# Mathematics II - Functions of multiple variables

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# V.1. $\mathbb{R}^n$ as a linear and metric space

### Definition

The set  $\mathbb{R}^n$ ,  $n \in \mathbb{N}$ , is the set of all ordered *n*-tuples of real numbers, i.e.

$$\mathbb{R}^n = \{ [x_1, \ldots, x_n] : x_1, \ldots, x_n \in \mathbb{R} \}.$$



https://en.wikipedia.org/wiki/File: Cartesian-coordinate-system.svg

### Exercise (2D)

### Sketch the following points and connect them.

$$(4,0), (0,3), (-4,0), (-6,2), (-5,0), (-6,-2), (-4,0), (-6,-2), (-4,0), (-6,-2), (-4,0), (-6,-2), (-$$

$$(0, -2), (4, 0),$$

and add one point:

(2, 1).

https: //www.geogebra.org/calculator/bbsahf43

#### Exercise (3D)

https://www.geogebra.org/classic/ydu8a7t7

# Which picture(s) plots the point (2, 1, 1) correctly?



https://www.cpp.edu/conceptests/question-library/
mat214.shtml

# Which picture(s) plots the point (2, 1, 1) correctly?



A, C

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# V.1. $\mathbb{R}^n$ as a linear and metric space

## Definition

For 
$$\mathbf{x} = [x_1, \dots, x_n] \in \mathbb{R}^n$$
,  $\mathbf{y} = [y_1, \dots, y_n] \in \mathbb{R}^n$  and  $\alpha \in \mathbb{R}$  we set

$$\boldsymbol{x} + \boldsymbol{y} = [x_1 + y_1, \dots, x_n + y_n], \qquad \alpha \boldsymbol{x} = [\alpha x_1, \dots, \alpha x_n].$$

Further, we denote 
$$\boldsymbol{o} = [0, \dots, 0]$$
 – the origin.

### Exercise

Find

A 
$$(1, 2, 3, 4) + (-2, 0, 3, -1)$$
  
B  $-2(1, 2, 3, 4)$ 

# V.1. $\mathbb{R}^n$ as a linear and metric space

## Definition

For 
$$\mathbf{x} = [x_1, \dots, x_n] \in \mathbb{R}^n$$
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Further, we denote 
$$\boldsymbol{o} = [0, \dots, 0]$$
 – the origin.

### Exercise

# Find A (1,2,3,4) + (-2,0,3,-1)B -2(1,2,3,4)

The Euclidean metric (distance) on  $\mathbb{R}^n$  is the function  $\rho \colon \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$  defined by

$$\rho(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

The number  $\rho(\mathbf{x}, \mathbf{y})$  is called the distance of the point  $\mathbf{x}$  from the point  $\mathbf{y}$ .



Find the distance of the points



https://www.summitlearning.org/guest/
focusareas/862919

B 
$$(1, -2, 3), (0, -3, -2)$$
  
C  $(-1, 0, 3, 2), (1, -1, 2, -3)$ 

Find the distance of the points



**C** 
$$(-1, 0, 3, 2), (1, -1, 2, -3)$$

 $\sqrt{52}, \sqrt{27}, \sqrt{31}$ 

# **A** $\rho((1,2),(1,2))$









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### Theorem 1 (properties of the Euclidean metric)

*The Euclidean metric*  $\rho$  *has the following properties:* 

Let  $\mathbf{x} \in \mathbb{R}^n$ ,  $r \in \mathbb{R}$ , r > 0. The set  $B(\mathbf{x}, r)$  defined by

$$B(\boldsymbol{x},r) = \{ \boldsymbol{y} \in \mathbb{R}^n; \ 
ho(\boldsymbol{x}, \boldsymbol{y}) < r \}$$

is called an open ball with radius r centred at x or the neighbourhood of x.



http://www.science4all.org/article/topology/

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https://commons. wikimedia.org/wiki/File: 4dSphere.jpg

https://en.wikipedia.
 org/wiki/N-sphere



https: //www.tinyepiphany.com/ 2011/12/ visualizing-4-dimensions. html



https://cs.wikipedia. org/wiki/%C4%8Ctvrt%C3% BD\_rozm%C4%9Br

Let  $M \subset \mathbb{R}^n$ . We say that  $x \in \mathbb{R}^n$  is an interior point of M, if there exists r > 0 such that  $B(x, r) \subset M$ .

The set of all interior points of M is called the interior of M and is denoted by Int M.

The set  $M \subset \mathbb{R}^n$  is open in  $\mathbb{R}^n$ , if each point of M is an interior point of M, i.e. if M = Int M.



# Find the interior



# Solution



#### Theorem 2 (properties of open sets)

- (i) The empty set and  $\mathbb{R}^n$  are open in  $\mathbb{R}^n$ .
- (ii) Let  $G_{\alpha} \subset \mathbb{R}^n$ ,  $\alpha \in A \neq \emptyset$ , be open in  $\mathbb{R}^n$ . Then  $\bigcup_{\alpha \in A} G_{\alpha}$  is open in  $\mathbb{R}^n$ .
- (iii) Let  $G_i \subset \mathbb{R}^n$ , i = 1, ..., m, be open in  $\mathbb{R}^n$ . Then  $\bigcap_{i=1}^m G_i$  is open in  $\mathbb{R}^n$ .

### Remark

(ii) A union of an arbitrary system of open sets is an open set.(iii) An intersection of a finitely many open sets is an open set.



Find the interior

1. 
$$\{[x, y] \in \mathbb{R}^2 : x^2 + y^2 \le 4\}$$
  
2.  $\{[x, y] \in \mathbb{R}^2 : 1 \le x < 4, |y| \ge 3\}$   
3.  $\{[x, y] \in \mathbb{R}^2 : x^2 + 3y^2 \ge 1, x + y > 2\}$ 

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1.  $\{[x, y] \in \mathbb{R}^2 : x^2 + y^2 < 4\}$   
2.  $\{[x, y] \in \mathbb{R}^2 : 1 < x < 4, |y| > 3\}$   
3.  $\{[x, y] \in \mathbb{R}^2 : x^2 + 3y^2 > 1, x + y > 2\}$ 

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Let  $M \subset \mathbb{R}^n$  and  $x \in \mathbb{R}^n$ . We say that x is a boundary point of M if for each r > 0

 $B(\mathbf{x},r) \cap M \neq \emptyset$  and  $B(\mathbf{x},r) \cap (\mathbb{R}^n \setminus M) \neq \emptyset$ .

The boundary of M is the set of all boundary points of M (notation bd M).



The closure of *M* is the set  $M \cup \operatorname{bd} M$  (notation  $\overline{M}$ ).

A set  $M \subset \mathbb{R}^n$  is said to be closed in  $\mathbb{R}^n$  if it contains all its boundary points, i.e. if  $\operatorname{bd} M \subset M$ , or in other words if  $\overline{M} = M$ .

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#### Exercise

Decide, if the set is closed or open, find the interior, the boundary, the closure.

$$M = \{ [x, y] \in \mathbb{R}^2 : 1 < x \le 2, 3 \le y \le 5 \}.$$

## The closure of *M* is the set $M \cup \operatorname{bd} M$ (notation $\overline{M}$ ).

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Decide, if the set is closed or open, find the interior, the boundary, the closure.

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#### Exercise

Find the boundary

1. 
$$\{[x, y] \in \mathbb{R}^2 : x^2 + y^2 \le 4\}$$
  
2.  $\{[x, y] \in \mathbb{R}^2 : 1 \le x < 4, |y| \ge 3\}$   
3.  $\{[x, y] \in \mathbb{R}^2 : x^2 + 3y^2 \ge 1, x + y \ge 2\}$ 

Let  $x^j \in \mathbb{R}^n$  for each  $j \in \mathbb{N}$  and  $x \in \mathbb{R}^n$ . We say that a sequence  $\{x^j\}_{j=1}^{\infty}$  converges to x, if

$$\lim_{j\to\infty}\rho(\boldsymbol{x},\boldsymbol{x}^j)=0.$$

The vector  $\mathbf{x}$  is called the limit of the sequence  $\{\mathbf{x}^j\}_{i=1}^{\infty}$ .

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### Exercise

$$\lim_{j \to \infty} \left(\frac{1}{j}, \frac{2j+1}{j}\right)$$

### Theorem 3 (convergence is coordinatewise)

Let  $\mathbf{x}^{j} \in \mathbb{R}^{n}$  for each  $j \in \mathbb{N}$  and let  $\mathbf{x} \in \mathbb{R}^{n}$ . The sequence  $\{\mathbf{x}^{j}\}_{j=1}^{\infty}$  converges to  $\mathbf{x}$  if and only if for each  $i \in \{1, ..., n\}$  the sequence of real numbers  $\{x_{i}^{j}\}_{j=1}^{\infty}$  converges to the real number  $x_{i}$ .

#### Remark

Theorem 3 says that the convergence in the space  $\mathbb{R}^n$  is the same as the "coordinatewise" convergence. It follows that a sequence  $\{x^j\}_{j=1}^{\infty}$  has at most one limit. If it exists, then we denote it by  $\lim_{j\to\infty} x^j$ . Sometimes we also write simply  $x^j \to x$  instead of  $\lim_{j\to\infty} x^j = x$ .

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$$\lim_{j \to \infty} \left( 1 + \frac{1}{j}, 3 - \frac{2}{j^2}, e^{-j} \right)$$
$$\lim_{j \to \infty} \left( (-1)^j, \arctan(j^3) \right)$$

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$$\lim_{j \to \infty} \left( 1 + \frac{1}{j}, 3 - \frac{2}{j^2}, e^{-j} \right)$$
$$\lim_{j \to \infty} \left( (-1)^j, \arctan(j^3) \right)$$

 $(1, 3, 0), \not\exists$
#### Theorem 4 (characterisation of closed sets)

Let  $M \subset \mathbb{R}^n$ . Then the following statements are equivalent:

- (i) *M* is closed in  $\mathbb{R}^n$ .
- (ii)  $\mathbb{R}^n \setminus M$  is open in  $\mathbb{R}^n$ .
- (iii) Any  $\mathbf{x} \in \mathbb{R}^n$  which is a limit of a sequence from M belongs to M.

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#### Exercise

Decide, if the sets are closed or open (or nothing)

(0, 1) in ℝ
 (0, ∞) in ℝ
 (-3, 2] in ℝ

4.  $(-\infty, 2]$  in  $\mathbb{R}$ 5.  $x^2 + y^2 < 4$  in  $\mathbb{R}^2$ 6.  $x^2 + y^2 \ge 2$  in  $\mathbb{R}^2$ 

#### Theorem 4 (characterisation of closed sets)

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- 1. (0,1) in  $\mathbb{R}$  4.  $(-\infty,2]$  in  $\mathbb{R}$
- 2.  $(0,\infty)$  in  $\mathbb{R}$  5.  $x^2 + y^2 <$
- 3. (-3, 2] in  $\mathbb{R}$

- 5.  $x^2 + y^2 < 4$  in  $\mathbb{R}^2$ 6.  $x^2 + y^2 \ge 2$  in  $\mathbb{R}^2$
- 1. open3. nothing5. open2. open4. closed6. closed

#### Theorem 5 (properties of closed sets)

- (i) The empty set and the whole space  $\mathbb{R}^n$  are closed in  $\mathbb{R}^n$ .
- (ii) Let  $F_{\alpha} \subset \mathbb{R}^{n}$ ,  $\alpha \in A \neq \emptyset$ , be closed in  $\mathbb{R}^{n}$ . Then  $\bigcap_{\alpha \in A} F_{\alpha}$  is closed in  $\mathbb{R}^{n}$ .
- (iii) Let  $F_i \subset \mathbb{R}^n$ , i = 1, ..., m, be closed in  $\mathbb{R}^n$ . Then  $\bigcup_{i=1}^m F_i$  is closed in  $\mathbb{R}^n$ .

#### Remark

# (ii) An intersection of an arbitrary system of closed sets is closed.

(iii) A union of finitely many closed sets is closed.



We say that the set  $M \subset \mathbb{R}^n$  is bounded if there exists r > 0such that  $M \subset B(o, r)$ . A sequence of points in  $\mathbb{R}^n$  is bounded if the set of its members is bounded.



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#### Theorem 6

A set  $M \subset \mathbb{R}^n$  is bounded if and only if its closure  $\overline{M}$  is bounded.

Find bounded sets

A 
$$x \in [-1, 3], 0 < y \le 100$$
  
B  $x^2 + y^2 + z^2 \le 5$   
C  $x - y < 6$   
D  $|x + y| < 6$ 

Find bounded sets

A 
$$x \in [-1, 3], 0 < y \le 100$$
  
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C  $x - y < 6$   
D  $|x + y| < 6$   
A, B, D

We say that a set  $M \subset \mathbb{R}^n$  is compact if for each sequence of elements of M there exists a convergent subsequence with a limit in M.

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## Theorem 7 (characterisation of compact subsets of $\mathbb{R}^n$ )

The set  $M \subset \mathbb{R}^n$  is compact if and only if M is bounded and closed.

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## Theorem 7 (characterisation of compact subsets of $\mathbb{R}^n$ )

The set  $M \subset \mathbb{R}^n$  is compact if and only if M is bounded and closed.

#### Exercise

Find compact sets

A (0, 1)  
B 
$$[1, 2] \times [-1, -3]$$
  
C  $1 < x^2 + (y - 3)^2 + z^2 \le 4$   
D  $xyz \le 1$ 

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## Theorem 7 (characterisation of compact subsets of $\mathbb{R}^n$ )

The set  $M \subset \mathbb{R}^n$  is compact if and only if M is bounded and closed.

#### Exercise

B

Find compact sets

A (0, 1)  
B 
$$[1, 2] \times [-1, -3]$$
  
C  $1 < x^2 + (y - 3)^2 + z^2 \le 4$   
D  $xyz \le 1$ 

Map game

We define a function of two variables as a mapping  $f : M \to \mathbb{R}$ , where  $M \subset \mathbb{R}^2$ .

## Example

$$\begin{split} f(x,y) &= x^2 + y^2, & [x,y] \in \mathbb{R}^2 \\ f(x,y) &= \arccos y \, \cdot \arcsin x, & D_f = [-1,1] \times [-1,1] \\ f(x,y) &= \ln(xy), & D_f = \{(x > 0 \land y > 0) \lor (x < 0 \land y < 0)\} \\ f(x,y) &= x^3, & [x,y] \in \mathbb{R}^2 \\ f(x,y) &= 5, & [x,y] \in \mathbb{R}^2 \end{split}$$

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## Example

$$f(x,y) = \frac{x^2}{x^2 + y^2}$$

$$f(x, y) = \sin x \cos y$$



## Example

$$f(x,y) = \sqrt{x^2 + y^2}$$

$$f(x, y) = \sqrt{4 - (x^2 + y^2)}$$



## Find the graph for the contourlines





http://www.cpp.edu/~conceptests/question-library/
mat214.shtml

## Find the graph for the contourlines





http://www.cpp.edu/~conceptests/question-library/
mat214.shtml
A

## Find the contourlines for the graph.





В

## Find the contourlines for the graph.





### Connect the contourlines and the functions



Figure: Hughes Hallett et al c 2009, John Wiley & Sons A  $-x^2 + y^2$  C  $-x^2 - y^2$ B  $x^2 - y^2$  D  $x^2 + y^2$ 

### Connect the contourlines and the functions



Figure: Hughes Hallett et al c 2009, John Wiley & Sons A  $-x^2 + y^2$  C  $-x^2 - y^2$ B  $x^2 - y^2$  D  $x^2 + y^2$ 

I D, II B, III C, IV A

We define a function of multiple variables as a mapping  $\int M dx = \sum_{n=1}^{\infty} m^n dx$ 

 $f: M \to \mathbb{R}$ , where  $M \subset \mathbb{R}^n$ .

## Example

$$f(x) = x^3,$$
  $x \in \mathbb{R}$   
 $f(x, y) = y \sin x,$   $[x, y] \in \mathbb{R}^2$   
 $f(x, y, z) = x^2 + y^2 z,$   $[x, y, z] \in \mathbb{R}^3$   
 $f(x, y, z) = e^{xy} \arcsin z,$   $D_f = \mathbb{R} \times \mathbb{R} \times [-1, 1]$   
 $f(x, y, z, u) = 5,$   $[x, y, z] \in \mathbb{R}^3$   
 $f(x, y, z, u) = xe^{yz} \ln u,$   $D_f = \{[x, y, z, u] \in \mathbb{R}^4 : u > 0\}$ 

## Example

- Length of the day
- Length of your shadow.
- Compound interest.
- Storm radar.
- Drivers license tests.
- Google ads.



https://math.stackexchange.com/questions/703443/ best-way-to-plot-a-4-dimensional-meshgrid https://www.mathworks.com/matlabcentral/answers/ 224648-plotting-4d-with-3-vectors-and-1-matrix

## Note: Mathematica animation

We say that a function f of n variables has a limit at a point  $a \in \mathbb{R}^n$  equal to  $A \in \mathbb{R}^*$  if

 $\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \ \exists \delta \in \mathbb{R}, \delta > 0 \ \forall \mathbf{x} \in B(\mathbf{a}, \delta) \setminus \{\mathbf{a}\} : f(\mathbf{x}) \in B(A, \varepsilon).$ 



surface generated by the function f is contained between the two planes z = L + c and z = L - c.

#### 

#### Remark

- Each function has at a given point at most one limit. We write lim<sub>x→a</sub>f(x) = A.
- The function f is continuous at a if and only if  $\lim_{x\to a} f(x) = f(a)$ .
- For limits of functions of several variables one can prove similar theorems as for limits of functions of one variable (arithmetics, the sandwich theorem, ...).

## Note: Mathematica animation

1. 
$$\lim_{(x,y)\to(2,-1)} x^2 - 2xy + 3y^2 - 4x + 3y - 6$$

2. 
$$\lim_{(x,y)\to(2,-1)} \frac{2x+3y}{4x-3y}$$

3. 
$$\lim_{(x,y)\to(0,0)} \frac{x^2+xy}{x+y}$$

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## In the table there are values of a function f(x, y). Does there exist the limit

 $\lim_{(x,y)\to(0,0)}f(x,y)?$ 

$x \setminus y$	-1.0	-0.5	-0.2	0	0.2	0.5	1.0
-1.0	0.00	0.60	0.92	1.00	0.92	0.60	0.00
-0.5	-0.60	0.00	0.72	1.00	0.72	0.00	-0.6
-0.2	-0.92	-0.72	0.00	1.00	0.00	-0.72	-0.92
0	-1.00	-1.00	-1.00		-1.00	-1.00	-1.00
0.2	-0.92	-0.72	0.00	1.00	0.00	-0.72	-0.92
0.5	-0.60	0.00	0.72	1.00	0.72	0.00	-0.6
1.0	0.00	0.60	0.92	1.00	0.92	0.60	0.00

https://www.cpp.edu/conceptests/question-library/ mat214.shtml

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-0.2	-0.92	-0.72	0.00	1.00	0.00	-0.72	-0.92
0	-1.00	-1.00	-1.00		-1.00	-1.00	-1.00
0.2	-0.92	-0.72	0.00	1.00	0.00	-0.72	-0.92
0.5	-0.60	0.00	0.72	1.00	0.72	0.00	-0.6
1.0	0.00	0.60	0.92	1.00	0.92	0.60	0.00

https://www.cpp.edu/conceptests/question-library/ mat214.shtml

No limit. (Candidates are at leat 1 and -1.)

## V.2. Continuous functions of several variables

#### Definition

Let  $M \subset \mathbb{R}^n$ ,  $x \in M$ , and  $f : M \to \mathbb{R}$ . We say that f is continuous at x with respect to M, if we

 $\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \ \exists \delta \in \mathbb{R}, \delta > 0 \ \forall \mathbf{y} \in B(\mathbf{x}, \delta) \cap M \colon f(\mathbf{y}) \in B(f(\mathbf{x}), \varepsilon).$ 

We say that f is continuous at the point x if it is continuous at x with respect to a neighbourhood of x, i.e.

 $\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \; \exists \delta \in \mathbb{R}, \delta > 0 \; \forall \mathbf{y} \in B(\mathbf{x}, \delta) : f(\mathbf{y}) \in B(f(\mathbf{x}), \varepsilon).$ 

Let  $M \subset \mathbb{R}^n$  and  $f \colon M \to \mathbb{R}$ . We say that f is continuous on M if it is continuous at each point  $x \in M$  with respect to M.

#### Remark

The functions  $\pi_j \colon \mathbb{R}^n \to \mathbb{R}, \pi_j(\mathbf{x}) = x_j, 1 \le j \le n$ , are continuous on  $\mathbb{R}^n$ . They are called coordinate projections.

#### Theorem 8

Let  $M \subset \mathbb{R}^n$ ,  $\mathbf{x} \in M$ ,  $f: M \to \mathbb{R}$ ,  $g: M \to \mathbb{R}$ , and  $c \in \mathbb{R}$ . If fand g are continuous at the point  $\mathbf{x}$  with respect to M, then the functions cf, f + g a fg are continuous at  $\mathbf{x}$  with respect to M. If the function g is nonzero at  $\mathbf{x}$ , then also the function f/g is continuous at  $\mathbf{x}$  with respect to M.

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#### Theorem 9

Let  $r, s \in \mathbb{N}$ ,  $M \subset \mathbb{R}^s$ ,  $L \subset \mathbb{R}^r$ , and  $\mathbf{y} \in M$ . Let  $\varphi_1, \ldots, \varphi_r$  be functions defined on M, which are continuous at  $\mathbf{y}$  with respect to M and  $[\varphi_1(\mathbf{x}), \ldots, \varphi_r(\mathbf{x})] \in L$  for each  $\mathbf{x} \in M$ . Let  $f : L \to \mathbb{R}$ be continuous at the point  $[\varphi_1(\mathbf{y}), \ldots, \varphi_r(\mathbf{y})]$  with respect to L. Then the compound function  $F : M \to \mathbb{R}$  defined by

$$F(\mathbf{x}) = f(\varphi_1(\mathbf{x}), \ldots, \varphi_r(\mathbf{x})), \quad \mathbf{x} \in M,$$

is continuous at y with respect to M.
Where is continuous  $f(x, y) = \cos \frac{x}{y}$ ?

- A Everywhere except at the origin
- B Everywhere except along the *x*-axis.
- C Everywhere except along the *y*-axis.
- D Everywhere except along the line y = x.

Where is continuous  $f(x, y) = \cos \frac{x}{y}$ ?

- A Everywhere except at the origin
- B Everywhere except along the *x*-axis.
- C Everywhere except along the *y*-axis.
- D Everywhere except along the line y = x.

B

Where is continuous  $f(x, y) = \cos \frac{x}{y}$ ?

- A Everywhere except at the origin
- B Everywhere except along the *x*-axis.
- C Everywhere except along the *y*-axis.
- D Everywhere except along the line y = x.

В

#### Exercise

Where is continuous  $f(x, y) = \operatorname{sgn} xy$ ?

- A Everywhere except along the axes.
- B Everywhere except along the *x*-axis.
- C Everywhere except at the origin.
- D Everywhere except along the line y = x.

Where is continuous  $f(x, y) = \cos \frac{x}{y}$ ?

- A Everywhere except at the origin
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В

А

### Exercise

Where is continuous  $f(x, y) = \operatorname{sgn} xy$ ?

- A Everywhere except along the axes.
- B Everywhere except along the *x*-axis.
- C Everywhere except at the origin.
- D Everywhere except along the line y = x.

Find continuous functions (at  $\mathbb{R}^2$ )

A 
$$\ln(x^{2} + y^{2} + 1)$$
  
B  $\frac{x-y}{e^{xy}}$   
C  $\frac{\sqrt{y-1}}{x^{2}}$   
D  $\sin(2x) + x \cot(x^{3} + 2y)$   
E  $\operatorname{sgn}(x^{4} + y^{4})$ 

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E  $\operatorname{sgn}(x^{4} + y^{4})$   
A, B

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#### Theorem 10

Let f be a continuous function on  $\mathbb{R}^n$  and  $c \in \mathbb{R}$ . Then the following holds:

- (i) The set  $\{\mathbf{x} \in \mathbb{R}^n; f(\mathbf{x}) < c\}$  is open in  $\mathbb{R}^n$ .
- (ii) The set  $\{\mathbf{x} \in \mathbb{R}^n; f(\mathbf{x}) > c\}$  is open in  $\mathbb{R}^n$ .
- (iii) The set  $\{\mathbf{x} \in \mathbb{R}^n; f(\mathbf{x}) \leq c\}$  is closed in  $\mathbb{R}^n$ .
- (iv) The set  $\{x \in \mathbb{R}^n; f(x) \ge c\}$  is closed in  $\mathbb{R}^n$ .
- (v) The set  $\{x \in \mathbb{R}^n; f(x) = c\}$  is closed in  $\mathbb{R}^n$ .

#### Example

$$f(x,y) = x^2 + y^2,$$

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# Partial derivatives



https://www.wikihow.com/ Take-Partial-Derivatives http: //calcnet.cst.cmich.edu/ faculty/angelos/m533/ lectures/pderv.htm

### Animation.

# Definition

# Let *f* be a function, $a \in \mathbb{R}$ .

$$f'(a) = \lim_{t \to 0} \frac{f(a+t) - f(a)}{t}.$$

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#### Definition

Let *f* be a function,  $a \in \mathbb{R}$ .

$$f'(a) = \lim_{t \to 0} \frac{f(a+t) - f(a)}{t}$$

Set 
$$e^{j} = [0, ..., 0, \frac{1}{j^{\text{th coordinate}}}, 0, ..., 0].$$

#### Definition

Let *f* be a function of *n* variables,  $j \in \{1, ..., n\}$ ,  $a \in \mathbb{R}^n$ . Then the number

$$\frac{\partial f}{\partial x_j}(\boldsymbol{a}) = \lim_{t \to 0} \frac{f(\boldsymbol{a} + t\boldsymbol{e}^j) - f(\boldsymbol{a})}{t}$$
$$= \lim_{t \to 0} \frac{f(a_1, \dots, a_{j-1}, a_j + t, a_{j+1}, \dots, a_n) - f(a_1, \dots, a_n)}{t}$$

is called the partial derivative (of first order) of function f according to *j*th variable at the point a (if the limit exists).

Find 
$$\frac{\partial f}{\partial x}$$
, if  $f(x, y) = x^3 + 3x^2y - 5x - 7y^3 + y - 5$ 

A 
$$\frac{\partial f}{\partial x} = 3x^2 + 6xy - 5$$
  
B  $\frac{\partial f}{\partial x} = x^3 + 3 - 21y^2 + 1 - 5$   
C  $\frac{\partial f}{\partial x} = 3x^2 - 21y^2 + 1$   
D  $\frac{\partial f}{\partial x} = 3x^2 + 6xy - 5 - 7y^3 + y$ 

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Find 
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A

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Find 
$$\frac{\partial f}{\partial y}$$
, if  $f(x, y) = x^2 \ln(x^2 y)$   
A  $\frac{\partial f}{\partial y} = \frac{2x}{y}$   
B  $\frac{\partial f}{\partial y} = \frac{1}{y}$   
C  $\frac{\partial f}{\partial y} = \frac{x^2}{y}$   
D  $\frac{\partial f}{\partial y} = \frac{1}{x^2 y}$ 

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D  $\frac{\partial f}{\partial y} = \frac{1}{x^2 y}$ 

#### С

According to: https://www.wiley.com/college/ hugheshallett/0470089148/conceptests/concept.pdf

The values of a function f(x, y) are in the table. Which statement is most accurate? (In the left columnt there is x, in the first row there is y.)

A 
$$\frac{\partial f}{\partial x}(1,2) \approx -1$$
  
B  $\frac{\partial f}{\partial y}(1,2) \approx 2$   
C  $\frac{\partial f}{\partial x}(3,2) \approx 1$   
D  $\frac{\partial f}{\partial y}(3,2) \approx 4$ 

https://www.cpp.edu/conceptests/question-library/
mat214.shtml

$x \setminus y$	0	1	2	3
0	3	5	7	9
1	2	4	6	8
2	1	3	5	7
3	0	2	4	6

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A, B

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0	3	5	7	9
1	2	4	6	8
2	1	3	5	7
3	0	2	4	6



A  $\frac{\partial f}{\partial x} > 0, \frac{\partial f}{\partial y} > 0$ **B**  $\frac{\partial f}{\partial x} < 0, \frac{\partial f}{\partial y} > 0$ C  $\frac{\partial f}{\partial x} > 0, \frac{\partial f}{\partial y} < 0$ D  $\frac{\partial f}{\partial x} < 0, \frac{\partial f}{\partial y} < 0$ 

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$$A \quad \frac{\partial f}{\partial x} > 0, \ \frac{\partial f}{\partial y} > 0$$
$$B \quad \frac{\partial f}{\partial x} < 0, \ \frac{\partial f}{\partial y} > 0$$
$$C \quad \frac{\partial f}{\partial x} > 0, \ \frac{\partial f}{\partial y} < 0$$
$$D \quad \frac{\partial f}{\partial x} < 0, \ \frac{\partial f}{\partial y} < 0$$

### В

https://www.cpp.edu/conceptests/question-library/
mat214.shtml

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1. Let  $f(x, y, z) = x^2 + z + 3$ . Then the partial derivative  $\frac{\partial f}{\partial y}$  is not defined, because there is no y in the function.

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2. Is there a function f(x, y) such that  $\frac{\partial f}{\partial y} = 3y^2$  and  $\frac{\partial f}{\partial x} = 3x^2$ ?

- 1. Let  $f(x, y, z) = x^2 + z + 3$ . Then the partial derivative  $\frac{\partial f}{\partial y}$  is not defined, because there is no y in the function. False,  $\frac{\partial f}{\partial y} = 0$ .
- 2. Is there a function f(x, y) such that  $\frac{\partial f}{\partial y} = 3y^2$  and  $\frac{\partial f}{\partial x} = 3x^2$ ? Yes. For example  $f(x, y) = x^3 + y^3$ .

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#### Exercise

Find a function, which is not constant, but  $\frac{\partial f}{\partial x} = 0$  for every *x*.

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#### Exercise

Find a function, which is not constant, but  $\frac{\partial f}{\partial x} = 0$  for every *x*. For example  $f(x, y) = y^2 + 4$ .

### Definition

Let  $G \subset \mathbb{R}^n$  be a non-empty open set. If a function  $f: G \to \mathbb{R}$  has all partial derivatives continuous at each point of the set G (i.e. the function  $\mathbf{x} \mapsto \frac{\partial f}{\partial x_j}(\mathbf{x})$  is continuous on G for each  $j \in \{1, \ldots, n\}$ ), then we say that f is of the class  $\mathcal{C}^1$  on G. The set of all of these functions is denoted by  $\mathcal{C}^1(G)$ .

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#### Remark

If  $G \subset \mathbb{R}^n$  is a non-empty open set and  $\operatorname{and} f, g \in \mathcal{C}^1(G)$ , then  $f + g \in \mathcal{C}^1(G), f - g \in \mathcal{C}^1(G)$ , and  $fg \in \mathcal{C}^1(G)$ . If moreover  $g(\mathbf{x}) \neq 0$  for each  $\mathbf{x} \in G$ , then  $f/g \in \mathcal{C}^1(G)$ .

Find functions, which are  $C^1(\mathbb{R}^2)$ .

A 
$$e^{xy}$$
  
B  $\sqrt[3]{x^2 + y^2}$   
C  $\frac{\sin(x-2y)}{2+x^2+y^2}$   
D  $\ln \frac{y}{x}$ 

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Find functions, which are  $C^1(\mathbb{R}^2)$ . A  $e^{xy}$  C  $\frac{\sin(x-2y)}{2+x^2+y^2}$ B  $\sqrt[3]{x^2+y^2}$  D  $\ln \frac{y}{x}$ A, C

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# Example

$$f(x, y) = \sqrt{100 - x^2 - y^2}$$



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### Example

$$f(x, y) = x^2 + y^2$$



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# Example

$$f(x,y) = 5\sqrt{x^2 + y^2}$$



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### Definition

Suppose that the function f has a finite derivative at a point  $a \in \mathbb{R}$ . The line

$$T_a = \{ [x, y] \in \mathbb{R}^2; \ y = f(a) + f'(a)(x - a) \}$$

is called the tangent to the graph of f at the point [a, f(a)].

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### Definition

Let  $G \subset \mathbb{R}^n$  be an open set,  $a \in G$ , and  $f \in C^1(G)$ . Then the graph of the function

$$T: \mathbf{x} \mapsto f(\mathbf{a}) + \frac{\partial f}{\partial x_1}(\mathbf{a})(x_1 - a_1) + \frac{\partial f}{\partial x_2}(\mathbf{a})(x_2 - a_2) \\ + \dots + \frac{\partial f}{\partial x_n}(\mathbf{a})(x_n - a_n), \quad \mathbf{x} \in \mathbb{R}^n,$$

is called the tangent hyperplane to the graph of the function f at the point [a, f(a)].

## Find the tangent plane of a function f(x, y) = xy at the point (2, 3).

A 
$$z-6 = x(x-2) + y(y-3)$$
  
B  $z-6 = y(x-2) + x(y-3)$   
C  $z-6 = 2(x-2) + 3(y-3)$   
D  $z-6 = 3(x-2) + 2(y-3)$ 

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### Exercise

Find the tangent plane of a function  $f(x, y, z, u) = \ln(xy + z^2 - u)$  at the point a = (1, 0, 2, 3).

Find the tangent plane of a function f(x, y) = xy at the point (2, 3).

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$$z-6 = x(x-2) + y(y-3)$$
  
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C  $z-6 = 2(x-2) + 3(y-3)$   
D  $z-6 = 3(x-2) + 2(y-3)$ 

### Exercise

Find the tangent plane of a function  $f(x, y, z, u) = \ln(xy + z^2 - u)$  at the point a = (1, 0, 2, 3). v - 0 = 0(x - 1) + 1(y - 0) + 4(z - 2) - 1(u - 3)v = y + 4z - u - 5

### Theorem 11 (tangent hyperplane)

Let  $G \subset \mathbb{R}^n$  be an open set,  $a \in G$ ,  $f \in C^1(G)$ , and let T be a function whose graph is the tangent hyperplane of the function f at the point [a, f(a)]. Then

$$\lim_{\mathbf{x}\to\mathbf{a}}\frac{f(\mathbf{x})-T(\mathbf{x})}{\rho(\mathbf{x},\mathbf{a})}=0.$$

### Theorem 11 (tangent hyperplane)

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### Theorem 12

Let  $G \subset \mathbb{R}^n$  be an open non-empty set and  $f \in C^1(G)$ . Then f is continuous on G.

### Theorem 11 (tangent hyperplane)

Let  $G \subset \mathbb{R}^n$  be an open set,  $a \in G$ ,  $f \in C^1(G)$ , and let T be a function whose graph is the tangent hyperplane of the function f at the point [a, f(a)]. Then

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### Theorem 12

Let  $G \subset \mathbb{R}^n$  be an open non-empty set and  $f \in C^1(G)$ . Then f is continuous on G.

#### Remark

Existence of partial derivatives at *a* **does not** imply continuity at *a*.

### Theorem 13 (derivative of a composite function; chain rule)

Let  $r, s \in \mathbb{N}$  and let  $G \subset \mathbb{R}^s$ ,  $H \subset \mathbb{R}^r$  be open sets. Let  $\varphi_1, \ldots, \varphi_r \in C^1(G), f \in C^1(H)$  and  $[\varphi_1(\mathbf{x}), \ldots, \varphi_r(\mathbf{x})] \in H$  for each  $\mathbf{x} \in G$ . Then the compound function  $F \colon G \to \mathbb{R}$  defined by

$$F(\mathbf{x}) = f(\varphi_1(\mathbf{x}), \varphi_2(\mathbf{x}), \dots, \varphi_r(\mathbf{x})), \quad \mathbf{x} \in G$$

is of the class  $C^1$  on G. Let  $\mathbf{a} \in G$  and  $\mathbf{b} = [\varphi_1(\mathbf{a}), \dots, \varphi_r(\mathbf{a})]$ . Then for each  $j \in \{1, \dots, s\}$  we have

$$\frac{\partial F}{\partial x_j}(\boldsymbol{a}) = \sum_{i=1}^r \frac{\partial f}{\partial y_i}(\boldsymbol{b}) \frac{\partial \varphi_i}{\partial x_j}(\boldsymbol{a}).$$

### Remark

Let f(x, y, z) be a differentiable function, let  $x = g_1(u, v)$ ,  $y = g_2(u, v)$ ,  $z = g_3(u, v)$ , where  $g_1, g_2, g_3$  are differentiable functions. Then for  $h(u, v) = f(g_1(u, v), g_2(u, v), g_3(u, v))$  we have

$$\frac{\partial h}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$
$$\frac{\partial h}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}$$

$$\frac{d}{d} = \frac{d}{d} \times \frac{d}{d}$$

Let  $h(u, v) = \sin x \cos y$ , where  $x = (u - v)^2$  and  $y = u^2 - v^2$ . Find  $\partial h / \partial u$  a  $\partial h / \partial v$ .

Let  $h(u, v) = \sin x \cos y$ , where  $x = (u - v)^2$  and  $y = u^2 - v^2$ . Find  $\partial h / \partial u$  a  $\partial h / \partial v$ .

$$\frac{\partial h}{\partial u} = \cos(u-v)^2 \cos(u^2 - v^2) 2(u-v) - \sin(u-v)^2 \sin(u^2 - v^2) 2u$$
$$\frac{\partial h}{\partial v} = -\cos(u-v)^2 \cos(u^2 - v^2) 2(u-v) + \sin(u-v)^2 \sin(u^2 - v^2) 2v$$

Let h(u, v) = xy, where  $x = u \cos v$  and  $y = u \sin v$ . Then for  $\partial h / \partial v$  we have

A  $\frac{\partial h}{\partial v} = 0$ B  $\frac{\partial h}{\partial v} = u^2 \cos(2v)$ C  $\frac{\partial h}{\partial v} = -u^3 \sin^2 v \cos v + u^3 \sin v \cos^2 v$ D Something else.

Let h(u, v) = xy, where  $x = u \cos v$  and  $y = u \sin v$ . Then for  $\partial h / \partial v$  we have

A 
$$\frac{\partial h}{\partial v} = 0$$
  
B  $\frac{\partial h}{\partial v} = u^2 \cos(2v)$   
C  $\frac{\partial h}{\partial v} = -u^3 \sin^2 v \cos v + u^3 \sin v \cos^2 v$   
D Something else.

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# V.4. Implicit function theorem and Lagrange multiplier theorem



# V.4. Implicit function theorem and Lagrange multiplier theorem



## V.4. Implicit function theorem and Lagrange multiplier theorem





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### Theorem 14 (implicit function)

Let  $G \subset \mathbb{R}^{n+1}$  be an open set,  $F \colon G \to \mathbb{R}$ , and  $\tilde{\mathbf{x}} \in \mathbb{R}^n$ ,  $\tilde{y} \in \mathbb{R}$ such that  $[\tilde{\mathbf{x}}, \tilde{y}] \in G$ . Suppose that (i)  $F \in C^1(G)$ , (ii)  $F(\tilde{\mathbf{x}}, \tilde{y}) = 0$ , (iii)  $\frac{\partial F}{\partial \nu}(\tilde{\mathbf{x}}, \tilde{y}) \neq 0$ .

Then there exist a neighbourhood  $U \subset \mathbb{R}^n$  of the point  $\tilde{\mathbf{x}}$  and a neighbourhood  $V \subset \mathbb{R}$  of the point  $\tilde{y}$  such that for each  $\mathbf{x} \in U$  there exists a unique  $y \in V$  satisfying  $F(\mathbf{x}, y) = 0$ . If we denote this y by  $\varphi(\mathbf{x})$ , then the resulting function  $\varphi$  is in  $C^1(U)$  and

$$\frac{\partial \varphi}{\partial x_j}(\boldsymbol{x}) = -\frac{\frac{\partial F}{\partial x_j}(\boldsymbol{x},\varphi(\boldsymbol{x}))}{\frac{\partial F}{\partial y}(\boldsymbol{x},\varphi(\boldsymbol{x}))} \quad for \, \boldsymbol{x} \in U, \, j \in \{1,\ldots,n\}$$

### Theorem

Let  $G \subset \mathbb{R}^{n+1}$  be an open set,  $F \colon G \to \mathbb{R}$ , and  $\tilde{\mathbf{x}} \in \mathbb{R}^n$ ,  $\tilde{y} \in \mathbb{R}$ such that  $[\tilde{\mathbf{x}}, \tilde{y}] \in G$ . Suppose that (i)  $F \in C^1(G)$ , (ii)  $F(\tilde{\mathbf{x}}, \tilde{y}) = 0$ , (iii)  $\frac{\partial F}{\partial y}(\tilde{\mathbf{x}}, \tilde{y}) \neq 0$ .

Then there exists a neighbourhood ...

### Exercise

Consider these exercises. Which condition is NOT satisfied?

A 
$$x^2 + y^3 = 4$$
 at  $(2, 0)$   
B  $y - \frac{1}{2} \sin y = x$  at  $(\pi, \pi)$   
C  $\sin(xy) + x^2 + y^2 = 1$  at  $(0, 3)$   
D  $|x| + e^{x+y} = 1$  at  $(0, 0)$ 

### Theorem

Let  $G \subset \mathbb{R}^{n+1}$  be an open set,  $F \colon G \to \mathbb{R}$ , and  $\tilde{\mathbf{x}} \in \mathbb{R}^n$ ,  $\tilde{y} \in \mathbb{R}$ such that  $[\tilde{\mathbf{x}}, \tilde{y}] \in G$ . Suppose that (i)  $F \in C^1(G)$ , (ii)  $F(\tilde{\mathbf{x}}, \tilde{y}) = 0$ , (iii)  $\frac{\partial F}{\partial y}(\tilde{\mathbf{x}}, \tilde{y}) \neq 0$ .

Then there exists a neighbourhood ...

### Exercise

Consider these exercises. Which condition is NOT satisfied?

A 
$$x^{2} + y^{3} = 4$$
 at  $(2, 0)$   
B  $y - \frac{1}{2} \sin y = x$  at  $(\pi, \pi)$   
C  $\sin(xy) + x^{2} + y^{2} = 1$  at  
 $(0, 3)$   
D  $|x| + e^{x+y} = 1$  at  $(0, 0)$   
A iii,  
B all is ok  
C ii,  
D i

### Definition

Let  $G \subset \mathbb{R}^n$  be an open set,  $a \in G$ , and  $f \in C^1(G)$ . The gradient of f at the point a is the vector

$$abla f(\boldsymbol{a}) = \left[ \frac{\partial f}{\partial x_1}(\boldsymbol{a}), \frac{\partial f}{\partial x_2}(\boldsymbol{a}), \dots, \frac{\partial f}{\partial x_n}(\boldsymbol{a}) \right]$$

### Exercise

Find the gradient of  $f(x, y, z) = y \cos^3(x^2 z)$  at the point [2, 1, 0]:

A	(1/5, 0, 1/5)	С	(0,1,0)
B	(0, 1, 1/5)	D	(1, 0, 1/2)

### Definition

Let  $G \subset \mathbb{R}^n$  be an open set,  $a \in G$ , and  $f \in C^1(G)$ . The gradient of f at the point a is the vector

$$abla f(\boldsymbol{a}) = \left[ \frac{\partial f}{\partial x_1}(\boldsymbol{a}), \frac{\partial f}{\partial x_2}(\boldsymbol{a}), \dots, \frac{\partial f}{\partial x_n}(\boldsymbol{a}) \right]$$

### Exercise

Find the gradient of  $f(x, y, z) = y \cos^3(x^2 z)$  at the point [2, 1, 0]:

A	(1/5, 0, 1/5)	С	(0,1,0)
B	(0, 1, 1/5)	D	$\left(1,0,1/2\right)$

С

### Remark

The gradient of f at a points in the direction of steepest growth of f at a. At every point, the gradient is perpendicular to the contour of f.

### Exercise

The bicyclist is on a trip up the hill, which can be described as  $f(x, y) = 25 - 2x^2 - 4y^2$ . When she is at the point [1, 1, 19], it starts to rain, so she decides to go down the hill as steeply as possible (so that she is down quickly). In what direction will she start her decline?

A 
$$(-4x; -8y)$$
C  $(-4; -8)$ B  $(4x; 8y)$ D  $(4; 8)$ 

### Remark

The gradient of f at a points in the direction of steepest growth of f at a. At every point, the gradient is perpendicular to the contour of f.

### Exercise

D

The bicyclist is on a trip up the hill, which can be described as  $f(x, y) = 25 - 2x^2 - 4y^2$ . When she is at the point [1, 1, 19], it starts to rain, so she decides to go down the hill as steeply as possible (so that she is down quickly). In what direction will she start her decline?

A 
$$(-4x; -8y)$$
C  $(-4; -8)$ B  $(4x; 8y)$ D  $(4; 8)$ 



### Definition

Let  $M \subset \mathbb{R}^n$ ,  $x \in M$ , and let f be a function defined at least on M (i.e.  $M \subset D_f$ ). We say that f attains at the point x its

- maximum on M if  $f(\mathbf{y}) \leq f(\mathbf{x})$  for every  $\mathbf{y} \in M$ ,
- local maximum with respect to M if there exists  $\delta > 0$ such that  $f(\mathbf{y}) \le f(\mathbf{x})$  for every  $\mathbf{y} \in B(\mathbf{x}, \delta) \cap M$ ,
- strict local maximum with respect to *M* if there exists  $\delta > 0$  such that  $f(\mathbf{y}) < f(\mathbf{x})$  for every  $\mathbf{y} \in (B(\mathbf{x}, \delta) \setminus \{\mathbf{x}\}) \cap M$ .

The notions of a minimum, a local minimum, and a strict local minimum with respect to *M* are defined in analogous way.

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### Definition

Let  $M \subset \mathbb{R}^n$ ,  $x \in M$ , and let f be a function defined at least on M (i.e.  $M \subset D_f$ ). We say that f attains at the point x its

- maximum on M if  $f(\mathbf{y}) \leq f(\mathbf{x})$  for every  $\mathbf{y} \in M$ ,
- local maximum with respect to M if there exists  $\delta > 0$ such that  $f(\mathbf{y}) \le f(\mathbf{x})$  for every  $\mathbf{y} \in B(\mathbf{x}, \delta) \cap M$ ,
- strict local maximum with respect to *M* if there exists δ > 0 such that f(y) < f(x) for every y ∈ (B(x, δ) \ {x}) ∩ M.</li>

The notions of a minimum, a local minimum, and a strict local minimum with respect to *M* are defined in analogous way.

### Definition

We say that a function f attains a local maximum at a point  $x \in \mathbb{R}^n$  if x is a local maximum with respect to some neighbourhood of x.



### Theorem 15 (attaining extrema)

Let  $M \subset \mathbb{R}^n$  be a non-empty compact set and  $f: M \to \mathbb{R}$  a function continuous on M. Then f attains its maximum and minimum on M.

### Corollary

Let  $M \subset \mathbb{R}^n$  be a non-empty compact set and  $f: M \to \mathbb{R}$  a continuous function on M. Then f is bounded on M.

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### Theorem 16 (necessary condition of the existence of local extremum)

Let  $G \subset \mathbb{R}^n$  be an open set,  $a \in G$ , and suppose that a function  $f: G \to \mathbb{R}$  has a local extremum (i.e. a local maximum or a local minimum) at the point a. Then for each  $j \in \{1, ..., n\}$  the following holds:

The partial derivative  $\frac{\partial f}{\partial x_j}(\boldsymbol{a})$  either does not exist or it is equal to zero.



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#### Definition

Let  $G \subset \mathbb{R}^n$  be an open set,  $a \in G, f \in C^1(G)$ , and  $\nabla f(a) = o$ . Then the point *a* is called a stationary (or critical) point of the function *f*.

$$f(x, y) = x^2 + y^2$$



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$$f(x,y) = e^{-x^2 - y^2}$$



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$$f(x,y) = x + 2y - 4$$



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$$f(x,y) = \sqrt{x^2 + y^2}$$



$$f(x, y) = x^2 - y^2$$



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- 1. Consider the points A, B, C, D, E. Find the critical points.
- 2. Which of these points are probably points of
  - 2.1 local maximum,
  - 2.2 local minimum,
  - 2.3 saddle point?



Figure: Calculus, 6th Edition; Hughes-Hallett, Gleason, McCallum et

- 1. Consider the points A, B, C, D, E. Find the critical points.
- 2. Which of these points are probably points of
  - 2.1 local maximum B
  - 2.2 local minimum E, G
  - 2.3 saddle point C, D, F



Figure: Calculus, 6th Edition; Hughes-Hallett, Gleason, McCallum et

#### Definition

Let  $G \subset \mathbb{R}^n$  be an open set,  $f: G \to \mathbb{R}$ ,  $i, j \in \{1, ..., n\}$ , and suppose that  $\frac{\partial f}{\partial x_i}(\mathbf{x})$  exists finite for each  $\mathbf{x} \in G$ . Then the partial derivative of the second order of the function f according to *i*th and *j*th variable at a point  $\mathbf{a} \in G$  is defined by

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{a}) = \frac{\partial \left(\frac{\partial f}{\partial x_i}\right)}{\partial x_j}(\boldsymbol{a})$$

If i = j then we use the notation  $\frac{\partial^2 f}{\partial x_i^2}(\boldsymbol{a})$ .

Similarly we define higher order partial derivatives.

#### Exercise

Find the second partial derivatives of the function  $f(x, y) = x^2 + xy + y^2$ .

- Find  $\frac{\partial^2 f}{\partial x \partial y}$ , if  $f(x, y) = e^{xy}$ A  $e^{xy}$ B  $ye^{xy}$ 
  - **C**  $x^2 e^{xy}$
  - D  $e^{xy}(xy+1)$

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Find  $\frac{\partial^2 f}{\partial x \partial y}$ , if  $f(x, y) = e^{xy}$ A  $e^{xy}$ B  $y e^{xy}$ C  $x^2 e^{xy}$ D  $e^{xy}(xy + 1)$ D

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Find  $\frac{\partial^2 f}{\partial x \partial y}$ , if  $f(x, y) = e^{xy}$ A  $e^{xy}$ B  $y e^{xy}$ C  $x^2 e^{xy}$ 

D 
$$e^{xy}(xy+1)$$

D

#### Exercise

Find  $\frac{\partial^2 f}{\partial y \partial x}$ , if  $f(x, y) = e^{xy}$ A  $e^{xy}$ B  $y e^{xy}$ C  $x^2 e^{xy}$ D  $e^{xy}(xy + 1)$ 

Find  $\frac{\partial^2 f}{\partial x \partial y}$ , if  $f(x, y) = e^{xy}$ A  $e^{xy}$ B  $y e^{xy}$ C  $x^2 e^{xy}$ 

D 
$$e^{xy}(xy+1)$$

D

## Exercise

Find 
$$\frac{\partial^2 f}{\partial y \partial x}$$
, if  $f(x, y) = e^{xy}$   
A  $e^{xy}$   
B  $y e^{xy}$   
C  $x^2 e^{xy}$   
D  $e^{xy}(xy + 1)$   
D

## Remark

In general it is **not true** that 
$$\frac{\partial^2 f}{\partial x_i \partial x_i}(\boldsymbol{a}) = \frac{\partial^2 f}{\partial x_i \partial x_i}(\boldsymbol{a})$$
.

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#### Remark

In general it is **not true** that 
$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{a}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\boldsymbol{a})$$
.

#### Theorem 17 (interchanging of partial derivatives)

Let  $i, j \in \{1, ..., n\}$  and suppose that a function f has both partial derivatives  $\frac{\partial^2 f}{\partial x_i \partial x_j}$  and  $\frac{\partial^2 f}{\partial x_j \partial x_i}$  on a neighbourhood of apoint  $\mathbf{a} \in \mathbb{R}^n$  and that these functions are continuous at  $\mathbf{a}$ . Then

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{a}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\boldsymbol{a}).$$

# You follow the red route. Where is the highest point of your trip?



# Where is the minimum and maximum of the function f(x, y) = y along the curve?



https://www.cpp.edu/conceptests/
question-library/mat214.shtml

# Where is the minimum and maximum of the function f(x, y) = y along the curve?



https://www.cpp.edu/conceptests/
question-library/mat214.shtml min: A, max B

#### Theorem 18 (Lagrange multiplier theorem)

Let  $G \subset \mathbb{R}^2$  be an open set,  $f, g \in C^1(G)$ ,  $M = \{[x, y] \in G; g(x, y) = 0\}$  and let  $[\tilde{x}, \tilde{y}] \in M$  be a point of local extremum of f with respect to M. Then at least one of the following conditions holds:

(I)  $\nabla g(\tilde{x}, \tilde{y}) = \boldsymbol{o}$ ,

(II) there exists  $\lambda \in \mathbb{R}$  satisfying

$$rac{\partial f}{\partial x}( ilde{x}, ilde{y}) + \lambda rac{\partial g}{\partial x}( ilde{x}, ilde{y}) = 0, \ rac{\partial f}{\partial y}( ilde{x}, ilde{y}) + \lambda rac{\partial g}{\partial y}( ilde{x}, ilde{y}) = 0.$$

#### Remark

The number  $\lambda$  is called the Lagrange multiplier.

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## V.5. Concave and quasiconcave functions



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• **b** 

• a
b a



$$\boldsymbol{a} = 1 \cdot \boldsymbol{a} + 0 \cdot \boldsymbol{b} = \boldsymbol{a} + 0 \cdot (\boldsymbol{b} - \boldsymbol{a})$$



## $\boldsymbol{b} = 0 \cdot \boldsymbol{a} + 1 \cdot \boldsymbol{b} = \boldsymbol{a} + 1 \cdot (\boldsymbol{b} - \boldsymbol{a})$



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$$t \cdot \boldsymbol{a} + (1-t) \cdot \boldsymbol{b} = \boldsymbol{a} + (1-t) \cdot (\boldsymbol{b} - \boldsymbol{a})$$

## Definition

Let  $M \subset \mathbb{R}^n$ . We say that *M* is convex if

$$\forall \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{M} \; \forall t \in [0, 1] \colon t\boldsymbol{x} + (1 - t)\boldsymbol{y} \in \boldsymbol{M}.$$

## Exercise





## Definition

Let  $M \subset \mathbb{R}^n$ . We say that *M* is convex if

$$\forall \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{M} \; \forall t \in [0, 1] \colon t\boldsymbol{x} + (1 - t)\boldsymbol{y} \in \boldsymbol{M}.$$

## Exercise



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#### Exercise

## Find convex sets



#### Exercise

## Find convex sets



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## Concave and convex functions



https://math24.net/convex-functions.html

## Concave and convex functions

## Definition

Let  $M \subset \mathbb{R}^n$  be a convex set and f a function defined on M. We say that f is

• concave on M if

$$\forall \boldsymbol{a}, \boldsymbol{b} \in M \forall t \in [0, 1]: f(t\boldsymbol{a} + (1 - t)\boldsymbol{b}) \ge tf(\boldsymbol{a}) + (1 - t)f(\boldsymbol{b}),$$

• strictly concave on *M* if

$$\forall \boldsymbol{a}, \boldsymbol{b} \in M, \boldsymbol{a} \neq \boldsymbol{b} \ \forall t \in (0, 1):$$
  
 $f(t\boldsymbol{a} + (1 - t)\boldsymbol{b}) > tf(\boldsymbol{a}) + (1 - t)f(\boldsymbol{b}).$ 

#### Remark

By changing the inequalities to the opposite we obtain a definition of a *convex* and a *strictly convex* function.





A function f is convex (strictly convex) if and only if the function -f is concave (strictly concave). All the theorems in this section are formulated for concave and strictly concave functions. They have obvious analogies that hold for convex and strictly convex functions.

- If a function *f* is strictly concave on *M*, then it is concave on *M*.
- Let f be a concave function on M. Then f is strictly concave on M if and only if the graph of f "does not contain a segment", i.e.

$$\neg (\exists \boldsymbol{a}, \boldsymbol{b} \in M, \boldsymbol{a} \neq \boldsymbol{b}, \forall t \in [0, 1]:$$
$$f(t\boldsymbol{a} + (1 - t)\boldsymbol{b}) = tf(\boldsymbol{a}) + (1 - t)f(\boldsymbol{b}))$$



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## Theorem 19

Let f be a function concave on an open convex set  $G \subset \mathbb{R}^n$ . Then f is continuous on G.



Figure: https://math24.net/convex-functions.html

Theorem 20 (characterisation of concave functions of the class  $C^1$ )

Let  $G \subset \mathbb{R}^n$  be a convex open set and  $f \in C^1(G)$ . Then the function f is concave on G if and only if

$$\forall \mathbf{x}, \mathbf{y} \in G: f(\mathbf{y}) \leq f(\mathbf{x}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{x})(y_i - x_i).$$

Theorem 20 (characterisation of concave functions of the class  $C^1$ )

Let  $G \subset \mathbb{R}^n$  be a convex open set and  $f \in C^1(G)$ . Then the function f is concave on G if and only if

$$\forall \boldsymbol{x}, \boldsymbol{y} \in G: f(\boldsymbol{y}) \leq f(\boldsymbol{x}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_{i}}(\boldsymbol{x})(y_{i} - x_{i}).$$

#### Corollary 21

Let  $G \subset \mathbb{R}^n$  be a convex open set,  $f \in C^1(G)$ , and let  $\mathbf{a} \in G$  be a critical point of f (i.e.  $\nabla f(\mathbf{a}) = \mathbf{o}$ ). If f is concave on G, then  $\mathbf{a}$  is a maximum point of f on G. If f is strictly concave on G, then  $\mathbf{a}$  is a strict maximum point of f on G.

#### Theorem 22 (level sets of concave functions)

Let f be a function concave on a convex set  $M \subset \mathbb{R}^n$ . Then for each  $\alpha \in \mathbb{R}$  the set  $Q_\alpha = \{ \mathbf{x} \in M; f(\mathbf{x}) \ge \alpha \}$  is convex.



## Definition

Let  $M \subset \mathbb{R}^n$  be a convex set and let f be a function defined on M. We say that f is

• quasiconcave on M if

 $\forall \boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{M} \, \forall t \in [0, 1] : f(t\boldsymbol{a} + (1 - t)\boldsymbol{b}) \geq \min\{f(\boldsymbol{a}), f(\boldsymbol{b})\},\$ 

• strictly quasiconcave on *M* if

$$\forall \boldsymbol{a}, \boldsymbol{b} \in M, \boldsymbol{a} \neq \boldsymbol{b}, \forall t \in (0, 1):$$
$$f(t\boldsymbol{a} + (1 - t)\boldsymbol{b}) > \min\{f(\boldsymbol{a}), f(\boldsymbol{b})\}.$$

#### Remark

By changing the inequalities to the opposite and changing the minimum to a maximum we obtain a definition of a *quasiconvex* and a *strictly quasiconvex* function.

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# Not quasiconcave



A function f is quasiconvex (strictly quasiconvex) if and only if the function -f is quasiconcave (strictly quasiconcave). All the theorems in this section are formulated for quasiconcave and strictly quasiconcave functions. They have obvious analogies that hold for quasiconvex and strictly quasiconvex functions.

A function f is quasiconvex (strictly quasiconvex) if and only if the function -f is quasiconcave (strictly quasiconcave). All the theorems in this section are formulated for quasiconcave and strictly quasiconcave functions. They have obvious analogies that hold for quasiconvex and strictly quasiconvex functions.

#### Remark

- If a function *f* is strictly quasiconcave on *M*, then it is quasiconcave on *M*.
- Let *f* be a quasiconcave function on *M*. Then *f* is strictly quasiconcave on *M* if and only if the graph of *f* "does not contain a horizontal segment", i.e.

$$\neg \big(\exists \boldsymbol{a}, \boldsymbol{b} \in M, \boldsymbol{a} \neq \boldsymbol{b}, \forall t \in [0, 1] : f(t\boldsymbol{a} + (1 - t)\boldsymbol{b}) = f(\boldsymbol{a})\big).$$

Let  $M \subset \mathbb{R}^n$  be a convex set and f a function defined on M.

- If f is concave on M, then f is quasiconcave on M.
- If *f* is strictly concave on *M*, then *f* is strictly quasiconcave on *M*.

Let  $M \subset \mathbb{R}^n$  be a convex set and f a function defined on M.

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Theorem 23 (characterization of quasiconcave functions using level sets)

Let  $M \subset \mathbb{R}^n$  be a convex set and f a function defined on M. Then f is quasiconcave on M if and only if for each  $\alpha \in \mathbb{R}$  the set  $Q_\alpha = \{ \mathbf{x} \in M; f(\mathbf{x}) \ge \alpha \}$  is convex.

#### Exercise

## Find quasiconcave functions:








### Find quasiconcave functions:









В



### Theorem 24 (a uniqueness of an extremum)

Let f be a strictly quasiconcave function on a convex set  $M \subset \mathbb{R}^n$ . Then there exists at most one point of maximum of f.

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### Corollary

Let  $M \subset \mathbb{R}^n$  be a convex, closed, bounded and nonempty set and f a continuous and strictly quasiconcave function on M. Then f attains its maximum at exactly one point.

### Theorem 25 (sufficient condition for concave and convex functions in $\mathbb{R}^2$ )

Let  $G \subset \mathbb{R}^2$  be convex and  $f \in C^2(G)$ . If  $\frac{\partial^2 f}{\partial x^2} \leq 0$ ,  $\frac{\partial^2 f}{\partial y^2} \leq 0$ , and  $\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \geq 0$  hold on G, then f is concave on G.

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#### Exercise

Decide if the following functions are convex or concave on  $\mathbb{R}^2$ .

A 
$$f(x, y) = x^{2} + y^{2}$$
  
B  $f(x, y) = -x^{4} - y^{4}$   
C  $f(x, y) = -x^{2} + y^{2}$ 

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#### Exercise

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A convex, B concave, C neither convex, nor concave

э.

### Good luck in the exam period!