Mathematics I - Sequences

23/24

Why study sequences

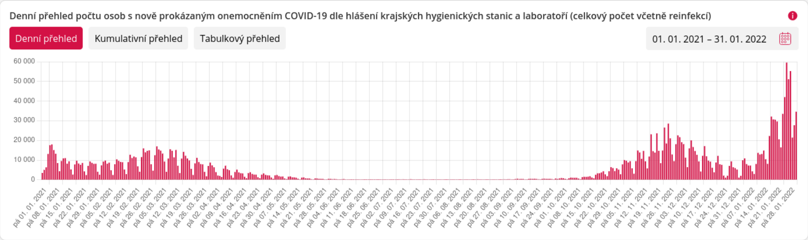

Figure: https://onemocneni-aktualne.mzcr.cz/covid-19

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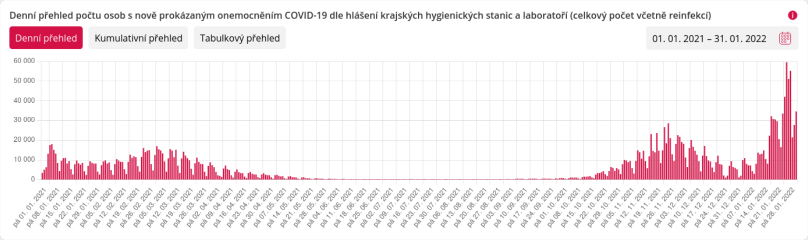

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https://www.gapminder.org/tools

Why study sequences

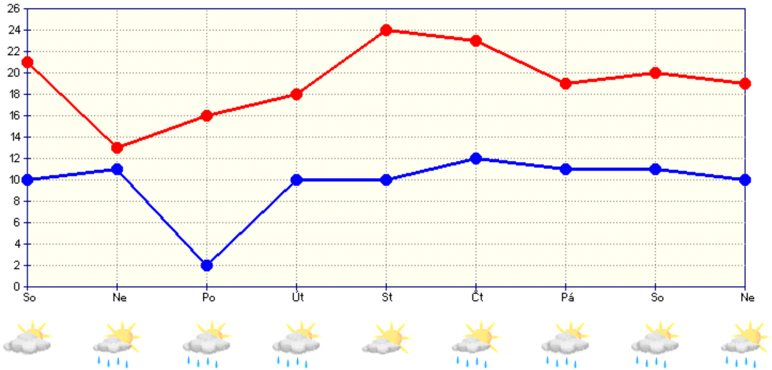

Figure: https://www.chmi.cz/predpovedi/predpovedi-pocasi/ceska-republika/tydenni-predpoved

II. Limit of a sequence

Definition

Suppose that to each natural number $n \in \mathbb{N}$ we assign a real number a_n. Then we say that $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers.

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A sequence $\{a_n\}_{n=1}^{\infty}$ is equal to a sequence $\{b_n\}_{n=1}^{\infty}$ if $a_n = b_n$ holds for every $n \in \mathbb{N}$.

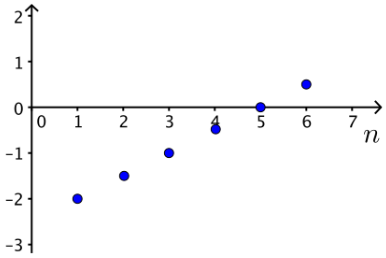
By the set of all members of the sequence $\{a_n\}_{n=1}^{\infty}$ we understand the set

$$\{x \in \mathbb{R};\ \exists n \in \mathbb{N}\colon a_n = x\}.$$

https://www.geogebra.org/calculator/q7vv3gjp

Exercise

Find the formula for a_n.



A. $a_n = (-\frac{1}{2})^n - \frac{3}{2}$

B. $a_n = \frac{1}{2}n + 5$

C. $a_n = \frac{1}{2}n - 2$

D. $a_n = -\frac{1}{2}n + \frac{5}{2}$

E. $a_n = \frac{1}{2}n - \frac{5}{2}$

Figure:
https://www.cpp.edu/conceptests/question-library/mat116.shtml

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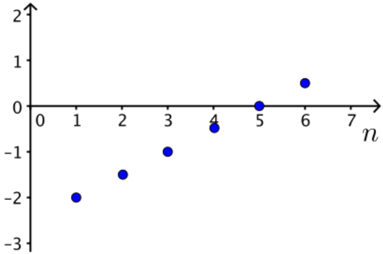

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E

Exercise

Find the first 4 terms of the sequences

A $a_n = \dfrac{(-1)^n}{n}$

B $a_n = \dfrac{n+1}{n}$

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$-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}$

$2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}$

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Exercise

Find the formula for the following sequences

A $1, \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \dfrac{1}{16}, \ldots$

B $-1, \dfrac{1}{2}, \dfrac{-1}{3}, \dfrac{1}{4}, \dfrac{-1}{5} \ldots$

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$\frac{1}{2^{n-1}}$

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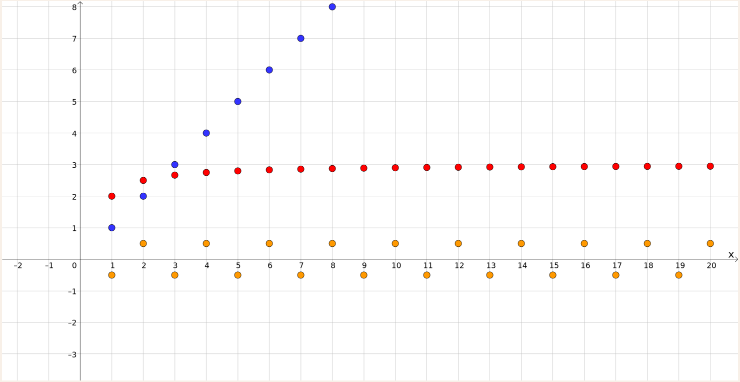
Exercise

Which of these sequences are bounded?



A blue

B red

C yellow

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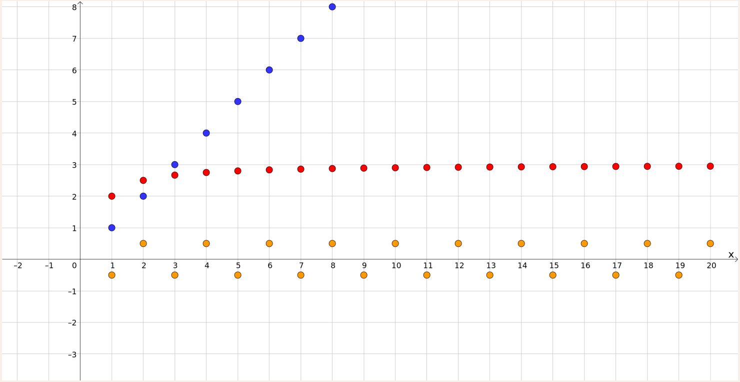

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B, C

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We say that a sequence $\{a_n\}$ is

- increasing if $a_n < a_{n+1}$ for every $n \in \mathbb{N}$,
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A sequence $\{a_n\}$ is monotone if it satisfies one of the conditions above.

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Exercise

Find non-decreasing sequences.

A $a_n = -4$

B $a_n = (-2)^n$

C $a_n = \dfrac{(-1)^n}{3^n}$

D $a_n = \log n$

E $a_n = e^{-n}$

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A, D

Exercise

Use the definition and check, if the sequence is monotone:

1. $a_n = \dfrac{n}{n+1}$
2. $a_n = \dfrac{n}{4+n^2}$

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$$\begin{aligned} ?\ a_n &\leq a_{n+1} \\ \frac{n}{n+1} &\leq \frac{n+1}{n+2} \\ n(n+2) &\leq (n+1)(n+1) \\ n^2+2n &\leq n^2+2n+1 \\ 0 &\leq 1 \end{aligned}$$

https://www.geogebra.org/calculator/w4twpbu2

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$$
\begin{aligned}
?\ a_n &\geq a_{n+1} \\
\frac{n}{4+n^2} &\geq \frac{n+1}{4+(n+1)^2} \\
n(4+n^2+2n+1) &\geq (n+1)(4+n^2) \\
4n+n^3+2n^2+n &\geq 4n+n^3+4+n^2 \\
n^2+n &\geq 4
\end{aligned}
$$

true for $n \geq 2$.

https://www.geogebra.org/calculator/w4twpbu2

Definition

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers.

- By the sum of sequences $\{a_n\}$ and $\{b_n\}$ we understand a sequence $\{a_n + b_n\}$.
- Analogously we define a difference and a product of sequences.
- Suppose all the members of the sequence $\{b_n\}$ are non-zero. Then by the quotient of sequences $\{a_n\}$ and $\{b_n\}$ we understand a sequence $\{\frac{a_n}{b_n}\}$.
- If $\lambda \in \mathbb{R}$, then by the λ-multiple of the sequence $\{a_n\}$ we understand a sequence $\{\lambda a_n\}$.

Exercise

Let $a_n = 1, 2, 3, 4, 5, \ldots$, $b_n = (-1)^n$. Find

A $a_n + b_n$

B a_n / b_n

C $3a_n$

Exercise

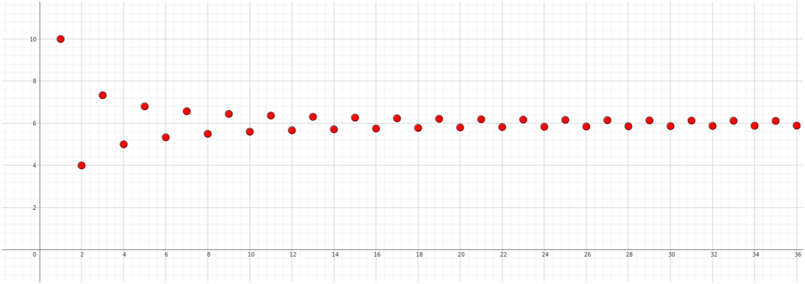
Let $a_n = 1, 2, 3, 4, 5, \ldots$, $b_n = (-1)^n$. Find

A $a_n + b_n$ B a_n/b_n C $3a_n$

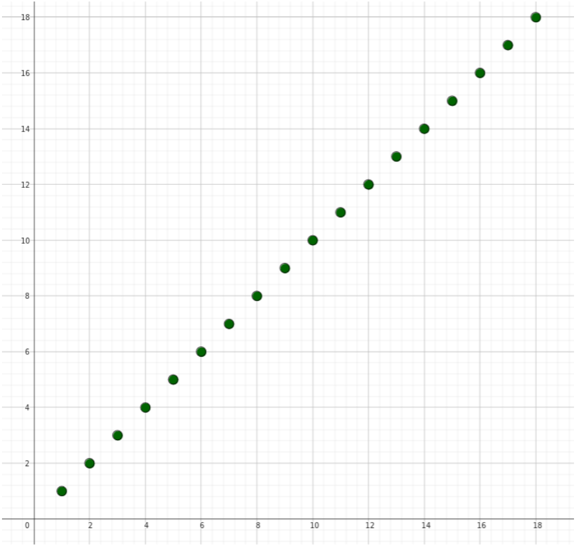
$a_n = 1, 2, 3, 4, 5 \ldots$
$b_n = -1, 1, -1, 1, -1 \ldots$

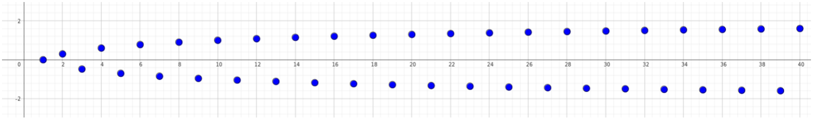
A: $0, 3, 2, 5, 4 \ldots$
B: $-1, 2, -3, 4, -5 \ldots$
C: $3, 6, 9, 12, 15 \ldots$

Limits



Limits




2
-2
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
32
34
36
38
40

Limits

Definition

We say that a sequence $\{a_n\}$ has a limit which equals to a number $A \in \mathbb{R}$ if to every positive real number ε there exists a natural number n_0 such that for every index $n \geq n_0$ we have $|a_n - A| < \varepsilon$, i.e.

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \; \exists n_0 \in \mathbb{N} \; \forall n \in \mathbb{N}, n \geq n_0 : \; |a_n - A| < \varepsilon.$$

Definition

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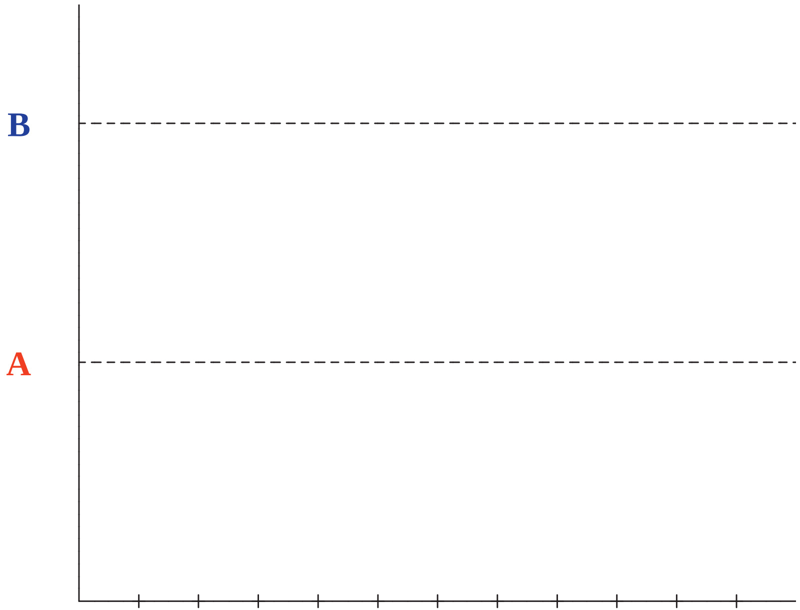
We say that a sequence $\{a_n\}$ is convergent if there exists $A \in \mathbb{R}$ which is a limit of $\{a_n\}$.

https://www.geogebra.org/m/GAcTpGCh

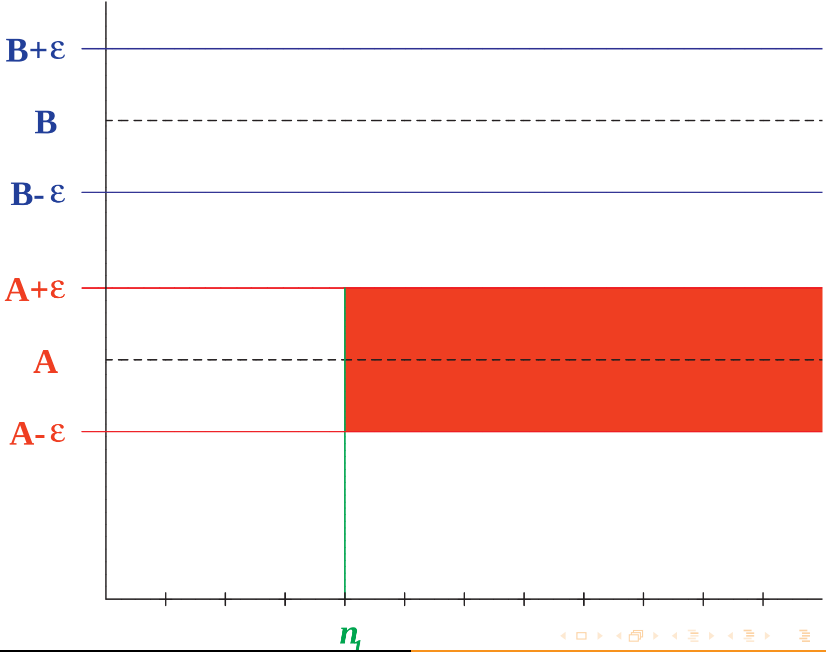
Theorem 1 (uniqueness of a limit)

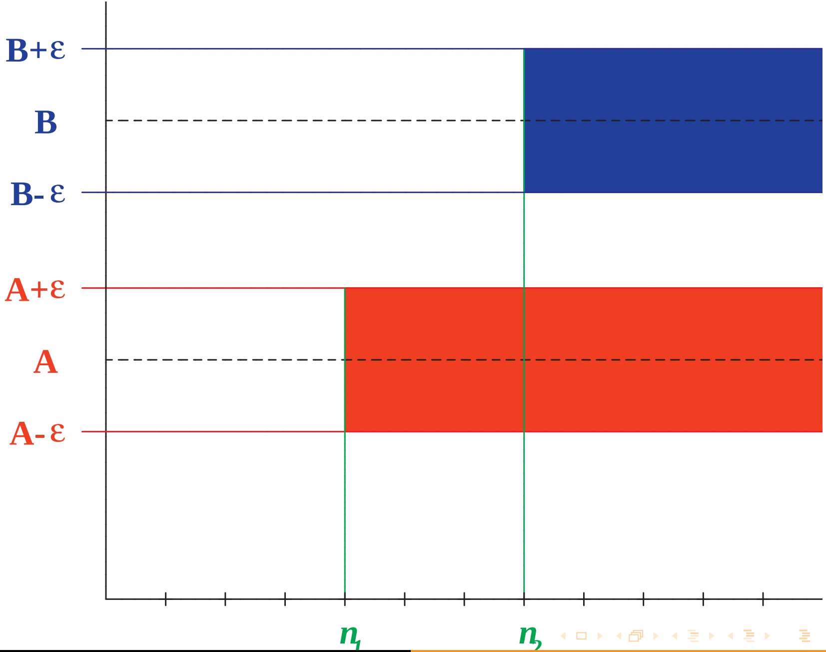
Every sequence has at most one limit.

We use the notation $\lim_{n\to\infty} a_n = A$ or simply $\lim a_n = A$.


B
A

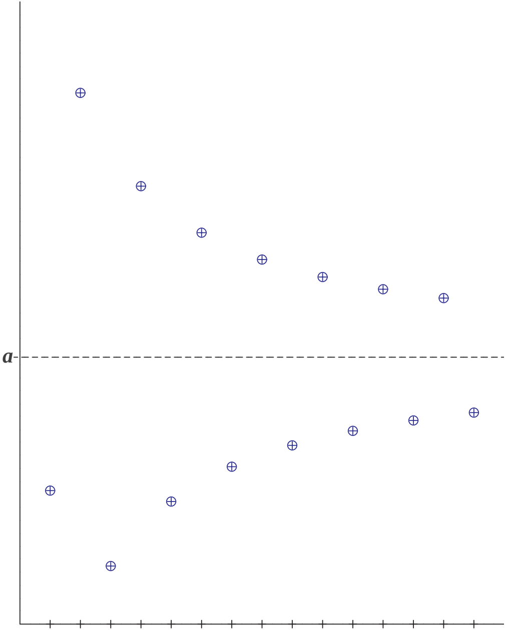

B+ε
B
B- ε
A+ε
A
A- ε

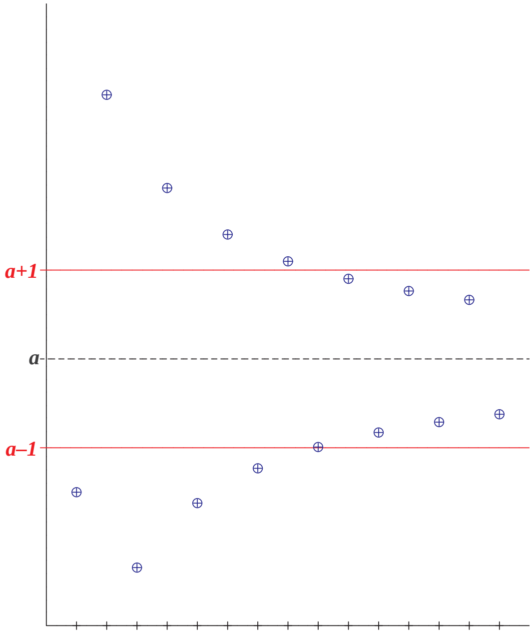

B+ε
B
B- ε
A+ε
A
A- ε
n_1

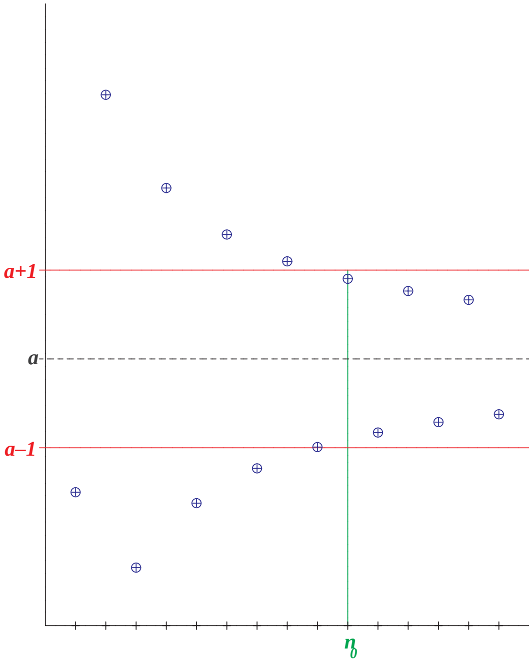

B+ε
B
B- ε
A+ε
A
A- ε
n_1
n_2

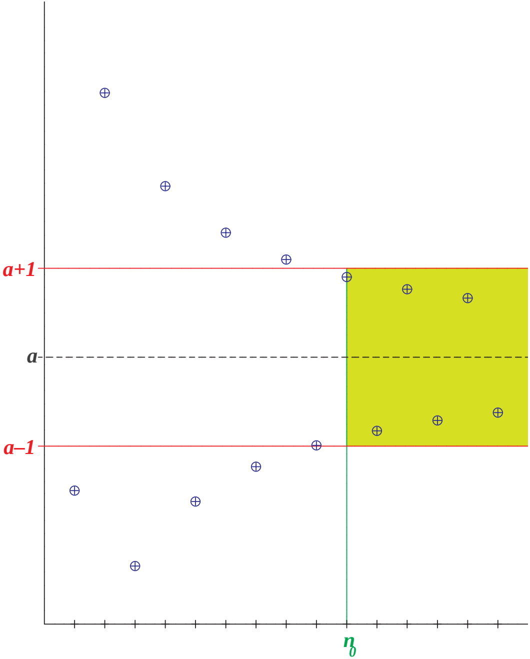
Theorem 2

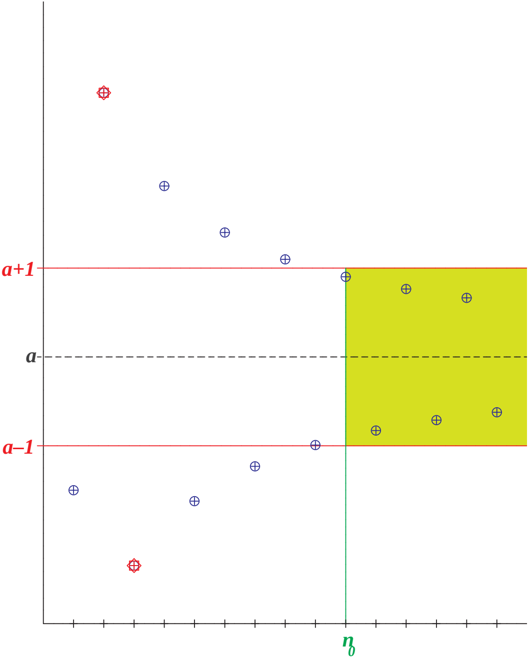
Every convergent sequence is bounded.

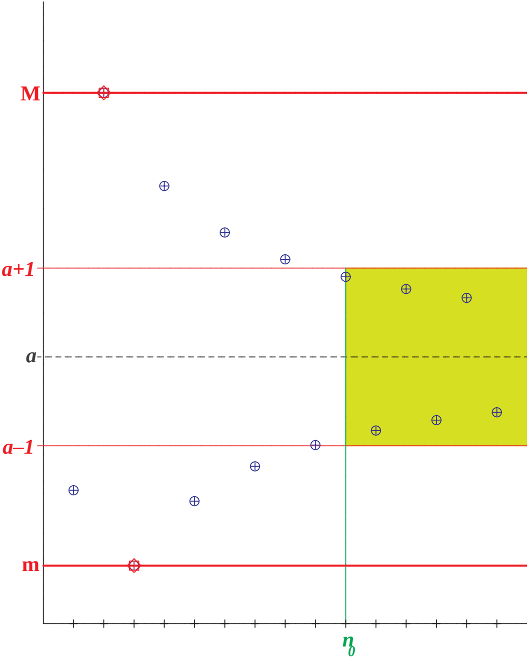

a


a+1
a
a–1


a+1
a
a–1
n
0


a+1
a
a–1
n_0


a+1
a
a–1
n0


M
a+1
a
a–1
m
n
0

Exercise

Find a sequence, which is

1. bounded and covergent
2. bounded and divergent
3. unbounded and covergent
4. unbounded and divergent

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1. $\frac{1}{n}$, $a_n = 42$
2. $a_n = (-1)^n$, $a_n = \sin n$
3. impossible
4. $a_n = n$, $a_n = (-1)^n n^2$

Definition

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. We say that a sequence $\{b_k\}_{k=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$ if there is an increasing sequence $\{n_k\}_{k=1}^{\infty}$ of natural numbers such that $b_k = a_{n_k}$ for every $k \in \mathbb{N}$.

https://www.geogebra.org/calculator/z4bnwfzr

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https://www.geogebra.org/calculator/z4bnwfzr

Exercise

Let $a_n = 3, 7, 4, 1/2, \pi, -1$. Find $b_n = a_{2n}$:

A $6, 14, 8 \ldots$

B $5, 9, 6 \ldots$

C $7, 1/2, -1 \ldots$

D $4, 1/2, \pi \ldots$

By: https://www.cpp.edu/conceptests/question-library/mat116.shtm

Theorem 3 (limit of a subsequence)

Let $\{b_k\}_{k=1}^{\infty}$ *be a subsequence of* $\{a_n\}_{n=1}^{\infty}$. *If* $\lim_{n\to\infty} a_n = A \in \mathbb{R}$, *then also* $\lim_{k\to\infty} b_k = A$.

Remark

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, $A \in \mathbb{R}$, $K \in \mathbb{R}$, $K > 0$. If

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \; \exists n_0 \in \mathbb{N} \; \forall n \in \mathbb{N}, n \geq n_0 : \; |a_n - A| < K\varepsilon,$$

then $\lim a_n = A$.

Theorem 4 (arithmetics of limits)

Suppose that $\lim a_n = A \in \mathbb{R}$ *and* $\lim b_n = B \in \mathbb{R}$*. Then*

(i) $\lim(a_n + b_n) = A + B$,

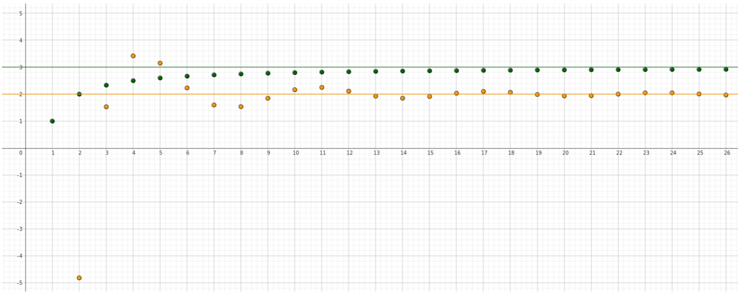
(ii) $\lim(a_n \cdot b_n) = A \cdot B$,

(iii) *if* $B \neq 0$ *and* $b_n \neq 0$ *for all* $n \in \mathbb{N}$*, then* $\lim(a_n/b_n) = A/B$.

Theorem 5 (limits and ordering)

Let $\lim a_n = A \in \mathbb{R}$ *and* $\lim b_n = B \in \mathbb{R}$.

(i) *Suppose that there is* $n_0 \in \mathbb{N}$ *such that* $a_n \geq b_n$ *for every* $n \geq n_0$. *Then* $A \geq B$.



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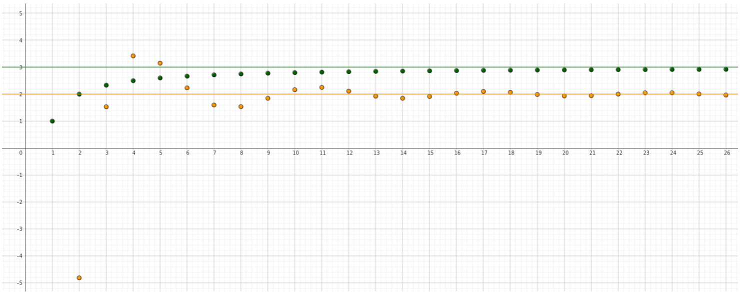
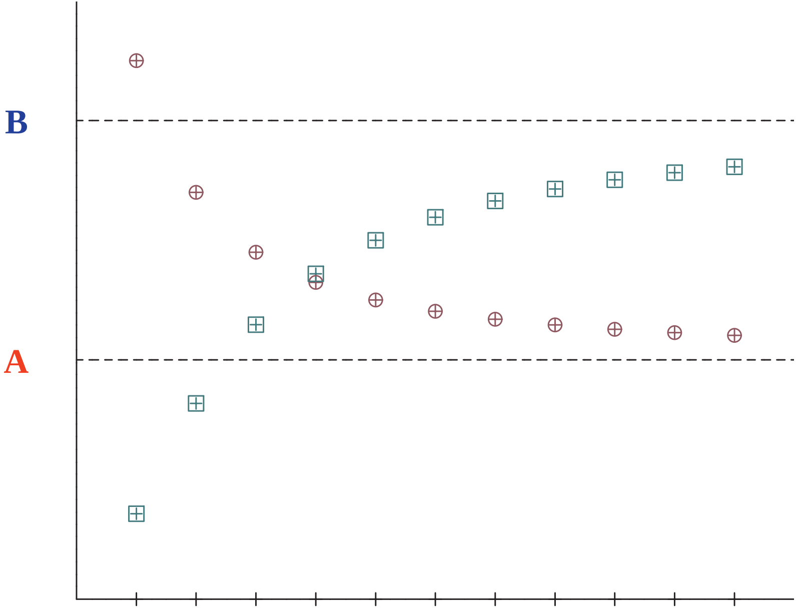
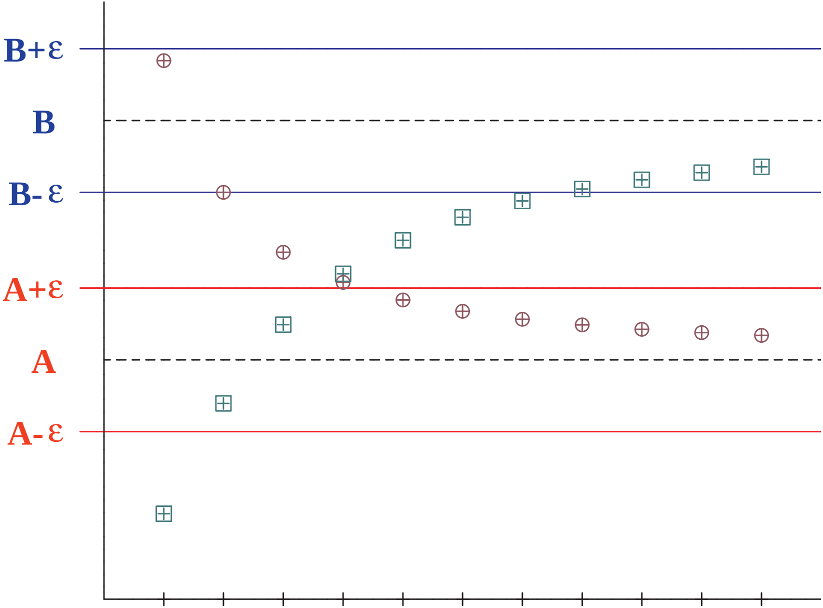
(i) *Suppose that there is* $n_0 \in \mathbb{N}$ *such that* $a_n \geq b_n$ *for every* $n \geq n_0$. *Then* $A \geq B$.

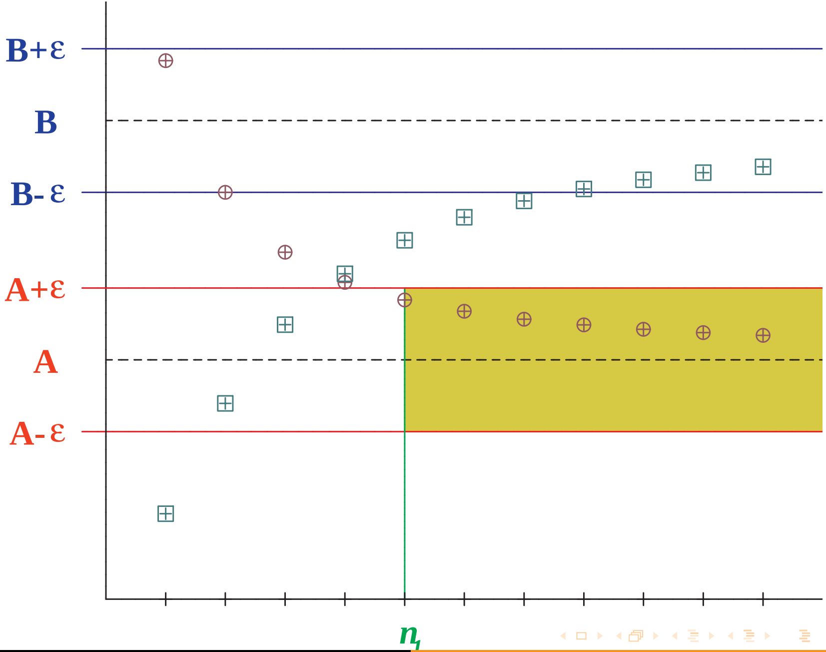
(ii) *Suppose that* $A < B$. *Then there is* $n_0 \in \mathbb{N}$ *such that* $a_n < b_n$ *for every* $n \geq n_0$.

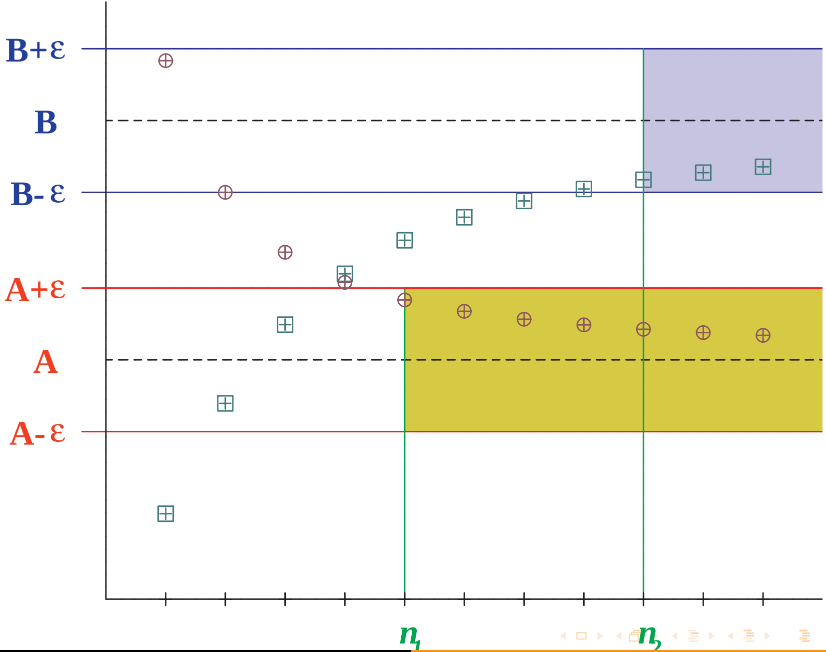


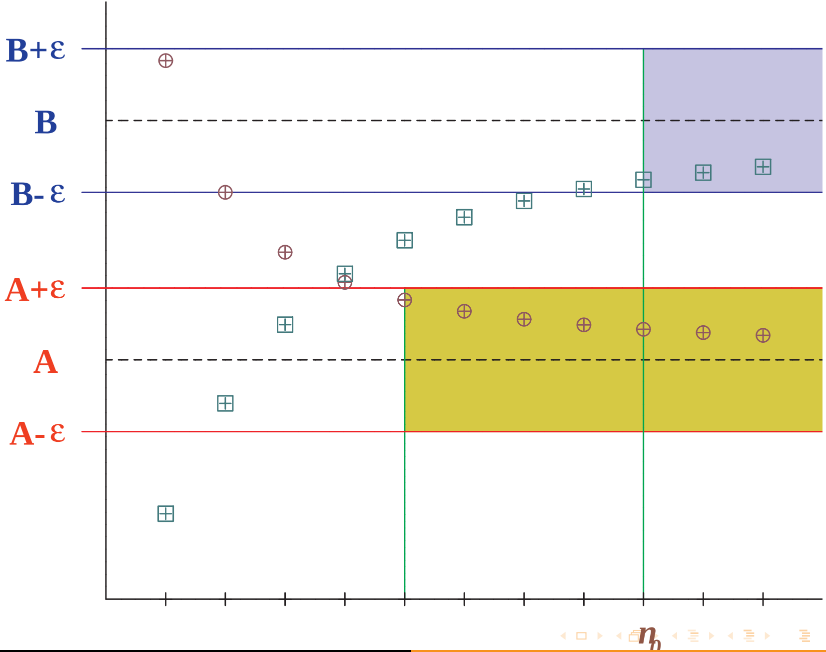


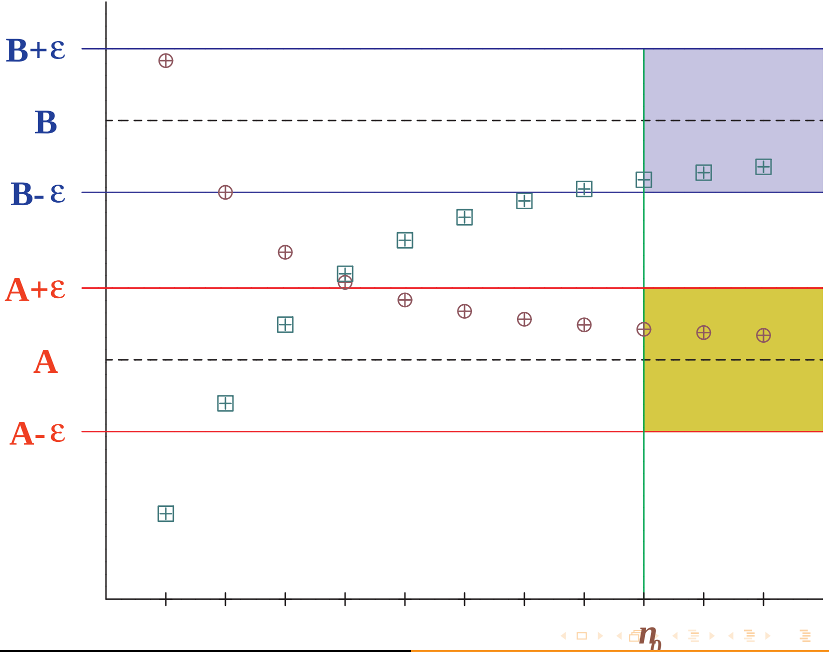
B
A


B+ε
B
B- ε
A+ε
A
A- ε


B+ε
B
B- ε
A+ε
A
A- ε
n


B+ε
B
B- ε
A+ε
A
A- ε
n_1
n_2


B+ε
B
B- ε
A+ε
A
A- ε
n_0


B+ε
B
B- ε
A+ε
A
A- ε
n_0

Theorem 6 (limits and ordering)

Let $\lim a_n = A \in \mathbb{R}$ *and* $\lim b_n = B \in \mathbb{R}$.

1. *Suppose that there is* $n_0 \in \mathbb{N}$ *such that* $a_n \geq b_n$ *for every* $n \geq n_0$. *Then* $A \geq B$.
2. *Suppose that* $A < B$. *Then there is* $n_0 \in \mathbb{N}$ *such that* $a_n < b_n$ *for every* $n \geq n_0$.

Exercise (True or false)

Let $\lim a_n = A \in \mathbb{R}$ and $\lim b_n = B \in \mathbb{R}$.
If $a_n < b_n$, then $A < B$.

Theorem 6 (limits and ordering)

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Exercise (True or false)

Let $\lim a_n = A \in \mathbb{R}$ and $\lim b_n = B \in \mathbb{R}$.
If $a_n < b_n$, then $A < B$. False. Consider $a_n = \frac{1}{n}$, $b_n = -\frac{1}{n}$.

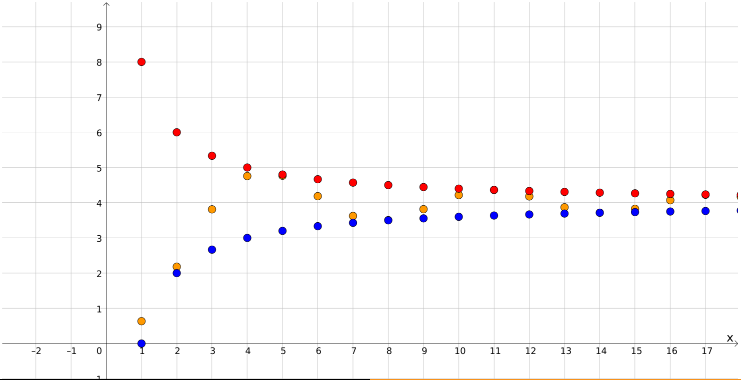
Theorem 7 (two policemen/sandwich theorem)

Let $\{a_n\}$, $\{b_n\}$ *be convergent sequences and let* $\{c_n\}$ *be a sequence such that*

(i) $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0 \colon a_n \leq c_n \leq b_n$,

(ii) $\lim a_n = \lim b_n$.

Then $\lim c_n$ *exists and* $\lim c_n = \lim a_n$.



Theorem 8 (two policemen/sandwich theorem)

Let $\{a_n\}$, $\{b_n\}$ *be convergent sequences and let* $\{c_n\}$ *be a sequence such that*

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Exercise

Find the sandwich for the sequence $a_n = \dfrac{\cos n}{n}$.

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Corollary 9

Suppose that $\lim a_n = 0$ *and the sequence* $\{b_n\}$ *is bounded. Then* $\lim a_n b_n = 0$.

Infinite Limits

https://www.geogebra.org/calculator/cpuzsnnh

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Definition

We say that a sequence $\{a_n\}$ has a limit $+\infty$ (plus infinity) if

$$\forall L \in \mathbb{R}\ \exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N}, n \geq n_0 : a_n > L.$$

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Theorem 1 on the uniqueness of a limit holds also for the limits $+\infty$ and $-\infty$. If $\lim a_n = +\infty$, then we say that the sequence $\{a_n\}$ diverges to $+\infty$, similarly for $-\infty$.

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Theorem 1 on the uniqueness of a limit holds also for the limits $+\infty$ and $-\infty$. If $\lim a_n = +\infty$, then we say that the sequence $\{a_n\}$ diverges to $+\infty$, similarly for $-\infty$. If $\lim a_n \in \mathbb{R}$, then we say that the limit is finite, if $\lim a_n = +\infty$ or $\lim a_n = -\infty$, then we say that the limit is infinite.

Theorem 2 does not hold for infinite limits. But:

Theorem 2'

- *Suppose that* $\lim a_n = +\infty$. *Then the sequence* $\{a_n\}$ *is not bounded from above, but is bounded from below.*
- *Suppose that* $\lim a_n = -\infty$. *Then the sequence* $\{a_n\}$ *is not bounded from below, but is bounded from above.*

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Give an example of $a_n \to \infty$ and find its lower bound.

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Give an example of $a_n \to \infty$ and find its lower bound.
$a_n = \log n$, $b = 0$.

Theorem 3 (limit of a subsequence) holds also for infinite limits.

Definition

We define the extended real line by setting $\mathbb{R}^* = \mathbb{R} \cup \{+\infty, -\infty\}$ with the following extension of operations and ordering from $\mathbb{R}$:

- $a < +\infty$ and $-\infty < a$ for $a \in \mathbb{R}$, $-\infty < +\infty$,
- $a + (+\infty) = (+\infty) + a = +\infty$ for $a \in \mathbb{R}^* \setminus \{-\infty\}$,
- $a + (-\infty) = (-\infty) + a = -\infty$ for $a \in \mathbb{R}^* \setminus \{+\infty\}$,
- $a \cdot (\pm\infty) = (\pm\infty) \cdot a = \pm\infty$ for $a \in \mathbb{R}^*$, $a > 0$,
- $a \cdot (\pm\infty) = (\pm\infty) \cdot a = \mp\infty$ for $a \in \mathbb{R}^*$, $a < 0$,
- $\frac{a}{\pm\infty} = 0$ pro $a \in \mathbb{R}$.

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- $\frac{a}{\pm\infty} = 0$ pro $a \in \mathbb{R}$.

Exercise

1. $2 + \infty$
2. $-\infty + 3$
3. $\pi\infty$
4. $-4(-\infty)$
5. -7∞
6. $\frac{\infty}{-3}$
7. $\frac{5}{\infty}$

The following operations are not defined:

- $(-\infty) + (+\infty)$, $(+\infty) + (-\infty)$, $(+\infty) - (+\infty)$, $(-\infty) - (-\infty)$,
- $(+\infty) \cdot 0$, $0 \cdot (+\infty)$, $(-\infty) \cdot 0$, $0 \cdot (-\infty)$,
- $\frac{+\infty}{+\infty}$, $\frac{+\infty}{-\infty}$, $\frac{-\infty}{-\infty}$, $\frac{-\infty}{+\infty}$, $\frac{a}{0}$ for $a \in \mathbb{R}^*$.

Theorem 4' (arithmetics of limits)

Suppose that $\lim a_n = A \in \mathbb{R}^*$ *and* $\lim b_n = B \in \mathbb{R}^*$. *Then*

(i) $\lim(a_n \pm b_n) = A \pm B$ *if the right-hand side is defined,*

Remark

Consider cases

1. $a_n = (-1)^n,\ b_n = (-1)^{n+1}$
2. $a_n = n,\ b_n = \frac{1}{n}$
3. $a_n = n,\ b_n = \frac{1}{n^2}$
4. $a_n = n^2,\ b_n = \frac{1}{n}$

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(ii) $\lim(a_n \cdot b_n) = A \cdot B$ *if the right-hand side is defined,*

(iii) $\lim a_n/b_n = A/B$ *if the right-hand side is defined.*

Remark

Consider cases

1. $a_n = (-1)^n$, $b_n = (-1)^{n+1}$
2. $a_n = n$, $b_n = \frac{1}{n}$
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Theorem 10

Suppose that $\lim a_n = A \in \mathbb{R}^*$, $A > 0$, $\lim b_n = 0$ *and there is* $n_0 \in \mathbb{N}$ *such that we have* $b_n > 0$ *for every* $n \in \mathbb{N}$, $n \geq n_0$. *Then* $\lim a_n/b_n = +\infty$.

https://www.geogebra.org/calculator/cpuzsnnh

Theorem 6 (limits and ordering) and Theorem 8 (sandwich theorem) hold also for infinite limits. Even the following modification holds:

Theorem 8' (one policeman)

Let $\{a_n\}$ and $\{b_n\}$ be two sequences.

- *If* $\lim a_n = +\infty$ *and there is* $n_0 \in \mathbb{N}$ *such that* $b_n \geq a_n$ *for every* $n \in \mathbb{N}$, $n \geq n_0$, *then* $\lim b_n = +\infty$.
- *If* $\lim a_n = -\infty$ *and there is* $n_0 \in \mathbb{N}$ *such that* $b_n \leq a_n$ *for every* $n \in \mathbb{N}$, $n \geq n_0$, *then* $\lim b_n = -\infty$.

Lemma 11

Let $M \subset \mathbb{R}$ be non-empty and $G \in \mathbb{R}^*$. Then the following statements are equivalent:

(1) $G = \sup M$.

(2) *The number G is an upper bound of M and there exists a sequence $\{x_n\}_{n=1}^{\infty}$ of members of M such that $\lim x_n = G$.*

Exercise

Find a sequence $\{x_n\}$ for a set $M = [2, 5)$.

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Exercise

Find a sequence $\{x_n\}$ for a set $M = [2, 5)$.
$x_n = 4, 4.5, 4\frac{2}{3}, 4.75 \ldots, x_n = 5 - \frac{1}{n}$

Theorem 12 (limit of a monotone sequence)

Every monotone sequence has a limit. If $\{a_n\}$ *is non-decreasing, then* $\lim a_n = \sup\{a_n;\ n \in \mathbb{N}\}$. *If* $\{a_n\}$ *is non-increasing, then* $\lim a_n = \inf\{a_n;\ n \in \mathbb{N}\}$.

Theorem 13 (Bolzano-Weierstraß)

Every bounded sequence contains a convergent subsequence.

https://www.geogebra.org/m/gnpqch7u

Exercise

Find a convergent subsequence:

A $a_n = (-1)^n$

B $a_n = \{0, 2, 0, 0, 2, 0, 0, 0, 2, 0, 0, 0, 0, 2, \dots\}$

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1. $1, 1, 1, \dots$
2. $0, 0, 0, \dots$