x}$, $x\in\mathbb{R}$

$$= e^{3x}\cos x + i\, e^{3x}\sin x$$

$$\frac{df}{dx}(x) = \frac{d}{dx}\left(e^{3x}\cos x\right) + i\,\frac{d}{dx}\left(e^{3x}\sin x\right)$$

$$= e^{3x}\cos x\,(3+i) + e^{3x}\sin x\,(3i\underbrace{-1}_{=i^2}) =$$

$$= (3+i)\,e^{(3+i)x} = f'(x)$$

+ CR-rovnice ↑   ↑ komplex. der.

Př. $(x \in \mathbb{R})$

$$\int \underbrace{e^{3x}\cdot\cos x}_{\mathrm{Re}\left(e^{(3+i)x}\right)}\,dx = \mathrm{Re}\int e^{(3+i)x}\,dx$$

$$= \mathrm{Re}\left(\frac{e^{(3+i)x}}{(3+i)}\right) = \frac{e^{3x}}{10}\left(3\cos x + \sin x\right)$$

$$\frac{1}{3+i} = \frac{3-i}{10}, \quad e^{(3+i)x} = e^{3x}\left(\cos x + i\cdot\sin x\right)$$

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

Př.

$x \in \mathbb{R}$

$a, b \in \mathbb{R}$

$b \neq 0$

$$\int \frac{dx}{x-(a+ib)} = \int \frac{x-a}{(x-a)^2+b^2}\,dx + i \int \frac{b\,dx}{(x-a)^2+b^2}$$

$$= \frac{1}{2}\log\left((x-a)^2+b^2\right) + i\cdot \operatorname{arctg}\left(\frac{x-a}{b}\right)$$

$$\frac{1}{(x-a)-ib} = \frac{x-a+ib}{(x-a)^2+b^2}$$

$$= \log|x-(a+ib)| + i \operatorname{arctg}\left(\frac{x-a}{b}\right), \quad x \in \mathbb{R}$$

LÉPE: $F(x) = \log(x-(a+ib))$, $x \in \mathbb{R}$

$$\frac{dF}{dx}(x) = F'(x) = \frac{1}{x-(a+ib)}, \text{ neboli}$$

CR-rovn. komplex. derivace

$$\int \frac{dx}{x-(a+ib)} = F(x), \quad x \in \mathbb{R}$$

Pozn: Toto platí i pro $b=0$ na $x \neq a$