VZOR

$$I = \int_{-\infty}^{\infty} \frac{dx}{1+x^6}$$

(1) prodloužení $\quad f(z) = \dfrac{1}{1+z^6}$

(2) na $\mathbb{R}$ ose nemá f žádné singularity

$z^6 = -1 \iff z = e^{it}$ $\quad 6it = i\pi + 2k\pi i \quad k \in \mathbb{Z}$

tedy $\quad z_k = e^{i\left(\frac{\pi}{6} + \frac{k\pi}{3}\right)} \quad k = 0, \dots, 5$

f jednoduché póly (2. pravidlo)

(3) $\operatorname{res}_{z_k} f = \dfrac{1}{6 z_k^5} = \dfrac{z_k}{6 z_k^6} = \dfrac{z_k}{6\cdot(-1)} = -\dfrac{z_k}{6}$

(4) $\operatorname{Im} z_k > 0 \quad k = 0, 1, 2$

bereme jen 3 z_k

$$-\frac{\pi}{3}(z_0 + z_1 + z_2) = -\frac{\pi i}{3}\left(\frac{\sqrt{3}}{2} + \frac{i}{2} + i - \frac{\sqrt{3}}{2} + \frac{i}{2}\right) = \frac{2\pi}{3} = I_R + J_R$$

pro $R > 1$, kde

$$I_R = \int_{-R}^{R} \frac{dx}{x^6+1} \qquad J_R = \int_0^{\pi} \frac{R i e^{it}}{R^6 e^{6it} + 1}\, dt = \int_{\Gamma_R} \frac{1}{z^6+1}\, dz$$

(5) $\lim\limits_{R\to\infty} I_R = I$

(6) $\lim\limits_{R\to\infty} J_R = 0$

$f(z) = \dfrac{1}{z^6+1} \qquad \alpha = 0$

Jordanovo lemma

(7) $I = \frac{2\pi}{3}$

$$I = \int_0^\infty \frac{x \sin(ax)}{x^2+b^2} \qquad a, b > 0$$

VZOR

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x}{x^2+b^2} \sin \alpha x \qquad R(x) = \frac{x}{x^2+b^2}$$

kořeny $x_1 = ib$, $x_2 = -ib$

komplexní $\rightarrow$

$$= \frac{1}{2} \operatorname{Im} \int_{-\infty}^{\infty} \frac{x}{x^2+b^2} e^{iax}\, dx$$

$$= \frac{1}{2} \operatorname{Im} \left[2\pi i \operatorname{Res}_{ib} \frac{z e^{i\alpha z}}{z^2+b^2} \right] = \frac{1}{2} \operatorname{Im} \left[2\pi i \frac{ib\, e^{iaib}}{2ib} \right]$$

$$= \frac{1}{2} \pi e^{-ab}$$

// J lemma, 1. případ $\alpha > 0$

$$\boxed{\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+5}}$$

$P(x) = 1$

$Q(x) = x^2+2x+5$

kořeny $z_1 = -1+2i \rightarrow$ horní polorovina

$z_2 = -1-2i$ dolní

} póly řádu 1

$z_1, z_2 \notin \mathbb{R}$

tedy $I = 2\pi i \operatorname{res}_{-1+2i} f = 2\pi i \lim_{z \to -1+2i} \frac{z-z_1}{(z-z_2)(z-z_1)} =$

$$= 2\pi i \frac{1}{-1+2i+1+2i} = 2\pi i \frac{1}{4i} = \frac{\pi}{2}$$

$$\boxed{\int_{-\infty}^{\infty} \frac{dx}{x^4+5x^2+4}}$$

$P(x) = 1$

$Q(x) = x^4+5x^2+4 = (x^2+1)(x^2+4)$

kořeny $z_1 = i$

$z_2 = -i$ póly řádu 1

$z_3 = 2i$

$z_4 = -2i$

$$\operatorname{res}_{z_1} f = \lim_{z \to i} \frac{1}{(z^2+4)(z+i)} = \frac{1}{2i(-1+4)} = \frac{1}{6i}$$

$$\operatorname{res}_{z_3} = \lim_{z \to 2i} \frac{1}{(z^2+1)(z+2i)} = -\frac{1}{12i}$$

celkem

$$I = 2\pi i \left(\frac{1}{6i} - \frac{1}{12i} \right) = \frac{\pi}{\cancel{12}\,6}$$

$$\boxed{\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+9)}}$$

$$Q(x) = (x-i)(x+i)(x+3i)(x-3i)$$

žádný kořen není $\mathbb{R}$

$$z_1 = i$$
$$z_2 = 3i$$

$$\operatorname{res}_i = \frac{1}{(i+i)(i+3i)(i-3i)} = \frac{1}{2i\,4i\,(-2i)} = \frac{1}{16i}$$

$$\operatorname{res}_{3i} = \frac{1}{(3i-i)(3i+i)(3i+3i)} = \frac{1}{2i\cdot 4i\cdot 6i} = -\frac{1}{48i}$$

$$I = 2\pi i\left(\frac{1}{16i} - \frac{1}{48i}\right) = \boxed{\frac{2\pi}{24}} = \boxed{\frac{\pi}{12}}$$

$$\boxed{\int_{0}^{\infty} \frac{dx}{(x^2+1)(x^2+9)}} = \frac{1}{2}\, I \text{ výše} = \boxed{\frac{\pi}{24}}$$

$$\boxed{\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx}$$

$$Q(x) = (x-i)(x+i)(x+2i)(x-2i)$$

$$z_1 = i$$
$$z_2 = 2i$$

$$\operatorname{res}_i = \frac{x^2}{(x^2+4)(x+i)}(i) = \frac{i^2}{(-1+4)2i} = \frac{-1}{6i}$$

$$\operatorname{res}_{2i} = \frac{(2i)^2}{((2i)^2+1)(2i+2i)} = \frac{-4}{-3\cdot 4i} = \frac{-4}{-12i}$$

$$I = 2\pi i\left(-\frac{1}{6i} + \frac{4}{12i}\right) = \frac{\pi}{3}$$

$$\boxed{\int_{-\infty}^{\infty} \frac{\cos x}{(x^2-4x+5)^2}\,dx} \qquad \boxed{\int_{-\infty}^{\infty} \frac{\sin x}{x^2-4x+5}}$$

pól 2. řádu $z_1 = 2+i$

$$\operatorname{res}_{z_1} \frac{e^{iz}}{(z^2-4x+5)^2} = \lim_{z \to 2+i} \left(\frac{e^{iz}}{(z-z_2)^2} \right)' = \lim \frac{e^{iz}\left(i\,(z-z_2)-2\right)}{(z-z_2)^3}$$

$$= \frac{e^{i(2+i)}\,(i2i-2)}{(2i)^3} = \frac{e^{-1}\,e^{2i}\,(-4)}{-8i} = \frac{4}{e}\,\frac{\cos 2 + i\sin 2}{8i}$$

$$I = 2\pi i \cdot \frac{4}{e}\,\frac{\cos 2 + i \sin 2}{8i} = \frac{\pi}{e}\,(\cos 2 + i \sin 2).$$

$$\boxed{\begin{aligned} I_1 &= \frac{\pi}{e}\cos 2 \\ I_2 &= \frac{\pi}{e}\sin 2 \end{aligned}}$$

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+2x+2}\,dx \qquad \int_{-\infty}^{\infty} \frac{\sin x}{x^2+2x+2}\,dx$$

$P = 1$ $\qquad f = e^{iz}$

$Q = x^2+2x+2$

$$z_{1,2} = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

jedn. pól

$$\operatorname{res}_{-1+i} f = \lim_{z \to -1+i} \frac{e^{iz}}{z+1+i} = \frac{e^{i(-1+i)}}{2i}$$

Tedy

$$\int \frac{e^{iz}}{z^2+2z+2} = 2\pi i \, \frac{e^{-1-i}}{2i} = \frac{\pi}{e}\left(\underbrace{\cos 1}_{Re} - \underbrace{i \sin 1}_{Im}\right)$$

$I_1 = \frac{\pi}{e} \cos 1$

$I_2 = -\frac{\pi}{e} \sin 1$

$$\boxed{\int_{-\infty}^{\infty} \frac{\cos x}{x^2-2x+5}\, dx}$$

$$f(x) = \frac{1}{x^2-2x+5} = \frac{1}{(x-1-2i)(x-1+2i)}$$

$$x_{1,2} = \frac{2 \pm \sqrt{4-4\cdot 5}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$z_1 = 1+2i$ kořen v horní polorovině
pól 1. řádu

$$\operatorname{Res}_{z_1} \frac{e^{iz}}{z^2-2z+5} = \lim_{z \to z_1} \frac{e^{iz}}{z-1+2i} = \frac{e^{i(1+2i)}}{4i}$$

Pak $\displaystyle\int_{-\infty}^{\infty} \frac{e^{iz}}{z^2-2z+5}\, dz = 2\pi i \frac{e^{i(1+2i)}}{4i} = \frac{\pi e^{-2+i}}{2} =$

$$= \frac{\pi}{2e^2}(\cos 1 + i \sin 1) =: *$$

$$I = \operatorname{Re} * = \underline{\underline{\frac{\pi}{2e^2}\cos 1}}$$

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2-2x+5}\, dx$$

Zounaváld příkladu nad tím

$$I = \operatorname{Im} * = \frac{\pi}{2e^2} \sin 1$$

$$\boxed{\int_0^\infty \frac{\cos x}{x^4+4}\,dx}$$

$x^4 = -4$ $\qquad 4\cdot(-1) = 4e^{i\pi}$

$z^4 = |z|^4 e^{4it} = 4e^{i\pi}$

$Q(x) = x^4+4$

$(x^2-2i)(x+2i)$

$(x-\sqrt{2}\sqrt{i})(x+\sqrt{2}\sqrt{i})$

$(x+i\sqrt{2i})(x-i\sqrt{2i})$

$z_1 = 1+i$

$z_2 = -1+i$

$z_3 = -1-i$

$z_4 = 1-i$

$$\operatorname{res}_{z_k}\left(\frac{e^{iz}}{z^4+4}\right) = \lim_{z\to z_k} \frac{(z-z_k)e^{iz}}{z^4+4} \overset{L'H}{=} \lim \frac{e^{iz}}{4z^3} =$$

$$= \frac{e^{iz_k}}{4z_k^3} = \frac{z_k e^{iz_k}}{z_k^4} = \frac{z_k e^{iz}}{-16} \qquad k = 1,\dots,4$$

Pak

$$\int \frac{e^{iz}}{z^4+4} = 2\pi i\left(\frac{(1+i)e^{i(1+i)}}{-16} + \frac{(-1+i)e^{i(-1+i)}}{-16}\right) =$$

$$= -\frac{\pi i}{8}\left((1+i)e^{-1+i} + (-1+i)e^{-1-i}\right) =$$

$$= \frac{\pi i}{4e}\left(\frac{e^i-e^{-i}}{2} + \frac{i(e^i+e^{-i})}{2}\right) = -\frac{\pi i}{4e}\left(\frac{i(e^i-e^{-i})}{2i} + \frac{i(e^i+e^{-i})}{2}\right)$$

$$= \frac{\pi}{4e}(\sin 1 + \cos 1) \quad (*)$$

tedy

$$I = \frac{1}{2}(\operatorname{Re}(*)) = \frac{\pi}{8e}(\sin 1 + \cos 1)$$