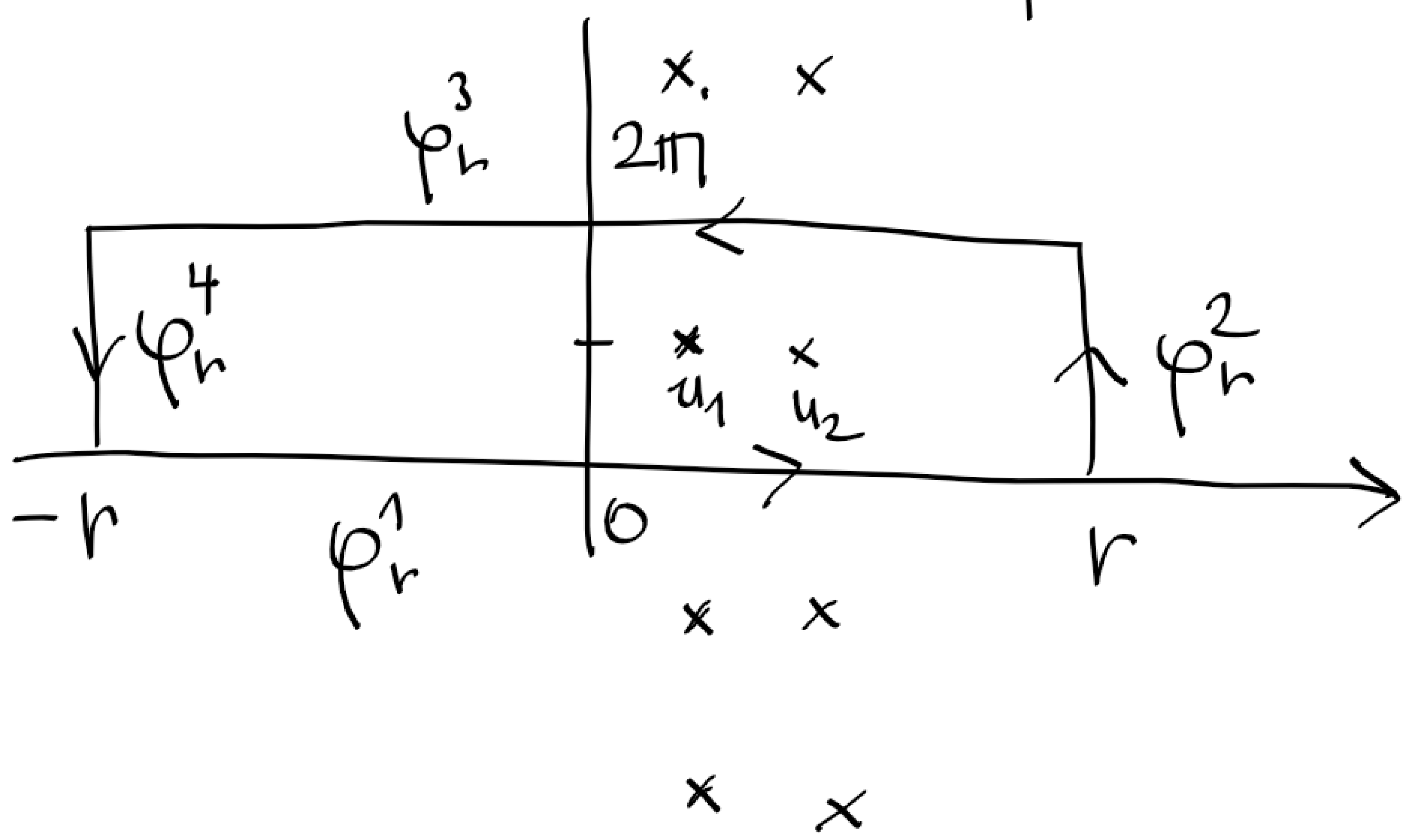
Př. 1

$$I = \int_0^{+\infty} \frac{\sqrt[6]{x^7}\,dx}{x^2+5x+6} \underset{\substack{x=e^u \\ dx = e^u\,du}}{=} \int_{-\infty}^{+\infty} \underbrace{\frac{e^{7u/6}\,du}{e^{2u}+5e^u+6}}_{f(u)}$$

$x = -2, -3$



$e^u = -2 \quad u_1 = \ln 2 + i\pi$

$ = -3 \quad u_2 = \ln 3 + i\pi$

$$f(t+2\pi i) = f(t)\, e^{14\pi i/6}, \quad e^{14\pi i/6} = e^{2\pi i/6}$$

$$e^{t+2\pi i} = e^t$$

$$2\pi i \cdot \left(\operatorname{res}_{u_1} f + \operatorname{res}_{u_2} f\right) \overset{RV}{=} \int_{\varphi_r} f = \int_{\varphi_r^1} f + \int_{\varphi_r^2} f + \int_{\varphi_r^3} f + \int_{\varphi_r^4} f$$

$r \to +\infty$: $\int_{\varphi_r^1} f \to I$, $\int_{\varphi_r^2} f \to 0$, $\int_{\varphi_r^3} f \to \,?$, $\int_{\varphi_r^4} f \to 0$

$$\int_{\varphi_3^r} f = -\int_{-r}^{r} f \cdot e^{\pi i/3} \xrightarrow{r \to +\infty} -e^{\pi i/3} I$$

$-\varphi_3^r(t) := t + 2\pi i, \; t \in [-r, r]$

$$\operatorname{res}_{u_j} f = \frac{e^{7u_j/6}}{2e^{2u_j} + 5e^{u_j}}$$

$$I \cdot (1 - e^{\pi i/3}) = 2\pi i \cdot \left(\operatorname{res}_{u_1} f + \operatorname{res}_{u_2} f\right)$$

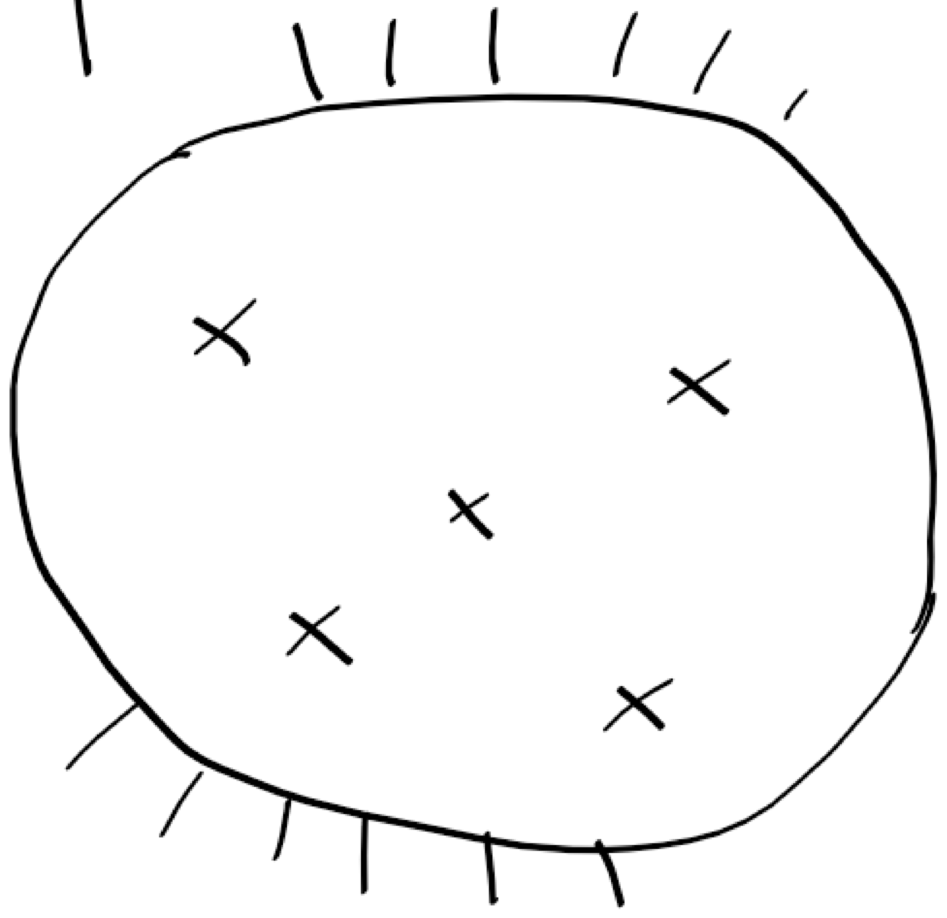
$$I = -\left(\frac{2\pi i}{1 - e^{\pi i/3}}\right) e^{\pi i/6} \left(\sqrt[6]{3} - \sqrt[6]{2}\right) = +\pi \cdot \frac{2i}{\underbrace{(-e^{-\pi i/6}) + e^{\pi i/6}}_{= \frac{1}{\sin(\pi/6)}}} \cdot (\dots) = 2\pi\left(\sqrt[6]{3} - \sqrt[6]{2}\right)$$

$u_1 = \ln 2 + \pi i$: $\quad \operatorname{res}_{u_1} f = \dfrac{-2 \cdot \sqrt[6]{2}\, e^{\pi i/6}}{-2} = \sqrt[6]{2}\, e^{\pi i/6}$

$u_2 = \ln 3 + \pi i$: $\quad = \dfrac{-3 \cdot \sqrt[6]{3}\, e^{\pi i/6}}{2 \cdot 9 - 15} = -\sqrt[6]{3}\, e^{\pi i/6}$

Reziduum v ∞

VĚTA: Nechť f je holomorfní funkce na $\mathbb{C}\setminus K$, kde K je konečná. Nechť $K\subset U(0,R)$. Potom



$$f(z)=\sum_{n=-\infty}^{+\infty} a_n z^n, \quad z\in P(0,R,+\infty) \quad \text{a}$$

platí, že

$$a_{-1}=\sum_{s\in K} \operatorname{res}_s f \qquad (*)$$

DŮKAZ: Nechť $\varphi(t):=2Re^{it}$, $t\in[0,2\pi]$. Potom

$$\sum_{s\in K} \operatorname{res}_s f \overset{RV}{=} \frac{1}{2\pi i}\int_\varphi f(z)\,dz = \frac{1}{2\pi i}\int_\varphi \sum_{-\infty}^{+\infty} a_n z^n\,dz =$$

$$= \sum_{-\infty}^{+\infty} a_n \cdot \frac{1}{2\pi i}\underbrace{\int_\varphi z^n\,dz}_{=0,\ n\neq -1} = a_{-1}\cdot \underbrace{\operatorname{ind}_\varphi 0}_{=1} . \qquad \blacksquare$$

DEF. Nechť f je holomorfní funkce na $P(\infty,r) = P(0,\frac{1}{r},+\infty)$ a $f(z)=\sum_{n=-\infty}^{+\infty} a_n z^n$, $z\in P(\infty,r)$.

Položme $\operatorname{res}_\infty f := -a_{-1}$.

Př. POZOR: Funkce $f(z) := \frac{1}{z}$, $z\in\mathbb{C}\setminus\{0\}$; $:= 0$, $z=\infty$ je holom. v $S\setminus\{0\}$, ale $\operatorname{res}_\infty f = -1$

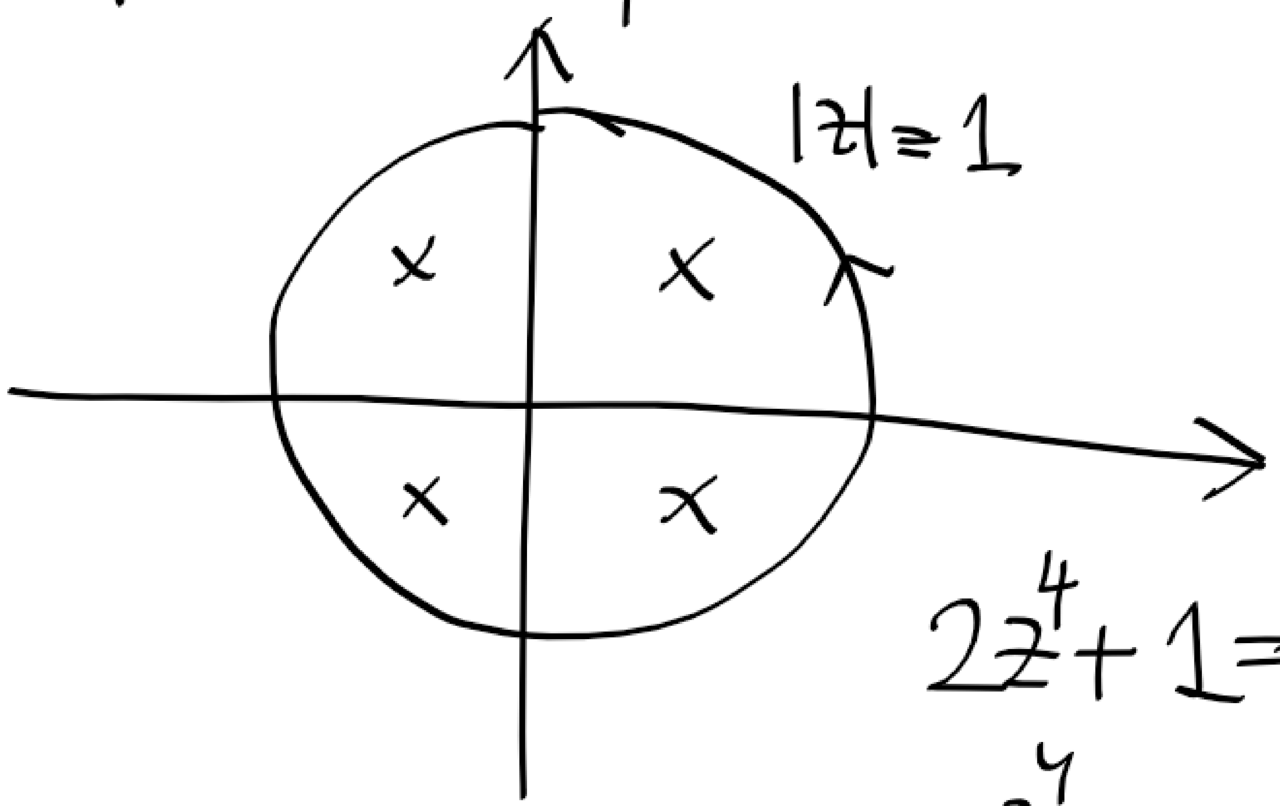
Pom: Vo VETE (*) znamene, za $\sum_{s \in K \cup \{\infty\}} \operatorname{res}_s f = 0.$

Pr. $I = \int_{|z|=1} \frac{z^3\,dz}{2z^4+1} \overset{RV}{=} 2\pi i \sum_{s^4=-\frac{1}{2}} \operatorname{res}_s f \underset{(*)}{=} -2\pi i \operatorname{res}_\infty f$

$\underbrace{2z^4+1}_{f(z)}$

$= \pi i$, pri tem

integriramo po $\varphi(t) := e^{it}$, $t \in [0, 2\pi]$



$2z^4+1=0$

$z^4 = -\frac{1}{2}$

$f(z) = \frac{z^3}{2z^4+1} = \frac{1}{2z} \cdot \frac{1}{1+\frac{1}{2z^4}}$

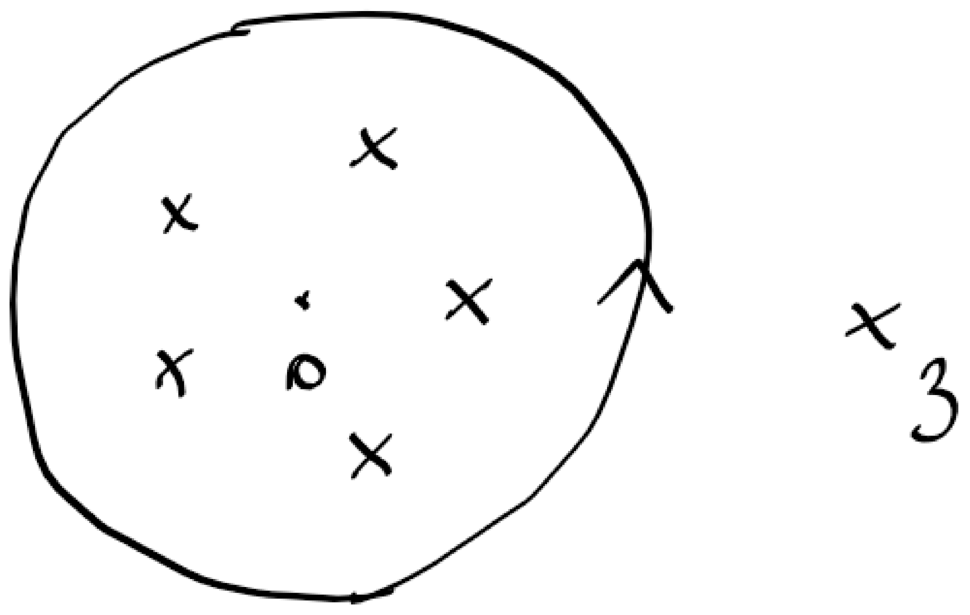
$= \sum_{n=0}^{+\infty} (-1)^n \frac{1}{2^{n+1} z^{4n+1}}, \quad |z| > \frac{1}{\sqrt[4]{2}}$

$\operatorname{res}_\infty f = -\frac{1}{2}$

Pr. $I = \int_{|z|=2} \frac{dz}{(z-3)(z^5-1)} \overset{RV}{=} 2\pi i \sum_{s^5=1} \operatorname{res}_s R \overset{(*)}{=} -2\pi i (\operatorname{res}_3 R + \operatorname{res}_\infty R) =$

$\underbrace{(z-3)(z^5-1)}_{R(z)}$

$= -\frac{\pi i}{121}$



$z^5 = 1$

$z = 3$

(i) $\operatorname{res}_3 R = \frac{1}{3^5-1} = \frac{1}{242}$

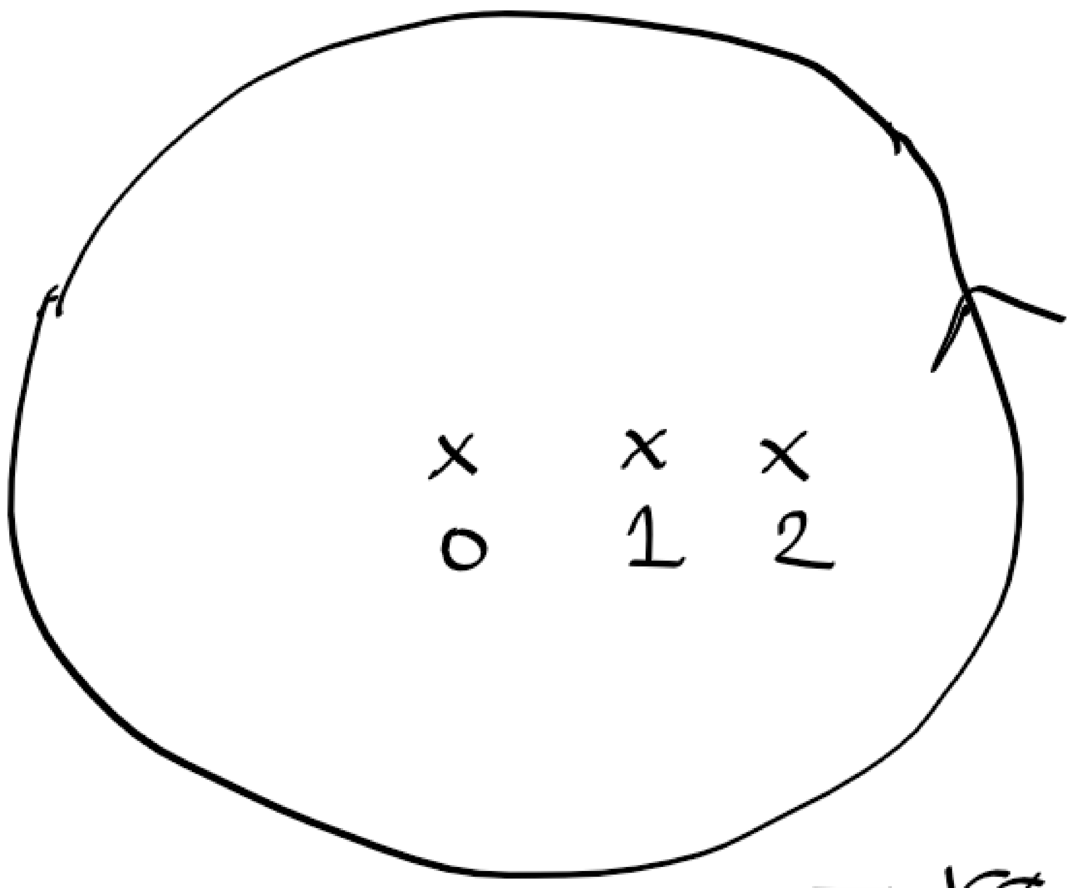
(ii) $\operatorname{res}_\infty R = 0$, pri tem

$R(z) = \frac{1}{z^6} \cdot \frac{1}{1-\frac{3}{z}} \cdot \frac{1}{1-\frac{1}{z^5}} =$

$= \frac{1}{z^6} \left(\sum_{n=0}^{\infty} \frac{3^n}{z^n} \right) \cdot \left(\sum_{m=0}^{\infty} \frac{1}{z^{5m}} \right), \quad |z| > 3$

$= \frac{1}{z^6} + \frac{1}{z^7}(3) + \cdots$

Pr. $I = \int_{|z|=3} \underbrace{(1+z+z^2)\cdot\left(e^{\frac{1}{z}} + e^{\frac{1}{z-1}} + e^{\frac{1}{z-2}}\right)}_{f(z) :=} dz \overset{RV}{=}$



$= 2\pi i \cdot \left(\operatorname{res}_0 f + \operatorname{res}_1 f + \operatorname{res}_2 f\right)$

$= -2\pi i \cdot \left(\operatorname{res}_\infty f\right)$

(i) $\operatorname{res}_0 f = \operatorname{res}_0 (1+z+z^2)\, e^{1/z} =$

$= \operatorname{res}_0 (1+z+z^2)\cdot\left(1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots\right)$

$= 1 + \frac{1}{2} + \frac{1}{6}$

(ii) $\operatorname{res}_1 f = \operatorname{res}_1 \left(1+z+z^2\right) e^{1/(z-1)} =$

$= \operatorname{res}_1 \left(3 + 3(z-1) + (z-1)^2\right)\left(1 + \frac{1}{z-1} + \frac{1}{2(z-1)^2} + \frac{1}{6(z-1)^3} + \cdots\right)$

$= 3 + \frac{3}{2} + \frac{1}{6}$

(iii) $\operatorname{res}_2 f = \ldots$