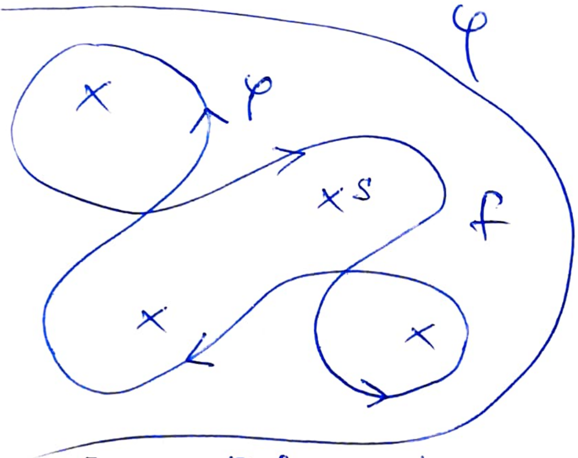
10. cvičení

Aplikace reziduové věty I.

VÍME: (RV) $\displaystyle\int_\varphi f = 2\pi i \cdot \sum_{s\in M} \operatorname{res}_s f \cdot \operatorname{ind}_\varphi s$

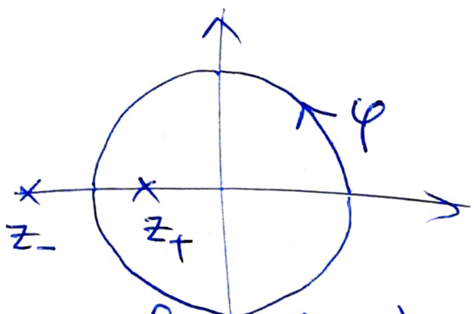


$G \subset \mathbb{C}$ hvězdovitá oblast

(Př1) Spočti $\displaystyle I := \int_\varphi \frac{z\,dz}{\underbrace{(z^2+4z+1)^2}_{= f(z)}}$, kde $\varphi(t) = e^{it}$, $t \in [0, 2\pi]$.

- Z minula víme: $z_\pm = -2 \pm \sqrt{3}$

  $\operatorname{res}_{z_+} f = \dfrac{1}{6\sqrt{3}}$



- Potom $I \overset{(RV)}{=} 2\pi i \cdot \left(\operatorname{res}_{z_+} f \cdot \underbrace{\operatorname{ind}_\varphi z_+}_{=1} + \operatorname{res}_{z_-} f \cdot \underbrace{\operatorname{ind}_\varphi z_-}_{=0}\right)$

$$= \underline{\underline{\frac{\pi i}{3\sqrt{3}}}}$$

Integrály typu $\int_0^{2\pi} R(\cos t, \sin t)\,dt$, $\int_0^{2\pi} R(e^{it})\,dt$

Převedeme na křivkový integrál podél $\varphi(t) := e^{it}$, $t \in [0, 2\pi]$!

Skutečně, z definice platí

$$\int_\varphi f(z)\,dz \underset{\substack{z = e^{it} \\ dz = ie^{it}dt}}{=} i \int_0^{2\pi} f(e^{it})\, e^{it}\,dt.$$

Tudíž $\int_0^{2\pi} R(e^{it}) \underset{\frac{ie^{it}}{ie^{it}}}{\wedge} dt = \int_\varphi R(z) \frac{dz}{iz}.$

Př. $I = \int_0^{2\pi} \frac{dt}{(2+\cos t)^2} = \int_\varphi \frac{dz}{\left(2 + \frac{1}{2}\left(z + \frac{1}{z}\right)\right)^2 \cdot i \cdot z} = ⊛$

$\cos t = \frac{e^{it} + e^{-it}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right)$, je-li $z = e^{it}$

$$⊛ = \frac{4}{i} \int_\varphi \frac{z\,dz}{(z^2+4z+1)^2} \overset{\text{dvůr}}{\underset{(RV)}{=}} \frac{4}{i} \cdot \frac{\pi i}{3\sqrt{3}} = \underline{\underline{\frac{4\pi}{3\sqrt{3}}}}$$

[Kopáček, př. 431–447]

Integrály typu $\int_{-\infty}^{+\infty} R(x)\,dx$

VĚTA: Nechť $R = P/Q$, kde P, Q jsou polynomy, které nemají společné kořeny a splňují

① $Q \neq 0$ na $\mathbb{R}$,

② $\operatorname{st} Q \geq \operatorname{st} P + 2$, kde $\operatorname{st} Q$ je stupeň polynomu Q. Potom

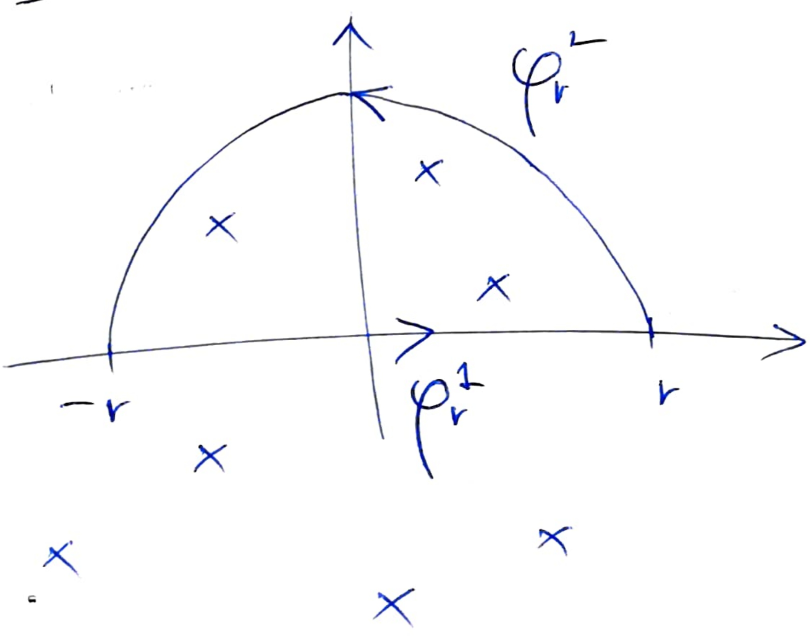
$$I := \int_{-\infty}^{+\infty} R(x)\,dx = 2\pi i \sum_{\substack{Q(s)=0 \\ \operatorname{Im} s > 0}} \operatorname{res}_s R \qquad (1)$$

Pozn: Ukažte, že integrál v (1) konverguje, pokud platí ①, ②. DCV

DŮKAZ: Nechť $r > 0$ a $\varphi_r := \varphi_r^1 \dotplus \varphi_r^2$, kde

$\varphi_r^1(t) := t$, $t \in [-r, r]$ a

$\varphi_r^2(t) := re^{it}$, $t \in [0, \pi]$.



Nechť $r > 0$ je tak velké, aby uvnitř (půlkruhu) φ_r ležely všechny póly R z horní poloroviny.

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Potom dostaneme

$$(*)\quad 2\pi i \cdot \sum_{\substack{Q(s)=0\\ \operatorname{Im} s>0}} \operatorname{res}_s R \overset{(RV)}{=} \int_{\varphi_r} R = \int_{\varphi_r^1} R + \int_{\varphi_r^2} R,$$

protože $\operatorname{ind}_{\varphi_r} s = 1$, je-li $Q(s)=0$, $\operatorname{Im} s>0$,
$= 0$, — " — $\operatorname{Im} s<0$. (Δ)

Dále platí $\int_{\varphi_r^1} R = \int_{-r}^{r} R \longrightarrow I$ pro $r\to+\infty$.

Protože (x) $\int_{\varphi_r^2} R \longrightarrow 0$ pro $r\to+\infty$, dostaneme

z (*), pro $r\to+\infty$, že platí (1).

———— x ————

Ověření (x): Protože $\lim_{z\to\infty} z^2\cdot R(z) =$

$$\lim_{z\to\infty} z^2\cdot \frac{a_0 z^n+\dots+a_n}{\underset{\neq 0}{b_0} z^m+\dots+b_m} = \lim_{z\to\infty} z^{\overset{\leq 0}{n+2-m}} \cdot \underbrace{\frac{a_0+\dots+\frac{a_n}{z^n}}{b_0+\dots\frac{b_m}{z^m}}}_{\downarrow\ \frac{a_0}{b_0}} \in \mathbb{C},$$

potom ex. $C\in(0,+\infty)$, ex. $r_0>0$ tak, že

$$|R(z)| \leq \frac{C}{r^2}, \text{ je-li } |z|=r\geq r_0.$$

Tedy $\left|\int_{\varphi_r^2} R\right| \leq \pi\cdot r\cdot \frac{C}{r^2} = \frac{\pi\cdot C}{r} \longrightarrow 0$ $r\to+\infty$.

▨

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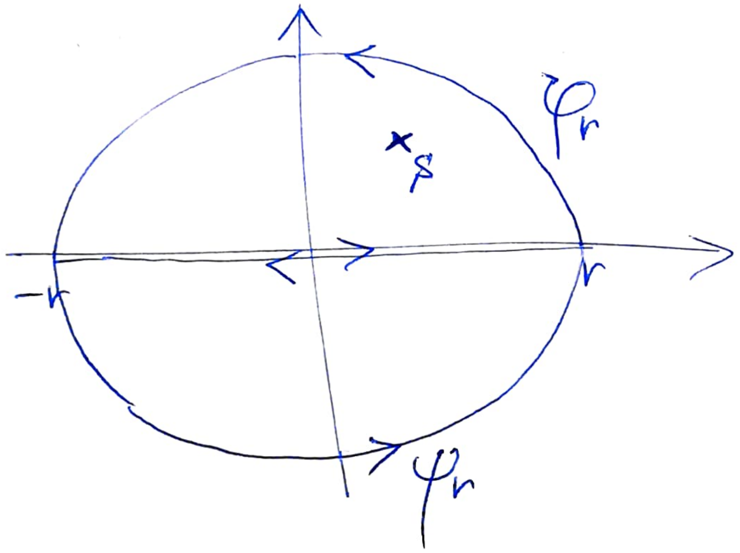
4.1

Pozn: (i) Ověření (△): zřejmě

$$1 = \operatorname{ind}_{\substack{\text{Kružnice}\\ \text{o poloměru}\\ r}} s = \operatorname{ind}_{(\varphi_r \dot{+} \psi_r)} s = \operatorname{ind}_{\varphi_r} s + \underbrace{\operatorname{ind}_{\psi_r} s}_{=0},$$

protože s leží vně ψ_r

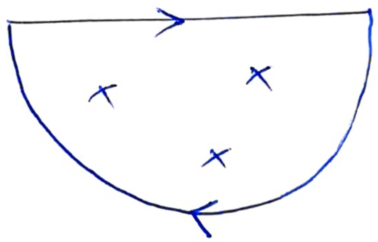


tudíž $\operatorname{ind}_{\varphi_r} s = 1$, viz OBR.

(ii) Ukažte, že $I = -2\pi i \cdot \sum\limits_{\substack{Q(s)=0\\ \operatorname{Im} s<0}} \operatorname{res}_s R.$

Dk.

Ukažte



Speciálně, pro R splňující (1.), (2.) tedy platí

$$\sum_{Q(s)=0} \operatorname{res}_s R = 0 \; !$$

Pří

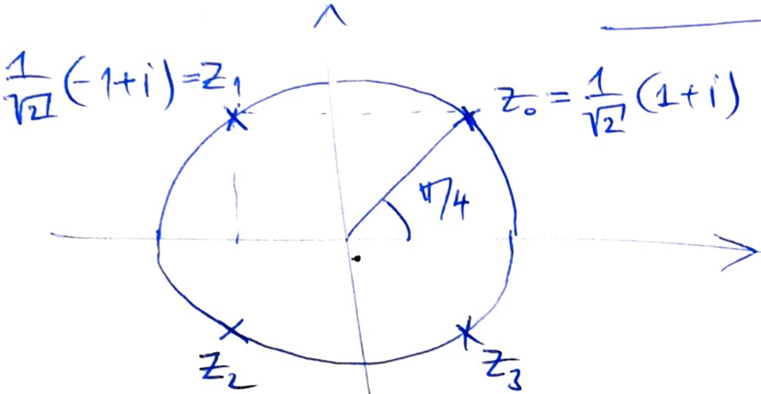
$$I := \int_0^{+\infty} \underbrace{\frac{x^2+1}{x^4+1}}_{\text{sudá}}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} R(x)\,dx,$$

Kde $R(z) := \dfrac{z^2+1}{z^4+1}$.

$z^4 = -1 = e^{i\pi}$

$z_k = e^{i\alpha_k}$

$\alpha_k := \frac{\pi}{4} + k\cdot \pi/2,\ k = 0,1,2,3$



Z RV VĚTY dostaneme, že

$$I = \frac{1}{2}\cdot 2\pi i\cdot\left(\operatorname{res}_{z_0}R + \operatorname{res}_{z_1}R\right).$$

- $\operatorname{res}_{z_k}R = \dfrac{z_k^2+1}{4z_k^3}\cdot\dfrac{z_k}{z_k} = -\dfrac{1}{4}\cdot(i+1)\cdot\dfrac{1}{\sqrt2}(1+i),\ k=0$

  Pravidlo (2.)

  $= -\dfrac{1}{4}(-i+1)\cdot\dfrac{1}{\sqrt2}(-1+i),\ k=1$

- $$I = -\frac{\pi i}{4\cdot\sqrt2}\cdot\Big(\underbrace{(1+i)^2-(1-i)^2}_{=4i}\Big) = \underline{\underline{\frac{\pi}{\sqrt2}}}$$

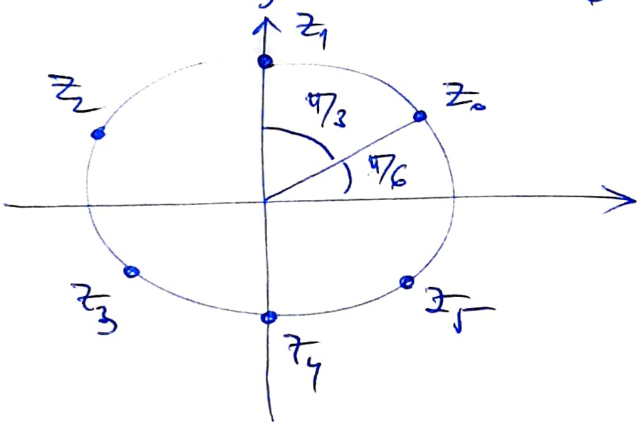
[Kopáček, 376–391]

(Př 1) Spočti $I = \int_0^{+\infty} \frac{dx}{1+x^6}$.

$$I = \frac{1}{2}\int_{-\infty}^{+\infty} R(x)\,dx, \quad \text{kde } R(z) = \frac{1}{1+z^6}$$

$z^6 = -1 = e^{\pi i}$

$z_k = e^{i\alpha_k}$, kde $\alpha_k = \frac{\pi}{6} + k\frac{\pi}{3}$, $k=0,\dots,5$

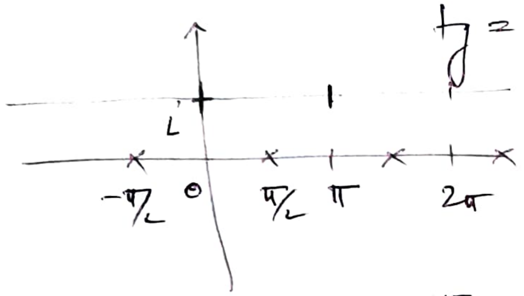


$$\operatorname{res}_{z_k} R = \frac{1}{6z_k^5}\cdot\frac{z_k}{z_k} = -\frac{z_k}{6}$$

Tedy
$$I = \frac{1}{2}\cdot 2\pi i\cdot\left(\operatorname{res}_{z_0} R + \operatorname{res}_{z_1} R + \operatorname{res}_{z_2} R\right)$$
$$= -\frac{\pi i}{6}\underbrace{(z_0 + z_1 + z_2)}_{2i} = \frac{\pi}{3}$$

Pr1 $I=\int_0^{\pi} \operatorname{tg}(x+i)\,dx = \frac{1}{2}\int_0^{2\pi} \operatorname{tg}(x+i)\,dx = (*)$

$\operatorname{tg} = \frac{\sin}{\cos}$ jest π-period.



$\cos z = 0 \iff z_k = \pi/2 + k\pi,\ k \in \mathbb{Z}$

$$\operatorname{tg}(x+i) = \frac{\sin(x+i)}{\cos(x+i)} = \frac{e^{ix-1} - e^{-ix+1}}{(e^{ix-1} + e^{-ix+1})\cdot i} = \qquad z = e^{ix}$$

$$= \frac{(z/e) - (e/z)}{((z/e) + (e/z))\,i} = \frac{z^2 - e^2}{(z^2+e^2)\cdot i}$$

$$(*) = -\frac{1}{2}\oint_{\varphi} \underbrace{\frac{z^2-e^2}{z^2+e^2}\cdot\frac{dz}{z}}_{=:\,f(z)} = -\pi i\cdot \operatorname{res}_0 f = \pi i, \text{ ponieważ}$$

φ jednost. okrąg

$z_0 := 0,\quad z_\pm := \pm ie$

$$\operatorname{res}_0 f = \left.\frac{z^2-e^2}{z^2+e^2}\right|_{z=0} = -1$$