(2)

Funkce: $G \subset \mathbb{C}$, $f: \mathbb{C} \to \mathbb{C}$

Def: · Je-li f definovaná na okolí $z_0 \in \mathbb{C}$, pak kompl. deri v z_0 nazveme kompl. č.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$ pokud lim vpravo $\exists$,

· Fce je komplexně diferencovatelná v $z_0 \in \mathbb{C}$
$\iff \exists$ $\mathbb{C}$ lin. zobr $L: \mathbb{C} \to \mathbb{C}$: $f(z_0 + h) = f(z_0) + Lh + \dots$

Takovéto Lh je rovno $Lh = f'(z_0) \cdot h$ $h \in \mathbb{C}$

Značení
$df(z_0) = L$ tot. diferenciál

$$\underset{\substack{h \in \mathbb{R}^2 \\ \mathbb{C}}}{df(z_0)h} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}(z_0) \cdot h$$

Věta (CR)

f je def na okolí z_0 a $f(z) = f_1 + i f_2$, f_1, f_2 $\mathbb{R}$ fce
Pak f je $\mathbb{C}$-diferencovatelná v z_0 $\iff$ $f = (f_1, f_2)$ je $\mathbb{R}$ diferencovatelná v z_0 a splňuje (CR) podmínky

$$\frac{\partial f_1}{\partial x}(z_0) = \frac{\partial f_2}{\partial y}(z_0) \qquad \frac{\partial f_2}{\partial x} = -\frac{\partial f_1}{\partial y}$$

Navíc (CR) platí $\iff$ $\frac{\partial f}{\partial x}(z_0) = -i \frac{\partial f}{\partial y}(z_0)$.

Existuje-li $f'(z_0)$, pak $df(z_0)h = f'(z_0)h$

$$f'(z_0) = \frac{\partial f}{\partial x} = -\frac{\partial f}{\partial y}$$

Limita

(1) $f(x,y) = \frac{x-y}{x+y}$

$$\lim_{x \to 0} \left(\lim_{y \to 0} f(x,y) \right) = 1$$

$$\lim_{y \to 0} \left(\lim_{x \to 0} f(x,y) \right) = -1$$

ale $\lim_{x,y \to 0} f(x,y)$ neexistuje

(2) $\frac{x^2y^2}{x^2y^2+(x-y)^2}$

$$\lim_{x \to 0} \lim_{y \to 0} (\quad) = \lim_{x \to 0} \frac{0}{x^2} = \underline{\underline{0}}$$

$$\lim_{y \to 0} \lim_{x \to 0} (\quad) = \lim_{y \to 0} \frac{0}{(-y)^2} = \underline{\underline{0}}$$

ale $\lim_{(x,y) \to (0,0)} \nexists$

zz. $x = y$

$$\lim \frac{x^2x^2}{x^2x^2+(x-x)^2} = \underline{\underline{1}}$$

Exp

$\exp(z) = e^x(\cos y - i \sin y)$

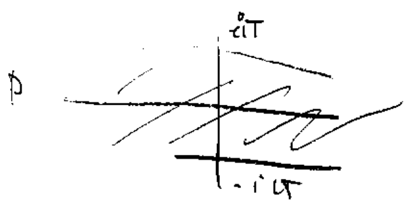
$(e^z)' = e^z$

exp není prostá, je 2π periodická

$e^z = e^w \iff \exists k \in \mathbb{Z} \; w = z + 2\pi k i$

$\exp|_P$ je prostá

$\exp(P) = \mathbb{C} \setminus \{0\}$



Mocniny

$\alpha \in \mathbb{C}, \quad z \in \mathbb{C} \setminus \{0\}$

$z^\alpha = \exp^\alpha \ln z$ → hlavní hodnota α-té mocniny z

$M_\alpha(z) = \{\exp(\alpha \cdot w),\; w \in \operatorname{Log} z\}$ → α-tá mocnina z

z^α holom. na $\mathbb{C} \setminus (-\infty, 0]$

$(z^\alpha)' = \alpha \cdot z^{\alpha - 1}$ na $\mathbb{C} \setminus (-\infty, 0]$

- $M_\alpha(z) = \{z^\alpha e^{2\alpha k \pi i}, k \in \mathbb{Z}\}$
- pro $\alpha \in \mathbb{Z}$: $M_\alpha(z) = \{z^\alpha\}$ a z^α je holom na $\mathbb{C} \setminus \{0\}$
- $\alpha \in \mathbb{Q} \Rightarrow \alpha = \frac{p}{q}$ $p \in \mathbb{Z}, q \in \mathbb{N}$, nesoudělné

pak $M_{\frac{p}{q}}(z) = \{z^{\frac{p}{q}} e^{2\frac{p}{q} k \pi i}, k = 0, \dots, q-1\}$

tvoří vrcholy prav. q-úhelníku se středem v $(0,0)$

Logaritmus

$\log = (\exp|_P)^{-1}$ hl. hodnota logaritmu

$z \in \mathbb{C} \setminus \{0\}$

$\mathrm{Log}\, z = \{ w \in \mathbb{C},\ \exp w = z \}$

$\log z = \ln |z| + i \arg z$

$\mathrm{Log}\, z = \{ \log z + 2k\pi i,\ k \in \mathbb{Z} \}$

log není spojitý někde na $(-\infty, 0]$

a $\log' z = \dfrac{1}{z}$

Gon. fce

$\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ na $\mathbb{R}$ splývají

$\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

$\sin' = \cos \qquad \cos' = -\sin$

2π periodické

$\sin(\mathbb{C}) = \cos(\mathbb{C}) = \mathbb{C}$

$\rightarrow$ <u>není</u> <u>omezené</u>

Mocniny v $\mathbb{C}$

Def Je-li $\alpha \in \mathbb{C}$ a $z \in \mathbb{C} \setminus \{0\}$, potom definujeme

$$\boxed{z^\alpha := \exp(\alpha \ln z)}$$

jako hlavní hodnotu α-té mocniny z a

$$M_\alpha(z) := \{ \exp(\alpha w),\ w \in \operatorname{Log} z \}$$

jako α-tou mocninu z

Vlastnosti

(1) z^α je holomorfní fce na $\mathbb{C} \setminus (-\infty, 0]$ a

$$(z^\alpha)' = \alpha z^{\alpha - 1} \quad \text{na } \mathbb{C} \setminus (-\infty, 0]$$

(2) $M_\alpha(z) = \{ z^\alpha e^{2\alpha k \pi i},\ k \in \mathbb{Z} \}$

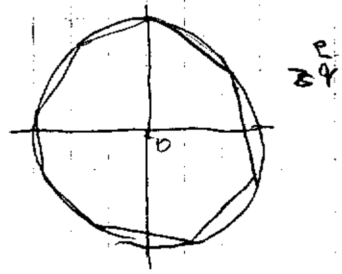
protože $\operatorname{Log} z = \{ \log z + 2k\pi i,\ k \in \mathbb{Z} \}$

(3) Je-li $\alpha \in \mathbb{Z}$, potom $M_\alpha(z) = \{ z^\alpha \}$ a z^α je holomorfní na $\mathbb{C} \setminus \{0\}$

(4) Je-li $\alpha \in \mathbb{Q}$ a $\alpha = \frac{p}{q}$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$, p, q jsou nesoudělná,

pak $M_{\frac{p}{q}}(z) = \{ z^{\frac{p}{q}} e^{2\frac{p}{q} k \pi i},\ k = 0, \dots, q-1 \}$

tvoří vrcholy ~~pravidelného~~ q-úhelníka se středem v $[0,0]$

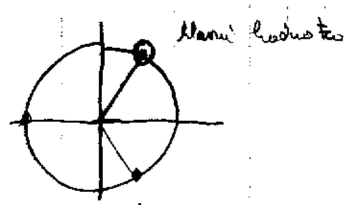


Př (1) $\sqrt{-1} = \exp(\frac{1}{2}\pi i) = i$

$M_{\frac{1}{2}}(-1) = \{ \pm i \}$

(2) $\sqrt[3]{-1} = \exp(\frac{1}{3}\pi i)$

$M_{1/3}(-1) = \{ -1, e^{\pm \frac{\pi}{3} i} \}$



! Pozor: $0 = \log(\underbrace{(-1)(-1)}_{=1}) \neq 2 \cdot \log(-1) = 2\pi i$