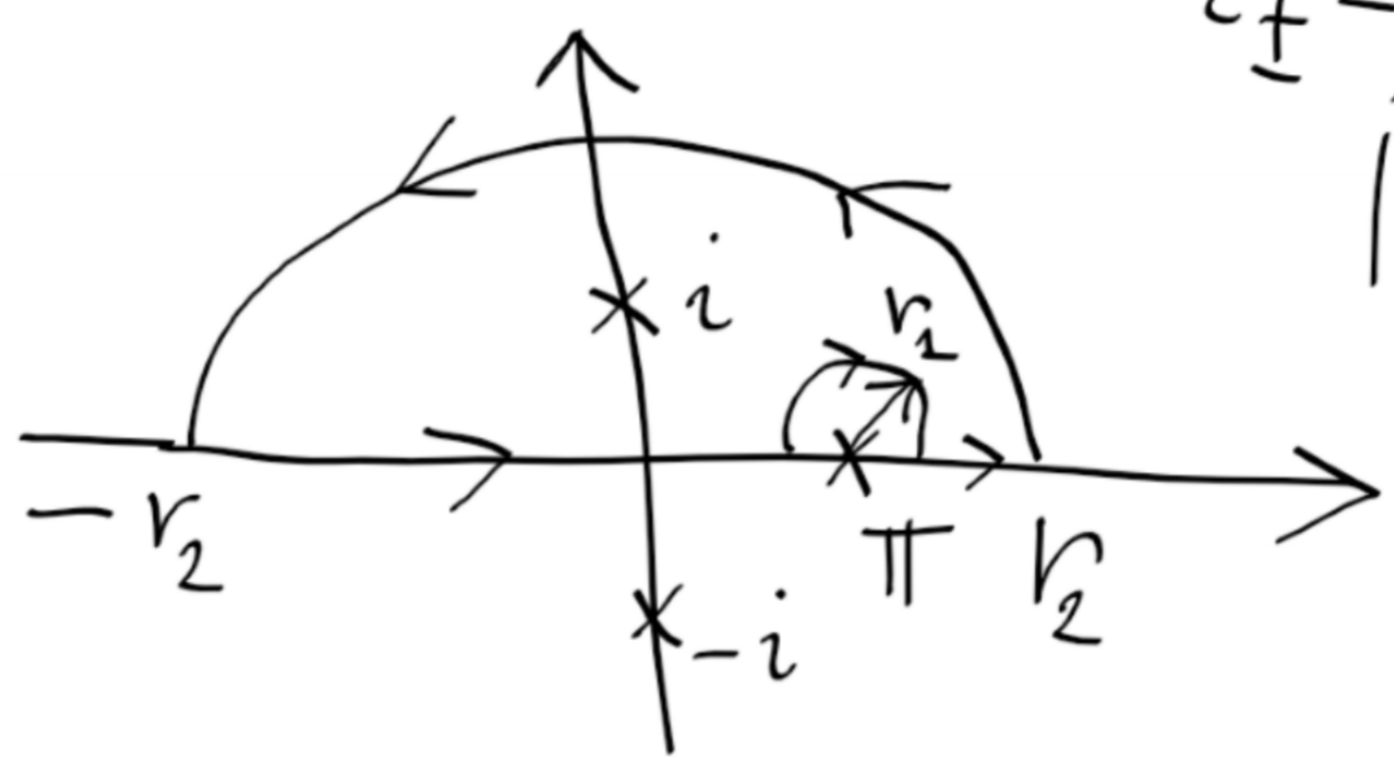
Př. $\quad I = \displaystyle\int_{-\infty}^{+\infty} \frac{\sin x \, dx}{(x^2+1)\cdot(x-\pi)}$

(i) Integrál I konverguje jako Newton. i Lebesg., ovšem ale
$$\int_{-\infty}^{+\infty} \frac{e^{ix}\,dx}{(x^2+1)(x-\pi)}$$

(ii) Integrujme $F(z) := \dfrac{e^{iz}}{(z^2+1)\cdot(z-\pi)}$ podél křivky



$z_{\pm} = \pm i$, $z_0 = \pi$ jednod. póly

$\varphi_{r_1 r_2} := [-r_2; \pi - r_1] \dotplus (\dot{-}\varphi_{r_1}) \dotplus [\pi + r_1; r_2] \dotplus \varphi_{r_2}$, kde

$\varphi_{r_1}(t) := \pi + r_1 e^{it}$, $t \in [0, \pi]$ a

$\varphi_{r_2}(t) := r_2 e^{it}$, $t \in [0, \pi]$.

Je-li $r_1 > 0$ dost malé a $r_2 > 0$ dost velké, potom
$$2\pi i \cdot \mathrm{res}_i F \overset{(RV)}{=} \int_{\varphi_{r_1 r_2}} F = \int_{-r_2}^{\pi - r_1} F + \int_{\pi + r_1}^{r_2} F - \int_{\varphi_{r_1}} F + \int_{\varphi_{r_2}} F, \text{ tudíž}$$
$$\mathrm{Im}\,(2\pi i \cdot \mathrm{res}_i F) = \underbrace{\int_{-r_2}^{\pi - r_1} \mathrm{Im} F + \int_{\pi + r_1}^{r_2} \mathrm{Im} F}_{\downarrow} - \mathrm{Im} \int_{\varphi_{r_1}} F + \mathrm{Im} \int_{\varphi_{r_2}} F \quad \text{a pro } \begin{matrix} r_1 \to 0+ \\ r_2 \to +\infty \end{matrix}$$

$\downarrow$ z JORDANOVA LEMMATU

je $\mathrm{Im}\,(2\pi i \cdot \mathrm{res}_i F) = I - \mathrm{Im}\,(\pi i \cdot \mathrm{res}_\pi F) + 0$.

Z LEMMATU o obcházení jednoduchého pólu totiž platí $\displaystyle\int_{\varphi_{r_1}} F \xrightarrow[r_1 \to 0+]{} \pi i \cdot \mathrm{res}_\pi F$

$$I = \operatorname{Im}\left(2\pi i \cdot \operatorname{res}_i F + \pi i \cdot \operatorname{res}_\pi F\right) = \circledast$$

$$F(z) = \frac{e^{iz}}{(z^2+1)(z-\pi)}$$

$$\operatorname{res}_i F = \frac{e^{-1}}{(i-\pi)\cdot 2i}, \quad \operatorname{res}_\pi F = \frac{e^{i\pi}}{(\pi^2+1)} = -\frac{1}{\pi^2+1}$$

$$= \frac{1}{2ie} \cdot \frac{-\pi - i}{1+\pi^2}$$

$$\circledast = \frac{\pi}{e}\frac{-1}{1+\pi^2} - \frac{\pi}{1+\pi^2} = \underline{\underline{-\frac{\pi}{1+\pi^2}\left(1+\frac{1}{e}\right)}}$$

<u>Integrály typu</u> $I = \int_0^{+\infty} R(x)\, x^{p-1}\, dx \underset{\substack{x=e^u \\ dx = e^u du}}{=} \int_{-\infty}^{+\infty} R(e^u)\, e^{pu}\, du$

<u>VĚTA</u>: Nechť $p \in (0,1)$ a R je racionální funkce. Potom integrál I konverguje, právě když $R = P/Q$, kde P, Q jsou polynomy splňující

1. $Q \neq 0$ na $[0, +\infty)$ a
2. $\operatorname{st} Q > \operatorname{st} P$.

Platí-li 1. a 2., potom

$$I = \frac{2\pi i}{1 - \exp(2\pi p i)} \cdot \sum_{0 < \operatorname{Im} s < 2\pi} \operatorname{res}_s \Big(\underbrace{R(e^z)\, e^{pz}}_{!!\; f(z)} \Big),$$

kde sčítáme přes všechny póly f v pásu $0 < \operatorname{Im} s < 2\pi$.

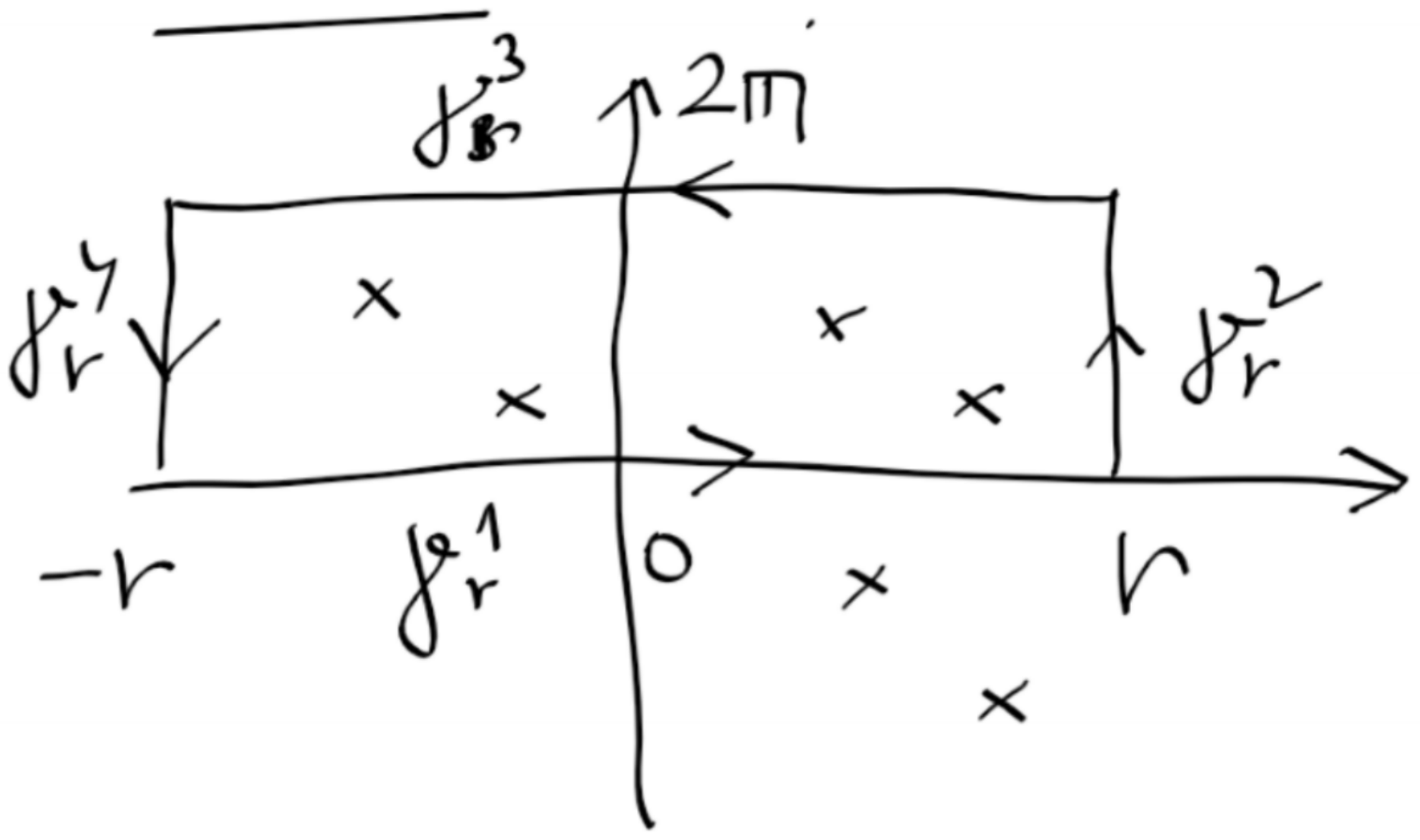
DŮKAZ: Nechť $r>0$ a položme $\gamma_r := \gamma_r^1 \dotplus \gamma_r^2 \dotplus \gamma_r^3 \dotplus \gamma_r^4$,

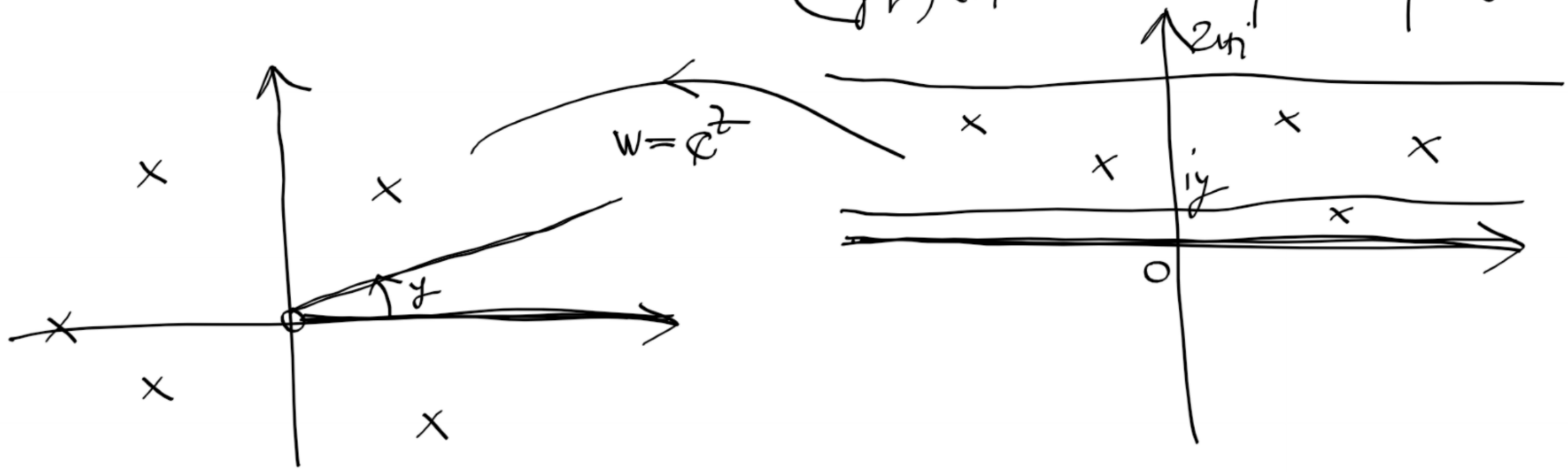
kde $\gamma_r^1(t) := t$, $t \in [-r,r]$

$\gamma_r^2(t) := r+it$, $t \in [0,2\pi]$

$(\dot{-}\gamma_r^3)(t) := 2\pi i + t$, $t \in [-r,r]$

$(\dot{-}\gamma_r^4)(t) := -r+it$, $t \in [0,2\pi]$





Je-li $r>0$ dost velké, potom

(1) $$\int_{\gamma_r} f \overset{(RV)}{=} 2\pi i \cdot \sum_{\substack{0<\operatorname{Im} s<2\pi \\ s \text{ je pól } f}} \operatorname{res}_s f \overset{\text{ozn.}}{=:} b.$$

Dále máme pro $r \to +\infty$

(2) $$\begin{array}{ccccccccc} \int_{\gamma_r} f & = & \int_{\gamma_r^1} f & + & \int_{\gamma_r^2} f & + & \int_{\gamma_r^3} f & + & \int_{\gamma_r^4} f \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ b & & I & & 0 & & ? & & 0 \end{array}$$

a $$\int_{\gamma_r^3} f = -\int_{-r}^{r} R(\underbrace{e^{t+2\pi i}}_{= e^t}) e^{pt} e^{p2\pi i}\,dt \xrightarrow[r\to+\infty]{} -e^{2\pi p i} I.$$

Z (1) a (2) máme $\quad b = I(1-e^{2\pi p i})$.

(2) platí: (i) z (2.) ex. $C>0$, $r_0>0$ tak, že

$$|R(z)| \le \frac{C}{|z|},\ |z| \ge r_0. \quad \text{Potom}$$

$$\left|\int_{\gamma_r^2} f\right| \le 2\pi \cdot \max_{z\in\langle\gamma_r^2\rangle} \left|\underbrace{R(e^z)\,e^z}\, e^{(p-1)z}\right| \le 2\pi C e^{(p-1)r} \xrightarrow{r\to+\infty} 0$$

$$|e^z| = e^r \to +\infty$$

(ii) z (1.) R je spojitá v 0, tudíž ex. $D>0$, $r_1>0$ tak, že $|R(z)| \le D$, $|z| \le r_1$. Potom

$$\left|\int_{\gamma_r^4} f\right| \le 2\pi \cdot \max_{z\in\langle\gamma_r^4\rangle} \left|R(e^z)\, e^{pz}\right| \le 2\pi D e^{-pr} \xrightarrow[r\to+\infty]{} 0$$

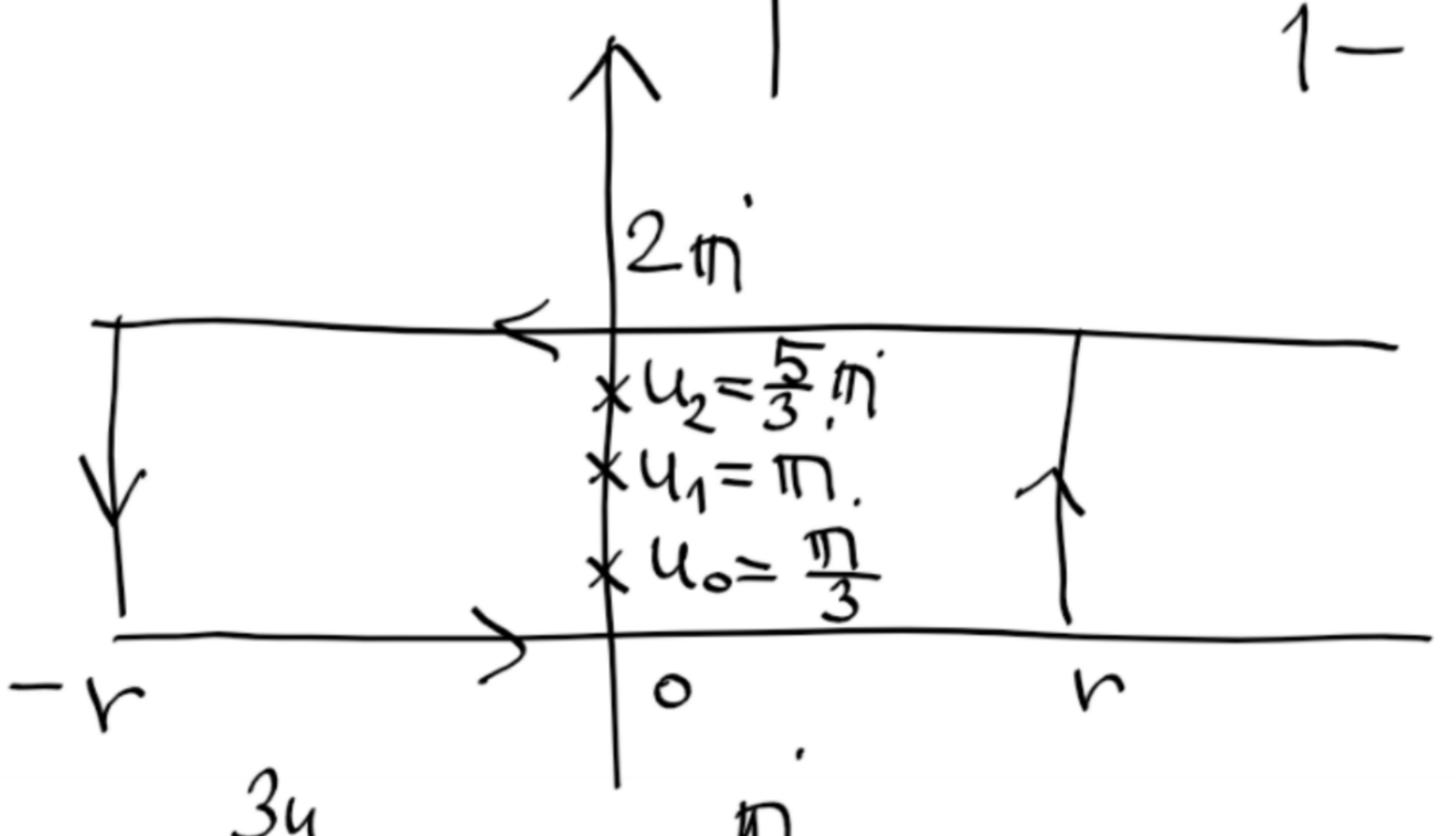
$$|e^z| = e^{-r} \to 0$$

∎

Př. $$I = \int_0^{+\infty} \frac{\sqrt{x}\,dx}{x^3+1} \underset{x=e^u}{=} \int_{-\infty}^{+\infty} \underbrace{\frac{e^{\frac{1}{2}u}\,e^u}{e^{3u}+1}}_{\overset{!!}{f(u)}} du$$

$$\boxed{R(x) = \frac{x}{x^3+1},\ p = 1/2}$$

Podle VĚTY je $$I = \frac{2\pi i}{1-\exp(\pi i)}\left(\operatorname{res}_{u_0} f + \operatorname{res}_{u_1} f + \operatorname{res}_{u_2} f\right) = \pi i\cdot\left(-\frac{i}{3}\right) = \underline{\underline{\frac{\pi}{3}}},$$ protože



$$\operatorname{res}_{u_k} f = \frac{e^{\frac{3}{2}u_k}}{3e^{3u_k}} = -e^{\frac{3}{2}u_k}\cdot\frac{1}{3}$$

$= -\frac{i}{3}$ pro $k=0$

$= +\frac{i}{3}$ $k=1$

$= -\frac{i}{3}$ $k=2$

$$e^{3u} = -1 = e^{\pi i}$$

$3u = \pi i + 2k\pi i$, $k\in\mathbb{Z}$ $\quad$ $\boxed{u_k = \frac{\pi i}{3} + k\cdot\frac{2}{3}\pi i,\ k\in\mathbb{Z}}$