$$\frac{1}{z(z-1)} \qquad z = 0$$

$$\frac{1}{z(z-1)} = \frac{1}{z} + \frac{1}{z-1}$$

(a) $|z| < 1$

$$\frac{1}{z-1} = \sum_{n=0}^{\infty} z^n$$

celkem $\sum_{n=-1}^{\infty} z^n$ $\qquad$ $a_{-1} = 1$

(b) $\qquad z = 1$

$|z-1| < 1$

$$\frac{1}{z} = \frac{1}{1-(1-z)} = \sum_{n=0}^{\infty} (1-z)^n = \sum_{n=0}^{\infty} (-1)^n (z-1)^n$$

tedy pro $0 < |z-1| < 1$

$$\frac{1}{z(1-z)} = \sum_{n=-1}^{\infty} (-1)^n (z-1)^n \qquad a_{-1} = 1$$

(c) $\qquad z = \infty$

pro $|z| > 1$

$$\frac{1}{1-z} = -\frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}} = -\frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n = -\sum_{n=1}^{\infty} \frac{1}{z^n}$$

celkem

$$\frac{1}{z(1-z)} = -\sum_{n=2}^{\infty} \frac{1}{z^n} \qquad -a_{-1} = 0$$

$$\boxed{f(z) = \frac{1}{(z+1)(z+2)}} \qquad z_0 = -2$$

$$\boxed{f(z) = \frac{1}{z+1} - \frac{1}{z+2}}$$

(a) $|z+2| < 1$

$$\frac{1}{z+1} = \frac{-1}{1-(z+2)} = -\sum_{n=0}^{\infty} (z+2)^n$$

$$\frac{1}{z+2} = \frac{1}{z+2}$$

celkem

$$-\sum_{k=-1}^{\infty} (z+2)^k$$

(b) $|z+2| > 1$

$$\frac{1}{z+1} = \frac{1}{z+2-1} = \frac{\frac{1}{z+2}}{1-\frac{1}{z+2}} =$$

$$= \frac{1}{z+2} \sum_{n=0}^{\infty} \frac{1}{(z+2)^n}$$

celkem $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(z+2)^n}$

$$\boxed{f(z) = \frac{1}{(z+1)(z+2)}} \qquad z_0 = -1$$

$$f(z) = \frac{1}{z+1} - \frac{1}{z+2}$$

(a) $0 < |z+1| < 1$

$$\frac{1}{z+2} = \frac{1}{1+\underbrace{z+1}_{z-(-1)}} = \frac{1}{1-(-(z+1))} = \sum_{n=0}^{\infty} (-1)^n (z+1)^n =$$

$$\frac{1}{z+1} = (z-(-1))^{-1}$$

dohromady $\quad f(z) = \sum_{k=-1}^{\infty} (-1)^{k+1} (z+1)^k$

(b) $|z+1| > 1$

$$\frac{1}{z+2} = \frac{1}{z+1+1} = \frac{\frac{1}{z+1}}{1+\frac{1}{z+1}} = \frac{\frac{1}{z+1}}{1-\left(\frac{-1}{z+1}\right)} =$$

$$= \frac{1}{z+1} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(z+1)^n} = \sum_{n=-1}^{-\infty} (-1)^{n+1} (z+1)^n.$$

dohromady

$$f(z) = \sum_{n=2}^{\infty} \frac{(-1)^n}{(z+1)^n}$$

$$\boxed{f(z) = \frac{1}{(z+1)(z+2)}} \qquad \underline{z_0 = 1}$$

nutno tam dostat výraz $(z-1)$

$$f(z) = \frac{1}{z+1} - \frac{1}{z+2}$$

(a) $$\frac{1}{z+1} = \frac{1}{2+z-1} = \frac{1}{2}\,\frac{1}{1-\left(-\frac{z-1}{2}\right)} =$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{-1}{2}\right)^n (z-1)^n \qquad \text{pro } \left|\frac{z-1}{2}\right| < 1$$

$$-\frac{1}{z+2} = -\frac{1}{3+z-1} = -\frac{1}{3}\,\frac{1}{1-\left(-\frac{z-1}{3}\right)} =$$

$$= -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{-1}{3}\right)^n (z-1)^n \qquad \text{pro } \left|\frac{z-1}{3}\right| < 1$$

celkem $$\underline{\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2^{n+1}} - \frac{1}{3^{n+1}}\right)(z-1)^n}, \quad \text{na } \underline{|z-1| < 2}$$

(b) $$\frac{1}{z+1} = \frac{\frac{1}{z-1}}{1-\frac{-2}{z-1}} = \frac{1}{z-1}\sum_{n=0}^{\infty}(-2)^n\frac{1}{(z-1)^n} =$$

$$= \sum \frac{-2^{n-1}}{(z-1)^n} \qquad u = \frac{-2}{z-1}$$

na $|u| < 1 \Rightarrow |z-1| > 2$

celkem
$$\underline{\sum_{n=-1}^{-\infty}(-2)^{-(n+1)}(z-1)^n + \sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}(z-1)^n}$$

na $\underline{2 < |z-1| < 3}$

(c) $$\frac{1}{z+2} = \frac{1}{z-1+3} = \frac{\frac{1}{z-1}}{1-\frac{-3}{z-1}} = \frac{1}{z-1}\sum_{n=0}^{\infty}(-3)^n\frac{1}{(z-1)^n} \qquad \text{na } |z-1| > 3$$

celkem $$\underline{\sum_{n=-1}^{-\infty}(-1)^{n+1}\left[2^{-(n+1)} - 3^{-(n+1)}\right](z-1)^n} \quad \text{na } \underline{|z-1| > 3}$$

$$\boxed{f(z) = \frac{1}{(z-2)(z-3)}} \qquad z_0 = 0$$

Rozklad na parc. zlomky

$$\frac{1}{z-3} - \frac{1}{z-2}$$

$$\frac{1}{z-3} = -\frac{1}{3}\,\frac{1}{1-\frac{z}{3}} = -\sum_{n=0}^{\infty} \frac{z^n}{3^{n+1}} \qquad |z| < 3$$

$$\frac{1}{z-2} = \frac{1}{z}\,\frac{1}{1-\frac{2}{z}} = \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} = \sum_{n=-\infty}^{-1} 2^{-n-1} z^n \qquad |z| > 2$$

Proč takhle

$\frac{1}{z-3}$ je holomorfní na $|z|<3$

a na $|z| > 3$

$\frac{1}{z-2}$ $|z|<2$

a na $|z| > 2$

mezikruží: $3 > |z| > 2$

Dohromady $\quad -\sum_{n=-\infty}^{-1} 2^{-n-1} z^n - \sum_{n=0}^{\infty} \frac{z^n}{3^{n+1}}$

Rozvinte v ∞

$$\boxed{f(z) = z^2 e^{\frac{1}{z}}}$$

$$z^2 e^{\frac{1}{z}} = z^2 \sum_{n=0}^{\infty} \frac{1}{n!\, z^n} = z^2 + z + \sum_{n=2}^{\infty} \frac{1}{n!\, z^{n-2}}$$

$$= z^2 + z + \sum_{n=0}^{\infty} \frac{1}{(n+2)!} \frac{1}{z^n}$$

$$\boxed{\frac{1}{z - z^3}}$$

$$\frac{1}{z - z^3} = \frac{1}{z(1-z)(1+z)}$$

$$\operatorname{res}_0 f(z) = \lim_{z \to 0} z \frac{1}{z(1-z^2)} = \underline{\underline{1}}$$

$$\operatorname{res}_1 = \lim_{z \to 1} (z-1) \frac{1}{z(1-z)(1+z)} = \lim \frac{-1}{z(1+z)} = \underline{\underline{-\frac{1}{2}}}$$

$$\operatorname{res}_{-1} = \lim_{z \to -1} (z+1) \frac{1}{z(1-z)(1+z)} = \lim \frac{1}{z(1-z)} = \underline{\underline{-\frac{1}{2}}}$$

$$\boxed{\frac{1}{z^5-4z^3}}$$

$$z^5-4z^3 = z^3(z-2)(z+2)$$

$$\operatorname{res}_2 = \lim_{z\to 2}\left[(z-2)\frac{1}{z^3(z-2)(z+2)}\right] = \underline{\underline{\frac{1}{32}}}$$

$$\operatorname{res}_{-2} = \lim_{z\to -2}\left[(z+2)\frac{1}{z^3(z-2)(z+2)}\right] = \underline{\underline{\frac{1}{32}}}$$

$$\operatorname{res}_0 = \lim_{z\to 0}\frac{1}{(3-1)!}\left[z^3\frac{1}{z^3(z-2)(z+2)}\right]^{(3-1)} =$$

$$= \lim \frac{1}{2}\left(\frac{1}{z^2-4}\right)^{(2)} =$$

$$= \lim \frac{1}{2}\left(\frac{-2z}{(z^2-4)^2}\right)' =$$

$$= \lim \frac{1}{2}\frac{-2(z^2-4)^2+8z^2(z^2-4)}{(z^2-4)^4} =$$

$$= -\underline{\underline{\frac{1}{16}}}$$

$$\boxed{\frac{\sin 2z}{(z+1)^3}}$$

$$z_1 = -1$$

$$\operatorname{res}_{-1} = \lim_{z\to -1} \frac{1}{(3-1)!} \left[(z+1)^3 \frac{\sin 2z}{(z+1)^3} \right]^{(3-1)} =$$

$$= \frac{1}{2} \lim \left(2\cos 2z \right)' = \frac{1}{2} \lim_{z\to -1} 2\cdot\left(-(\sin 2z)\right)\cdot 2 =$$

$$= 2 \; -(\sin(-2)) = -2 \sin(-2)$$

$$\boxed{\frac{z+1}{z^2+2z+2}}$$

$z_1 = -1+i$

$z_2 = -1-i$ 1. Fádu

$$\operatorname{res}_{-1+i} f = \lim_{z\to -1+i} \frac{z+1}{z+1+i} = \frac{\cancel{-1}+i+\cancel{1}}{\cancel{-1}+i+\cancel{1}+i} = \frac{i}{2i} = \underline{\underline{\frac{1}{2}}}$$

$$\operatorname{res}_{-1-i} f = \lim_{z\to -1-i} \frac{z+1}{z+1-i} = \frac{\cancel{-1}-i+\cancel{1}}{\cancel{-1}-i+\cancel{1}-i} = \frac{-i}{-2i} = \underline{\underline{\frac{1}{2}}}$$

$$\underline{\underline{\operatorname{res}_\infty = -1}}$$

$$\boxed{\frac{e^z}{z^2(z^2+9)}} = \frac{e^z}{z^2(z+3i)(z-3i)}$$

$e^z \to$ holom.

(a) res v bodě 0 – 2. nás. pól

$$\underline{\underline{\operatorname{res}_0 f}} = \lim_{z\to 0} \frac{1}{(2-1)!}\left[(z-0)^2 f(z)\right]^{(2-1)}$$

dco

$$= \lim_{z\to 0} \left(\frac{e^z}{(z^2+9)}\right)' = \lim_{z\to 0} \frac{e^z(z^2+9) - e^z 2z}{(z^2+9)^2}$$

$$= \frac{1\cdot 9 - 0}{9^2} = \underline{\underline{\frac{1}{9}}}$$

(b) bod $-3i$, $3i$ $\to$ 1 duchý pól

$$\underline{\underline{\operatorname{res}_{3i} f}} = \lim_{z\to 3i} \frac{1}{(1-1)!}\left[(z-3i)^1 \frac{e^z}{z^2(z+3i)(z-3i)}\right]^{(1-1)} =$$

$$= \lim_{z\to 3i} \frac{e^z}{z^2(z+3i)} = \frac{1}{-1\cdot 9\cdot 6i}\cdot e^{3i} =$$

$$= \frac{1}{54} i\,(\cos 3 + i\sin 3) = \underline{\frac{1}{54}(-\sin 3 + i\cos 3)}$$

(c) bod $3i$ 1 duchý pól

$$\operatorname{res}_{-3i} f = \lim_{z\to -3i} \frac{1}{(1-1)!}\left[(z+3i)^1 \frac{e^z}{z^2(z+3i)(z-3i)}\right]^{(1-1)} =$$

$$= \lim_{z\to -3i} \frac{e^z}{z^2(z-3i)} = \frac{e^{-3i}}{-9(-6i)} = -\frac{1}{54} i\,(\cos -3 + i\sin(-3))$$

$$= \underline{-\frac{1}{54}(\sin 3 + i\cos 3)}$$

(d) $r \to \infty$

$$\sum_{a \in A \cup \{\infty\}} \operatorname{res}_a f = 0$$

kdy $\operatorname{res}_0 + \operatorname{res}_{3i} + \operatorname{res}_{-3i} + \operatorname{res}_{\infty} = 0$

$$\Rightarrow \quad \operatorname{res}_{\infty} f = -\left(\frac{1}{9} + \frac{1}{54}\left(-\sin 3 + i\cos 3\right) - \frac{1}{54}\left(\sin 3 + i\cos 3\right)\right) =$$

$$= \underline{-\frac{1}{9} + \frac{1}{27}\sin 3}$$