Mathematics I - Introduction

23/24

https://www.hbo.com/chernobyl

Why study Math?

Goal of the course

- Preparation for **other courses** — Statistics, Microeconomics, …

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At the end of the course students should be able to

- compute limits and derivatives and investigate functions
- understand definitions (give positive and negative examples) and theorems (explain their meaning, neccessity of the assumptions, apply them in particular situations)
- perform mathematical proofs, give mathematically exact arguments, write mathematical formulae, use quantifiers

Content of Mathematics I

- Introduction
- Limit of a sequence
- Mappings
- Functions of one real variable

Textbooks

- **Hájková et al: Mathematics 1**
- Trench: Introduction to real analysis
- Ghorpade, Limaye: A course in calculus and real analysis
- Zorich: Mathematical analysis I
- Rudin: Principles of mathematical analysis

Sets - Notation

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We take a set to be a collection of definite and distinguishable objects into a coherent whole.

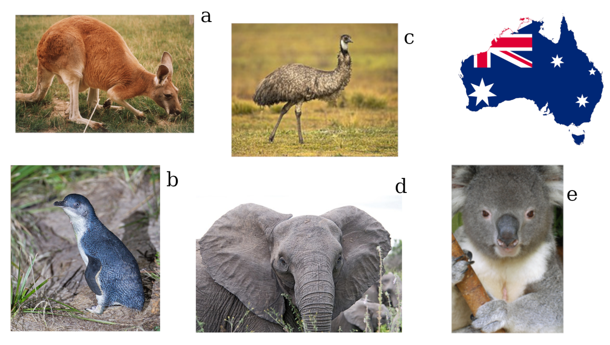
- $x \in A$ ... x is an element (or member) of the set A
- $x \notin A$ ... x is not a member of the set A

Sets

Exercise (True or false)

A - set of all animals living in Australia.

A $a \in A$ B $b \in A$ C $c \in A$ D $d \in A$ E $e \in A$



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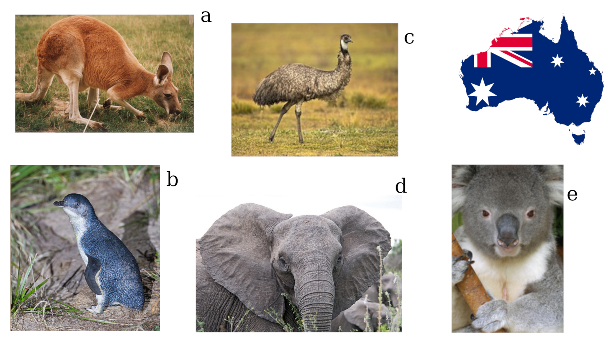

True: A, B, C, E

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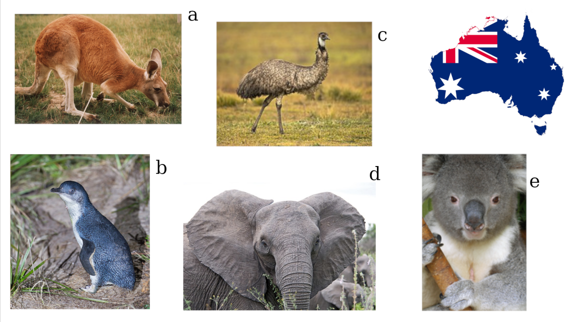

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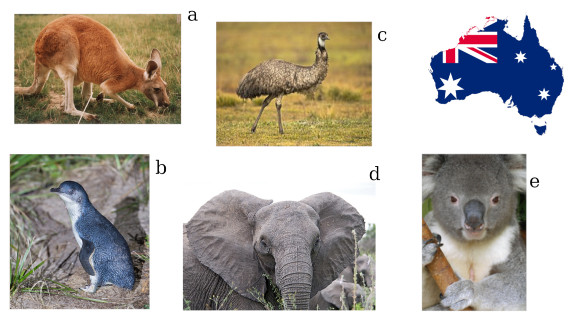

True: D

- $\emptyset$ ... an empty set
- A^c ... the complement of the set A
- $A \cup B$ ... the union of the sets A and B
- $A \cap B$ ... the intersection of the sets A and B
- disjoint sets ... A and B are disjoint if $A \cap B = \emptyset$
- $A \setminus B = \{x \in A;\ x \notin B\}$ ... a difference of the sets A and B

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Exercise

Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 3, 5, 7, 9\}$ and $B = \{1, 2, 3, 4, 5\}$. Find

1. $A \cup B$
2. $A \cap B$
3. A^c
4. $(B^c)^c$
5. $A \setminus B$
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1. $\{1, 2, 3, 4, 5, 7, 9\}$
2. $\{1, 3, 5\}$
3. $\{2, 4, 6, 8\}$
4. B
5. $\{7, 9\}$
6. $\{2, 4\}$

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- $A_1 \times \cdots \times A_m = \{[a_1, \ldots, a_m];\ a_1 \in A_1, \ldots, a_m \in A_m\}$
  … the Cartesian product

Sets

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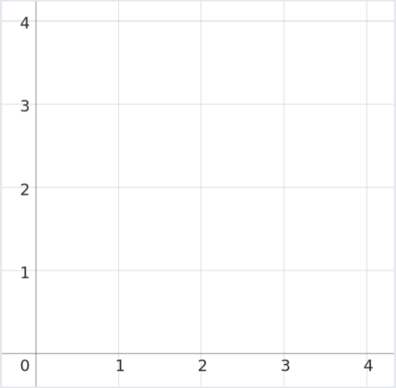
Exercise

Let $A = \{1, 2, 3\}$, $B = \{2, 4\}$. Find $A \times B$, $B \times B$ and sketch them.



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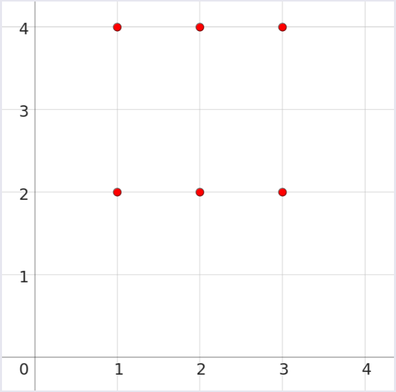

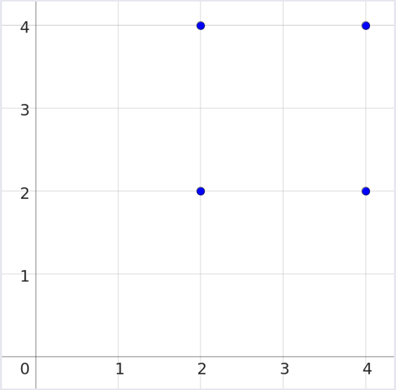

Sets

Let I be a non-empty set of indices and suppose we have a system of sets A_α, where the indices α run over I.

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Infinitely many sets: $A_1 \cup A_2 \cup A_3 \cup \ldots$ is equivalent to $\bigcup\limits_{i=1}^{\infty} A_i$, and also to $\bigcup\limits_{i \in \mathbb{N}} A_i$.

Exercise

Let $A_1 = \{0, 1\}, A_2 = \{0, 2\}, A_3 = \{0, 3\}$. Find

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$\{0, 1, 2, 3\}, \{0\}$

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- B We all live in a yellow submarine.
- C Is there anybody out there?
- D We don't need no education.
- E You can't always get what you want.

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B, D, E

Statements

- $\neg$, also non . . . negation
- & (also $\wedge$) . . . conjunction, logical “and”
- $\vee$ . . . disjuction (alternative), logical “or”
- $\Rightarrow$ . . . implication
- $\Leftrightarrow$ . . . equivalence; “if and only if”

A	B	$\neg A$	$\neg B$	$A \wedge B$	$A \vee B$	$A \Rightarrow B$	$A \Leftrightarrow B$
1	1	0	0	1	1	1	1
1	0	0	1	0	1	0	0
0	1	1	0	0	1	1	0
0	0	1	1	0	0	1	1

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Exercise

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B: Max likes lemon icecream.
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1. If it will be raining tomorrow, we will play board games.
2. We will play board games tomorrow if and only if it will be raining.

Time for the table of statements.

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- 7 is a prime number;
- 4 is a prime number;
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3. 8 is bigger than 3;
4. x is bigger than y.
5. x is bigger than y, $x, y \in \mathbb{R}$.

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$$V(x), x \in M$$

$$V(x_1, \ldots, x_n), x_1 \in M_1, \ldots, x_n \in M_n$$

Example

$V(x)$: x is even
$M = \{1, 2, 3, 4, 5\}$
$V(3)$ false, $V(4)$ true.

$V(x_1, x_2)$: $x_1 \cdot x_2 = 1$
$M = \{2, \frac{1}{2}, 3, 4\}$
$V(2, \frac{1}{2})$ true, $V(2, 3)$ false.

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Example

$\forall x \in \mathbb{R} : |x| \geq 0$
$\exists x \in \mathbb{Q} : x + 3 \leq 12$
$\exists ! x \in \mathbb{R}^+ : x^2 = 36$

If $A(x)$, $x \in M$ and $B(x)$, $x \in M$ are predicates, then

$$\forall x \in M, B(x) : A(x) \quad \text{means} \quad \forall x \in M : (B(x) \Rightarrow A(x)),$$

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$$\exists x \in M, B(x) : A(x) \quad \text{means} \quad \exists x \in M : (A(x) \;\&\; B(x)).$$

Example

$$\forall n \in \mathbb{N}, \forall x \in \mathbb{R}, x \geq -2 : 1 + nx \leq (1+x)^n$$

$$\exists x \in \mathbb{R}, x \geq 0 : x \geq x^2$$

https://www.geogebra.org/calculator/gajuueuy

Negations of the statements with quantifiers:

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Example

Find negation

$$\exists x \in \mathbb{R}, x \geq 0 : x \geq x^2$$

$$\forall x \in \mathbb{R}, x \geq -1 : 1 + 3x \leq (1+x)^3$$

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x \geq 0, y \geq 0 : \frac{x+y}{2} \geq \sqrt{xy}$$

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Example

Find negation

$$\forall x \in \mathbb{R}, x \geq 0 : x < x^2$$

$$\exists x \in \mathbb{R}, x \geq -1 : 1 + 3x > (1+x)^3$$

$$\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x \geq 0, y \geq 0 : \frac{x+y}{2} < \sqrt{xy}$$

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Exercise

For every $k \in \mathbb{N}$ we have

$$\sum_{j=1}^{k}(2j-1) = k^2$$

Numbers

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- The set of rational numbers

$$\mathbb{Q} = \left\{ \frac{p}{q};\ p \in \mathbb{Z}, q \in \mathbb{N} \right\},$$

where $\frac{p_1}{q_1} = \frac{p_2}{q_2}$ if and only if $p_1 \cdot q_2 = p_2 \cdot q_1$.

Real numbers

Real numbers

By the set of real numbers $\mathbb{R}$ we will understand a set on which there are operations of addition and multiplication (denoted by $+$ and $\cdot$), and a relation of ordering (denoted by $\leq$), such that it has the following three groups of properties.

I. The properties of addition and multiplication and their relationships.
II. The relationships of the ordering and the operations of addition and multiplication.
III. The infimum axiom.

The properties of addition and multiplication and their relationships:

The properties of addition and multiplication and their relationships:

- $\forall x, y \in \mathbb{R}\colon x + y = y + x$ (commutativity of addition),
- $\forall x, y, z \in \mathbb{R}\colon x + (y + z) = (x + y) + z$ (associativity),
- There is an element in $\mathbb{R}$ (denoted by 0 and called a zero element), such that $x + 0 = x$ for every $x \in \mathbb{R}$,
- $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\colon x + y = 0$ (y is called the negative of x, such y is only one, denoted by $-x$),
- $\forall x, y \in \mathbb{R}\colon x \cdot y = y \cdot x$ (commutativity),
- $\forall x, y, z \in \mathbb{R}\colon x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (associativity),
- There is a non-zero element in $\mathbb{R}$ (called identity, denoted by 1), such that $1 \cdot x = x$ for every $x \in \mathbb{R}$,
- $\forall x \in \mathbb{R} \setminus \{0\}\ \exists y \in \mathbb{R}\colon x \cdot y = 1$ (such y is only one, denoted by x^{-1} or $\frac{1}{x}$),
- $\forall x, y, z \in \mathbb{R}\colon (x + y) \cdot z = x \cdot z + y \cdot z$ (distributivity).

The relationships of the ordering and the operations of addition and multiplication:

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- $\forall x, y, z \in \mathbb{R}\colon (x \le y \ \ \& \ \ y \le z) \Rightarrow x \le z$ (transitivity),
- $\forall x, y \in \mathbb{R}\colon (x \le y \ \ \& \ \ y \le x) \Rightarrow x = y$ (weak antisymmetry),
- $\forall x, y \in \mathbb{R}\colon x \le y \vee y \le x$,
- $\forall x, y, z \in \mathbb{R}\colon x \le y \Rightarrow x + z \le y + z$,
- $\forall x, y \in \mathbb{R}\colon (0 \le x \ \ \& \ \ 0 \le y) \Rightarrow 0 \le x \cdot y$.

Definition

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Exercise

Which sets are bounded from below? Bounded from above? Bounded?

A $\mathbb{N}$

B $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\}$

C $\mathbb{R} \setminus \mathbb{Q} \cap (-3, 2]$

D $\{x \in \mathbb{R} : x < \pi\}$

E $(-\infty, -1) \cup \{0\} \cup [1, \infty)$

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Which sets are bounded from below? Bounded from above? Bounded?

- A $\mathbb{N}$
- B $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$
- C $\mathbb{R} \setminus \mathbb{Q} \cap (-3, 2]$
- D $\{x \in \mathbb{R} : x < \pi\}$
- E $(-\infty, -1) \cup \{0\} \cup [1, \infty)$

below: A, B, C; above: B, C, D; bounded: B, C

Definition

Let $M \subset \mathbb{R}$. We say that a is a maximum of the set M (denoted by $\max M$) if a is an upper bound of M and $a \in M$.
Analogously we define a minimum of M, denoted by $\min M$.

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Exercise

Find minimum and maximum:

1. $\{1, 2, 3, 4\}$
2. $[-2, 3]$
3. $(-2, 3]$
4. $[-2, -1) \cup (0, 25]$
5. $[0, \infty)$
6. $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$
7. $\mathbb{N}$
8. $(\mathbb{R} \setminus \mathbb{Q}) \cap [0, \pi]$

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1. $\min = 1$, $\max = 4$
2. $-2, 3$
3. $\nexists, 3$
4. $-2, 25$
5. $0, \nexists$
6. $\nexists, 1$
7. $1, \nexists$
8. $\nexists, \pi$

The infimum axiom:

Let M be a non-empty set bounded from below. Then there exists a unique number $g \in \mathbb{R}$ such that

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(i) $\forall x \in M : x \geq g$,

(ii) $\forall g' \in \mathbb{R}, g' > g\ \exists x \in M : x < g'$.

The number g is denoted by $\inf M$ and is called the infimum of the set M.

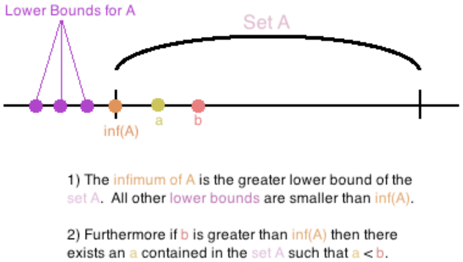

Figure:
https://mathspandorabox.wordpress.com/2016/03/11/the-difference-between-supremum-and-infimum-equivalent-and-equal-set/

Remark

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- The infimum of the set M is its greatest lower bound.
- The real numbers exist and are uniquely determined by the properties I–III.

The following hold:

(i) $\forall x \in \mathbb{R}\colon x \cdot 0 = 0 \cdot x = 0,$

(ii) $\forall x \in \mathbb{R}\colon\ -x = (-1) \cdot x,$

(iii) $\forall x, y \in \mathbb{R}\colon xy = 0 \Rightarrow (x = 0 \vee y = 0),$

(iv) $\forall x \in \mathbb{R}\ \forall n \in \mathbb{N}\colon x^{-n} = (x^{-1})^n,$

(v) $\forall x, y \in \mathbb{R}\colon (x > 0 \wedge y > 0) \Rightarrow xy > 0,$

(vi) $\forall x \in \mathbb{R}, x \geq 0\ \forall y \in \mathbb{R}, y \geq 0\ \forall n \in \mathbb{N}\colon x < y \Leftrightarrow x^n < y^n.$

Let $a, b \in \mathbb{R}$, $a \leq b$. We denote:

- An open interval $(a, b) = \{x \in \mathbb{R};\ a < x < b\}$,
- A closed interval $[a, b] = \{x \in \mathbb{R};\ a \leq x \leq b\}$,
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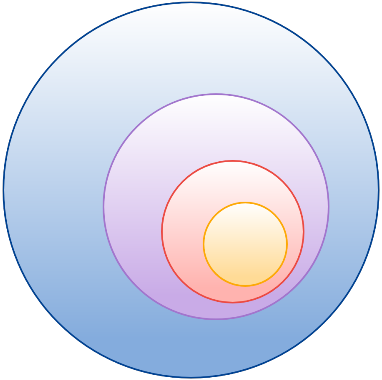
Unbounded intervals:

$$(a, +\infty) = \{x \in \mathbb{R};\ a < x\}, \quad (-\infty, a) = \{x \in \mathbb{R};\ x < a\},$$

analogically $(-\infty, a]$, $[a, +\infty)$ and $(-\infty, +\infty)$.

Label the Venn diagram with $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R} \setminus \mathbb{Q}$.



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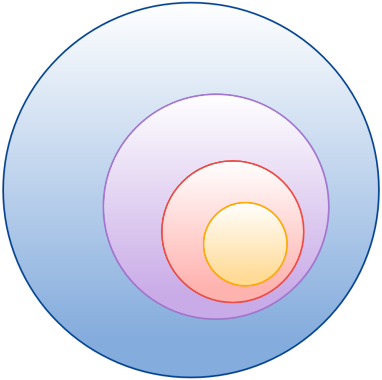

We have $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. If we transfer the addition and multiplication from $\mathbb{R}$ to the above sets, we obtain the usual operations on these sets.

A real number that is not rational is called irrational. The set $\mathbb{R} \setminus \mathbb{Q}$ is called the set of irrational numbers.

Definition

Let $M \subset \mathbb{R}$. A number $G \in \mathbb{R}$ satisfying

(i) $\forall x \in M : x \leq G$,

(ii) $\forall G' \in \mathbb{R}, G' < G \; \exists x \in M : x > G'$,

is called a supremum of the set M.

Consequences of the infimum axiom

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Theorem 1 (Supremum theorem)

Let $M \subset \mathbb{R}$ be a non-empty set bounded from above. Then there exists a unique supremum of the set M.

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Let $M \subset \mathbb{R}$ be a non-empty set bounded from above. Then there exists a unique supremum of the set M.

The supremum of the set M is denoted by $\sup M$.
The following holds: $\sup M = -\inf(-M)$.

Exercise

Find infimum, minimum, maximum and supremum:

1. $\{1, 2, 3, 4\}$
2. $\{-1, -2, -3, -4\}$
3. $[-2, 3]$
4. $(-2, 3)$
5. $(-2, 3]$
6. $[-2, -1) \cup (0, 25]$
7. $(-7, -0) \cup (1, 2)$
8. $[0, \infty)$
9. $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$
10. $\mathbb{N}$

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10. $\mathbb{N}$

1. $1, 1, 4, 4$
2. $-4, -4, -1, -1$
3. $-2, -2, 3, 3$
4. $-2, \nexists, \nexists, 3$
5. $-2, \nexists, 3, 3$
6. $-2, -2, 25, 25$
7. $-7, \nexists, \nexists, 2$
8. $0, 0, \nexists, \infty$
9. $0, \nexists, 1, 1$
10. $1, 1, \nexists, \infty$

Theorem 2 (Archimedean property)

For every $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ satisfying $n > x$.

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Theorem 3 (existence of an integer part)

For every $r \in \mathbb{R}$ there exists an integer part of r, i.e. a number $k \in \mathbb{Z}$ satisfying $k \leq r < k + 1$. The integer part of r is determined uniquely and it is denoted by $[r]$.

Theorem 4 (nth root)

For every $x \in [0, +\infty)$ and every $n \in \mathbb{N}$ there exists a unique $y \in [0, +\infty)$ satisfying $y^n = x$.

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Theorem 5 (density of $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$)

Let $a, b \in \mathbb{R}$, $a < b$. Then there exist $r \in \mathbb{Q}$ satisfying $a < r < b$ and $s \in \mathbb{R} \setminus \mathbb{Q}$ satisfying $a < s < b$.