Mathematics II - Integrals

23/24

- Antiderivative and the Riemann Integral
- Matrix calculus
- Functions of several variables

VII. Antiderivatives and Riemann integral

VII.1. Antiderivatives

Definition

Let f be a function defined on an open interval I. We say that a function $F\colon I \to \mathbb{R}$ is an antiderivative of f on I if for each $x \in I$ the derivative $F'(x)$ exists and $F'(x) = f(x)$.

Exercise

Connect the functions in the left column with their antiderivatives on the right.

1. 0
2. 1
3. x
4. $\cos x$
5. $\sin x$

A $-\cos x$

B $\sin x$

C x

D 1

E $\frac{x^2}{2}$

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2. 1	B $\sin x$
3. x	C x
4. $\cos x$	D 1
5. $\sin x$	E $\frac{x^2}{2}$

1D, 2C, 3E, 4B, 5A

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Remark

- An antiderivative of f is sometimes called a function *primitive* to f or *indefinite integral* of f.
- If F is an antiderivative of f on I, then F is continuous on I.

Definition

Let f be a function defined on an open interval I. We say that a function $F : I \to \mathbb{R}$ is an antiderivative of f on I if for each $x \in I$ the derivative $F'(x)$ exists and $F'(x) = f(x)$.

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Exercise

Find $\int e^x \, dx$:

A e^x

B $-e^x$

C $e^x + 3$

D $e^x + e^\pi$

E $2e^x + 2$

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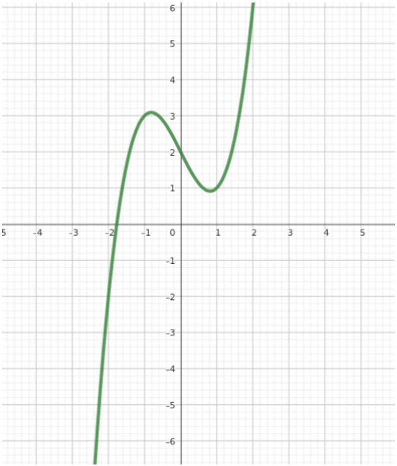
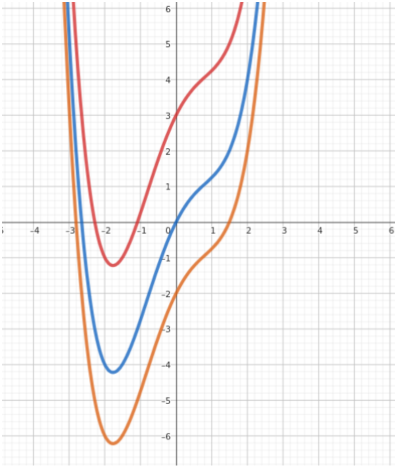
A, C, D

Theorem 1 (Uniqueness of an antiderivative)

Let F and G be antiderivatives of f on an open interval I. Then there exists $c \in \mathbb{R}$ such that $F(x) = G(x) + c$ for each $x \in I$.





Remark

The set of all antiderivatives of f on an open interval I is denoted by

$$\int f(x)\,\mathrm{d}x.$$

The fact that F is an antiderivative of f on I is expressed by

$$\int f(x)\,\mathrm{d}x \stackrel{c}{=} F(x), \quad x \in I.$$

Exercise

Find $\int x \sin x$.

A $F = \sin x + x \cos x$

B $F = \sin x - x \cos x$

C $F = x \sin x + \cos x$

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Exercise

Find F. You know that $F = \int 3x^2 + 2x \, \mathrm{d}x$ and $F(0) = 1$.

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Exercise (True or false?)

Let $F = \int \frac{1}{x^2} \, \mathrm{d}x$ and $F(1) = 1$. Then $F(-1) = 3$.

Exercise

Find F. You know that $F = \int 3x^2 + 2x \, \mathrm{d}x$ and $F(0) = 1$.
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Let $F = \int \frac{1}{x^2} \, \mathrm{d}x$ and $F(1) = 1$. Then $F(-1) = 3$.
False. `https://www.geogebra.org/calculator/mxkdt9vr`

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Exercise (True or false?)

- A If $f'(x) = g'(x)$, then $f(x) = g(x)$ (for all x).
- B If $\int f(x) = \int g(x)$, then $f(x) = g(x)$ (for all x).

http://www.math.cornell.edu/~GoodQuestions/GQbysection_pdfversion.pdf

Exercise

Find F. You know that $F = \int 3x^2 + 2x \, \mathrm{d}x$ and $F(0) = 1$.
$F = x^3 + x^2 + 1$.

Exercise (True or false?)

Let $F = \int \frac{1}{x^2} \, \mathrm{d}x$ and $F(1) = 1$. Then $F(-1) = 3$.
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False, True

Table of basic antiderivatives

- $\displaystyle\int x^n \,\mathrm{d}x \overset{c}{=} \frac{x^{n+1}}{n+1}$ on $\mathbb{R}$ for $n \in \mathbb{N} \cup \{0\}$; on $(-\infty, 0)$ and on $(0, \infty)$ for $n \in \mathbb{Z}$, $n < -1$,
- $\displaystyle\int x^\alpha \,\mathrm{d}x \overset{c}{=} \frac{x^{\alpha+1}}{\alpha+1}$ on $(0, +\infty)$ for $\alpha \in \mathbb{R} \setminus \{-1\}$,
- $\displaystyle\int \frac{1}{x} \,\mathrm{d}x \overset{c}{=} \log|x|$ on $(0, +\infty)$ and on $(-\infty, 0)$,
- $\displaystyle\int e^x \,\mathrm{d}x \overset{c}{=} e^x$ on $\mathbb{R}$,
- $\displaystyle\int \sin x \,\mathrm{d}x \overset{c}{=} -\cos x$ on $\mathbb{R}$,
- $\displaystyle\int \cos x \,\mathrm{d}x \overset{c}{=} \sin x$ on $\mathbb{R}$,

- $\displaystyle\int \frac{1}{\cos^2 x}\,\mathrm{d}x \overset{c}{=} \operatorname{tg} x$ on each of the intervals $(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi)$, $k \in \mathbb{Z}$,
- $\displaystyle\int \frac{1}{\sin^2 x}\,\mathrm{d}x \overset{c}{=} -\operatorname{cotg} x$ on each of the intervals $(k\pi, \pi + k\pi)$, $k \in \mathbb{Z}$,
- $\displaystyle\int \frac{1}{1+x^2}\,\mathrm{d}x \overset{c}{=} \operatorname{arctg} x$ on $\mathbb{R}$,
- $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x \overset{c}{=} \arcsin x$ on $(-1, 1)$,
- $\displaystyle\int -\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x \overset{c}{=} \arccos x$ on $(-1, 1)$.

Matching game:

`https://www.flippity.net/mg.php?k=1F5i3udTbsLHBaMpm3DcWBM3sc9757rctP-S9gM8Ejgc`

Theorem 2 (Existence of an antiderivative)

Let f be a continuous function on an open interval I. Then f has an antiderivative on I.

Exercise

Which of the following functions definitely have primitive function?

A $\frac{1}{x}, x \in \mathbb{R}$

B $\arctan x^2, x \in \mathbb{R}$

C $\log x, x \in (0, \infty)$

D $\frac{x^2}{x^3+1}, x \in \mathbb{R}$

E $\cot x, x \in (0, \pi)$

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E $\cot x, x \in (0, \pi)$

B, C, E

Theorem 3 (Existence of an antiderivative)

Let f be a continuous function on an open interval I. Then f has an antiderivative on I.

Remark

The following functions have antiderivatives, but it can not be easily expressed.

- $\int e^{x^2}\,\mathrm{d}x$
- $\int \log(\log x)\,\mathrm{d}x$
- $\int \sin x^2\,\mathrm{d}x$
- $\int \frac{\sin x}{x}\,\mathrm{d}x$
- $\int \sqrt{1-x^4}\,\mathrm{d}x$

Exercise

Find antiderivatives of the following functions:

A $\int 2 \cdot \frac{1}{x}\, \mathrm{d}x$

B $\int \frac{1}{1+x^2} + \cos x \, \mathrm{d}x$

C $\int 5e^x - 3\frac{1}{\sqrt{1-x^2}}\, \mathrm{d}x$

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$2\log x$, $\arctan x + \sin x$, $5e^x - 3\arcsin x$

Theorem 4 (Linearity of antiderivatives)

Suppose that f has an antiderivative F on an open interval I, g has an antiderivative G on I, and let $\alpha, \beta \in \mathbb{R}$. Then the function $\alpha F + \beta G$ is an antiderivative of $\alpha f + \beta g$ on I.

Example

$(\sin(x^2 + 3))'$

Example

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$\int \sin(x^2+3)\cdot 2x\,\mathrm{d}x$

Example

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Example

Find the following integrals

1. $\int e^{\sin x}\cos x\,\mathrm{d}x$
2. $\int (x^4-3x^2)^{17}(4x^3-6x)\,\mathrm{d}x$
3. $\int \frac{1}{1+(\log x)^2}\cdot\frac{1}{x}\,\mathrm{d}x$
4. $\int \frac{1}{x^3+7x}\cdot(3x^2+7)\,\mathrm{d}x$
5. $\int \frac{1}{\sqrt{1-(e^x)^2}}\cdot e^x\,\mathrm{d}x$

Theorem 5 (substitution)

(i) *Let F be an antiderivative of f on (a, b). Let $\varphi\colon (\alpha, \beta) \to (a, b)$ have a finite derivative at each point of (α, β). Then*

$$\int f\big(\varphi(x)\big)\varphi'(x)\,\mathrm{d}x \overset{c}{=} F\big(\varphi(x)\big) \quad \textit{on } (\alpha, \beta).$$

Example

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x$$

Theorem 6 (substitution)

(i) *Let F be an antiderivative of f on (a,b). Let $\varphi\colon(\alpha,\beta)\to(a,b)$ have a finite derivative at each point of (α,β). Then*

$$\int f\big(\varphi(x)\big)\varphi'(x)\,\mathrm{d}x \overset{c}{=} F\big(\varphi(x)\big) \quad \text{on } (\alpha,\beta).$$

(ii) *Let φ be a function with a finite derivative in each point of (α,β) such that the derivative is either everywhere positive or everywhere negative, and such that $\varphi\big((\alpha,\beta)\big)=(a,b)$. Let f be a function defined on (a,b) and suppose that*

$$\int f\big(\varphi(t)\big)\varphi'(t)\,\mathrm{d}t \overset{c}{=} G(t) \quad \text{on } (\alpha,\beta).$$

Then

$$\int f(x)\,\mathrm{d}x \overset{c}{=} G\big(\varphi^{-1}(x)\big) \quad \text{on } (a,b).$$

Theorem 7 (integration by parts)

Let I be an open interval and let the functions f and g be continuous on I. Let F be an antiderivative of f on I and G an antiderivative of g on I. Then

$$\int f(x)G(x)\,\mathrm{d}x = F(x)G(x) - \int F(x)g(x)\,\mathrm{d}x \quad \textit{on } I.$$

Remark

We can write also as $\int f'g = fg - \int fg'$ or $\int u'v = uv - \int uv'$.

https://cs.khanacademy.org/math/integralni-pocet/xbf9b4d9711003f1c:integracni-metody/xbf9b4d9711003f1c:integrace-per-partes/v/integral-of-ln-x

Example

$$\int x \cos x \, \mathrm{d}x \qquad \int x \arctan x \, \mathrm{d}x \qquad \int \log x \, \mathrm{d}x$$

Exercise

Find integrals, which should be solved by integration by parts

A $\int x e^{x^2} \, \mathrm{d}x$

B $\int x \sin x \, \mathrm{d}x$

C $\int 1 \cdot \arctan x \, \mathrm{d}x$

D $\int (x^2 + 4) \log x \, \mathrm{d}x$

E $\int \sin x \cos x \, \mathrm{d}x$

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D $\int (x^2 + 4) \log x \, \mathrm{d}x$

E $\int \sin x \cos x \, \mathrm{d}x$

B, C, D, E (E also by substitution)

Exercise

By parts or by substitution?

- A $\displaystyle\int \arcsin x \, \mathrm{d}x$
- B $\displaystyle\int \frac{x}{1+x^2} \, \mathrm{d}x$
- C $\displaystyle\int (x^2-3) \log x \, \mathrm{d}x$
- D $\displaystyle\int \frac{1}{x \log x} \, \mathrm{d}x$
- E $\displaystyle\int x^2 \cos 2x \, \mathrm{d}x$

Exercise

By parts or by substitution?

A $\int \arcsin x \, dx$

B $\int \frac{x}{1+x^2} \, dx$

C $\int (x^2 - 3) \log x \, dx$

D $\int \frac{1}{x \log x} \, dx$

E $\int x^2 \cos 2x \, dx$

By parts: A, C, E
Substitution: B, D

https://learningapps.org/34149679

Exercise

True or false?

- A $\int kf = k\int f$
- B $\int f+g = \int f + \int g$
- C $\int f-g = \int f - \int g$
- D $\int f\cdot g = \int f \cdot \int g$
- E $\int f/g = \int f / \int g$

Exercise

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- C $\int f - g = \int f - \int g$
- D $\int f \cdot g = \int f \cdot \int g$
- E $\int f/g = \int f / \int g$

True: A, B, C

Exercise

Find a mistake.

1. $$\int \frac{3x^2+1}{2x}\,\mathrm{d}x = \frac{x^3+x}{x^2} + c$$

2. $\forall a \in \mathbb{R}$
$$\int x^a\,\mathrm{d}x = \frac{x^{a+1}}{a+1} + c$$

Calculus: Single and Multivariable, 6th Edition, Deborah Hughes-Hallett and col.

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Integral of fraction is not a fraction of integral.

True only for $a \neq -1$.

Polynomials

Polynomials

- roots; conjugate roots;
- factorization;
- polynomial division;

Theorem ("fundamental theorem of algebra")

Let $n \in \mathbb{N}$, $a_0, \ldots, a_n \in \mathbb{C}$, $a_n \neq 0$. *Then the equation*

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$$

has at least one solution $z \in \mathbb{C}$.

Example

- $x^2 + x - 2$, $x_1 = 1$, $x_2 = -2$
- $x^2 + 1$, $x_1 = i$, $x_2 = -i$
- $x^3 - ix^2 + 2x$, $x_1 = 0$, $x_2 = 2i$, $x_3 = -i$

Polynomial division

Exercise

Consider $P = x^2 + 3x + 1$ and $Q = x + 1$. Find $(x^2 + 3x + 1) : (x + 1)$.

Polynomial division

Exercise

Consider $P = x^2 + 3x + 1$ and $Q = x + 1$. Find $(x^2 + 3x + 1) : (x + 1)$.
Then we can write

$$x^2 + 3x + 1 = (x + 2)(x + 1) - 1.$$

In other words

$$\frac{x^2 + 3x + 1}{x + 1} = x + 2 + \frac{-1}{x + 1}$$

Theorem 8 (factorisation into monomials)

Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial of degree $n \in \mathbb{N}$. Then there are numbers $x_1, \ldots, x_n \in \mathbb{C}$ such that

$$P(x) = a_n(x - x_1) \cdots (x - x_n), \quad x \in \mathbb{C}.$$

Example

$$x^2 - x - 6 = (x+2)(x-3)$$

$$x^3 + x = x(x-i)(x+i)$$

$$2x^5 + 10x^4 + 4x^3 - 40x^2 - 48x = 2x(x+2)^2(x-2)(x+3)$$

Definition

Let P be a polynomial that is not zero, $\lambda \in \mathbb{C}$, and $k \in \mathbb{N}$. We say that λ is a root of multiplicity k of the polynomial P if there is a polynomial S satisfying $S(\lambda) \neq 0$ and $P(x) = (x - \lambda)^k S(x)$ for all $x \in \mathbb{C}$.

Exercise

Find the multiplicity of $\lambda = -2$ of the polynomial $P(x) = (x^2 + x - 2)(x + 2)^3$.

A -2 B 1 C 2 D 3 E 4

Definition

Let P be a polynomial that is not zero, $\lambda \in \mathbb{C}$, and $k \in \mathbb{N}$. We say that λ is a root of multiplicity k of the polynomial P if there is a polynomial S satisfying $S(\lambda) \neq 0$ and $P(x) = (x - \lambda)^k S(x)$ for all $x \in \mathbb{C}$.

Exercise

Find the multiplicity of $\lambda = -2$ of the polynomial $P(x) = (x^2 + x - 2)(x + 2)^3$.

A -2 B 1 C 2 D 3 E 4

E, $(x^2 + x - 2)(x + 2)^3 = (x - 1)(x + 2)^4$.

Example

$$x^3 + x = x(x - i)(x + i)$$

$$x^2 + 4 = (x + 2i)(x - 2i)$$

Theorem 9 (roots of a polynomial with real coefficients)

Let P be a polynomial with real coefficients and $\lambda \in \mathbb{C}$ a root of P of multiplicity $k \in \mathbb{N}$. Then the also the conjugate number $\overline{\lambda}$ is a root of P of multiplicity k.

Theorem 10 (factorisation of a polynomial with real coefficients)

Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ be a polynomial of degree n with real coefficients. Then there exist real numbers $x_1, \ldots, x_k$, $\alpha_1, \ldots, \alpha_l$, $\beta_1, \ldots, \beta_l$ and natural numbers $p_1, \ldots, p_k$, $q_1, \ldots, q_l$ such that

- $P(x) = a_n (x - x_1)^{p_1} \cdots (x - x_k)^{p_k} (x^2 + \alpha_1 x + \beta_1)^{q_1} \cdots (x^2 + \alpha_l x + \beta_l)^{q_l}$,
- *no two polynomials from* $x - x_1, x - x_2, \ldots, x - x_k$, $x^2 + \alpha_1 x + \beta_1, \ldots, x^2 + \alpha_l x + \beta_l$ *have a common root,*
- *the polynomials* $x^2 + \alpha_1 x + \beta_1, \ldots, x^2 + \alpha_l x + \beta_l$ *have no real root.*

Example

$$2x^4 + 6x^3 - 8x^2 - 24x = 2x(x+2)^2(x-2)(x+3)$$

$$x^3 + x^2 + 4x + 4 = (x+1)(x^2+4) = (x+2i)(x-2i)(x+1)$$

Definition

A rational function is a ratio of two polynomials, where the polynomial in the denominator is not a zero polynomial.

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Exercise

Find rational functions.

A $\frac{3x-4+x^4}{x^2-2x+1}$

B x^6+5

C $\frac{x^5-8x+2}{3}$

D $\frac{\sqrt{2+5}}{1+\sqrt[3]{x^3-8}}$

E $\frac{(3x-4)(2x+5)}{(x-1)(x^2+2)}$

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E $\frac{(3x-4)(2x+5)}{(x-1)(x^2+2)}$

A, B, C, E

Decomposition to partial fractions

Example

$$\frac{3x}{x\left(x+2\right)\left(x-3\right)}=-\frac{3}{5\left(x+2\right)}+\frac{3}{5\left(x-3\right)}$$

$$\frac{x^2+4}{\left(x-2\right)\left(x^2+1\right)}=\frac{8}{5\left(x-2\right)}+\frac{-3x-6}{5\left(x^2+1\right)}$$

$$\frac{-x+2}{\left(x+1\right)^3\left(x^2+x+1\right)}=-\frac{1}{x+1}+\frac{2}{\left(x+1\right)^2}+\frac{3}{\left(x+1\right)^3}+\frac{x-2}{x^2+x+1}$$

Theorem 11 (decomposition to partial fractions)

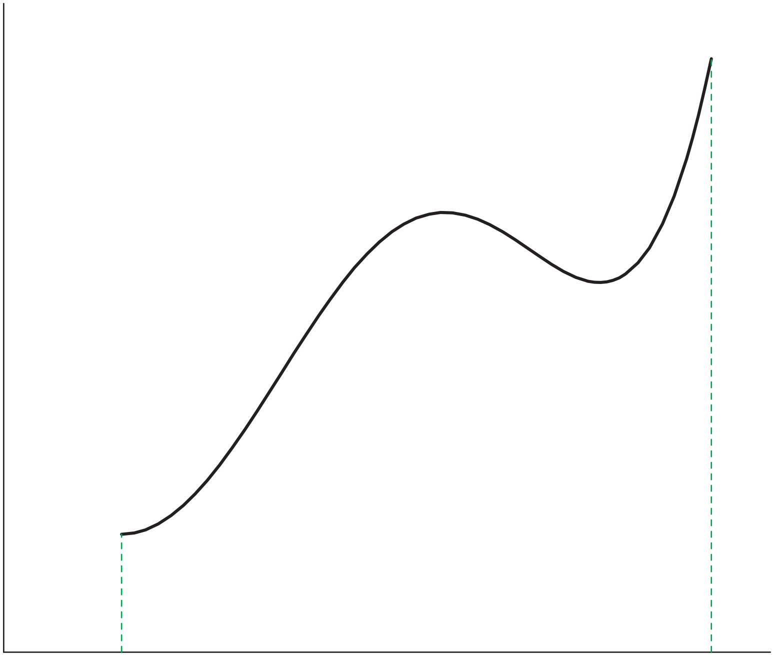
Let P, Q be polynomials with real coefficients such that $\deg P < \deg Q$ *and let*

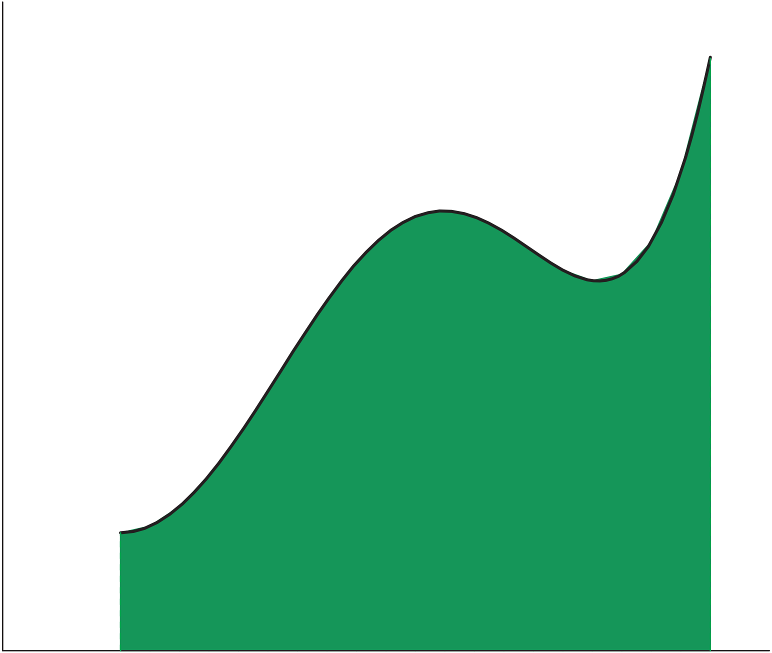
$$Q(x) = a_n(x-x_1)^{p_1}\cdots(x-x_k)^{p_k}(x^2+\alpha_1 x+\beta_1)^{q_1}\cdots(x^2+\alpha_l x+\beta_l)^{q_l}$$

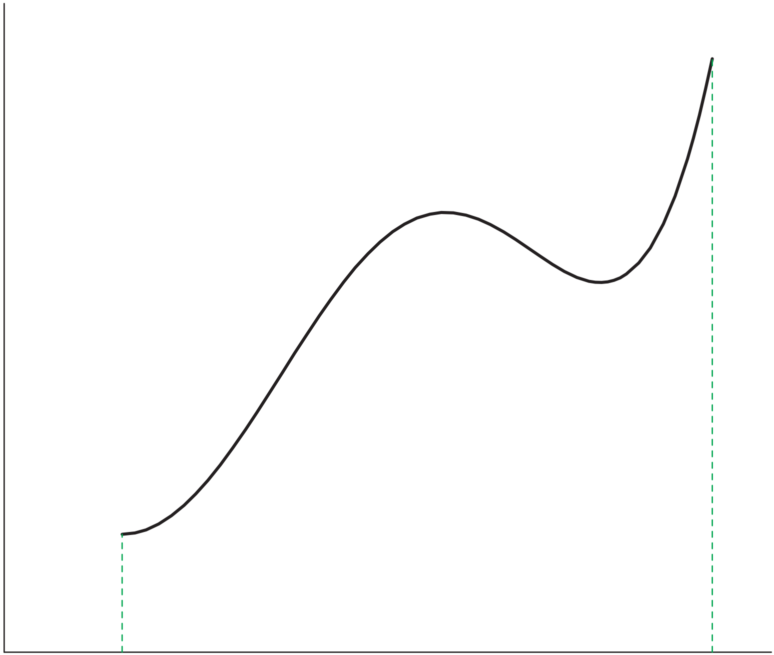
be a factorisation of from Theorem 10. Then there exist unique real numbers $A_1^1,\ldots,A_{p_1}^1,\ldots,A_1^k,\ldots,A_{p_k}^k$, $B_1^1, C_1^1,\ldots,B_{q_1}^1, C_{q_1}^1,\ldots,B_1^l, C_1^l,\ldots,B_{q_l}^l, C_{q_l}^l$ *such that*

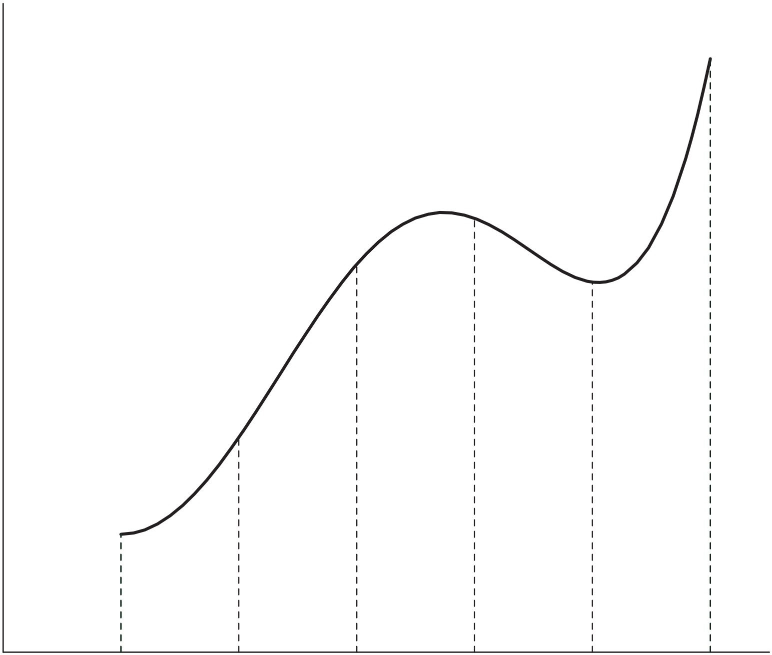
$$\frac{P(x)}{Q(x)} = \frac{A_1^1}{(x-x_1)} + \cdots + \frac{A_{p_1}^1}{(x-x_1)^{p_1}} + \cdots + \frac{A_1^k}{(x-x_k)} + \cdots + \frac{A_{p_k}^k}{(x-x_k)^{p_k}} +$$
$$+ \frac{B_1^1 x + C_1^1}{(x^2+\alpha_1 x+\beta_1)} + \cdots + \frac{B_{q_1}^1 x + C_{q_1}^1}{(x^2+\alpha_1 x+\beta_1)^{q_1}} + \cdots +$$
$$+ \frac{B_1^l x + C_1^l}{(x^2+\alpha_l x+\beta_l)} + \cdots + \frac{B_{q_l}^l x + C_{q_l}^l}{(x^2+\alpha_l x+\beta_l)^{q_l}, x\in\mathbb{R}\setminus\{x_1,\ldots,x_k\}.}$$

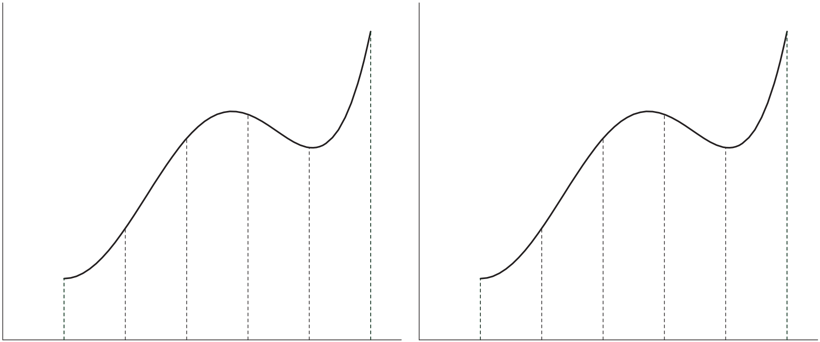
https://www.geogebra.org/calculator/aw2yjsjx

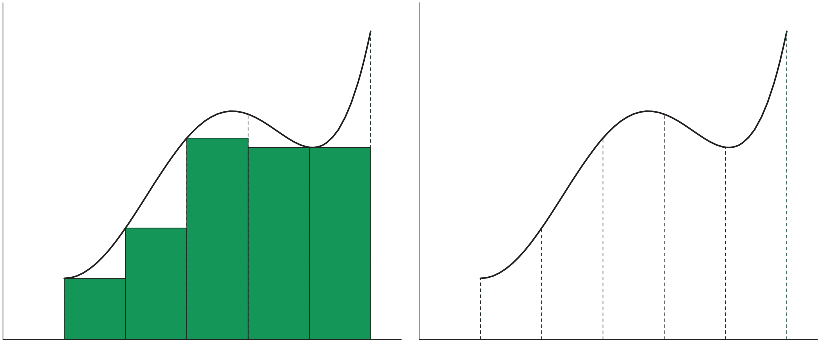


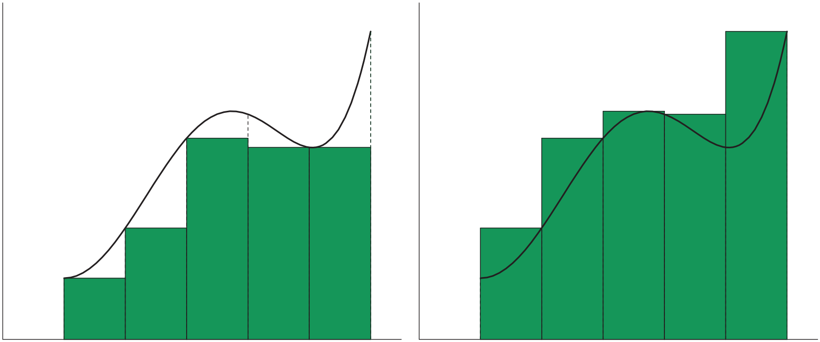


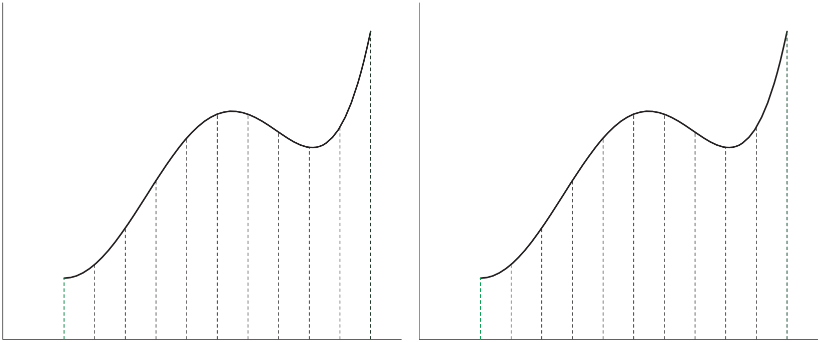


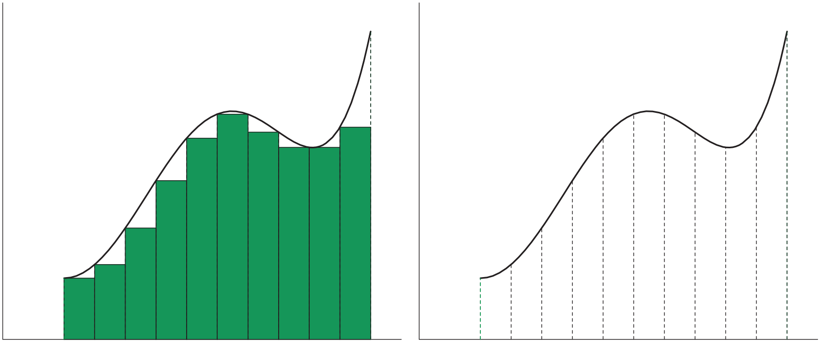


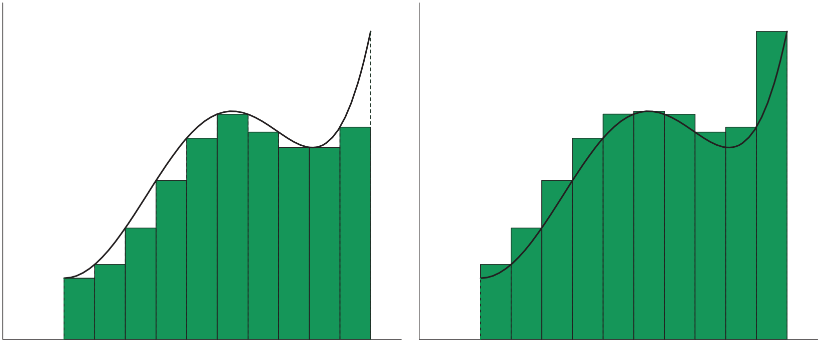












Definition

A finite sequence $\{x_j\}_{j=0}^n$ is called a partition of the interval $[a,b]$ if

$$a = x_0 < x_1 < \cdots < x_n = b.$$

The points $x_0, \ldots, x_n$ are called the partition points.

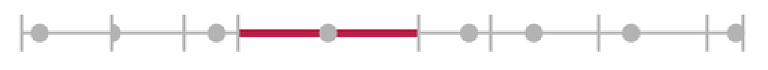


Figure: https://en.wikipedia.org/wiki/Integral

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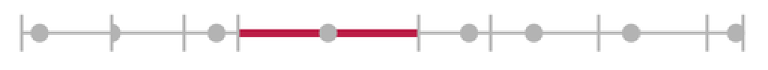

Figure: https://en.wikipedia.org/wiki/Integral

We say that a partition D' of an interval $[a, b]$ is a refinement of the partition D of $[a, b]$ if each partition point of D is also a partition point of D'.

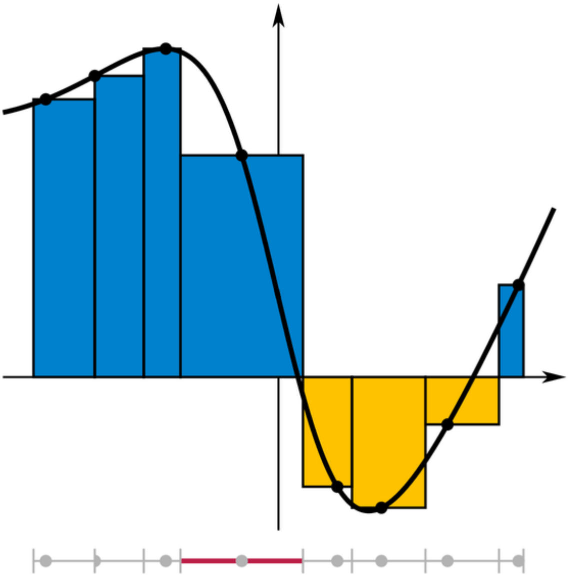

Figure: https://en.wikipedia.org/wiki/Integral

Definition

Suppose that $a, b \in \mathbb{R}$, $a < b$, the function f is bounded on $[a, b]$, and $D = \{x_j\}_{j=0}^{n}$ is a partition of $[a, b]$. Denote

$$\overline{S}(f, D) = \sum_{j=1}^{n} M_j(x_j - x_{j-1}), \text{ where } M_j = \sup\{f(x);\ x \in [x_{j-1}, x_j]\},$$

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We say that a function f has the Riemann integral over the interval $[a, b]$ if $\overline{\int_a^b} f = \underline{\int_a^b} f$.

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If $a > b$, then we define $\displaystyle\int_a^b f = -\int_b^a f$, and in case that $a = b$ we put $\displaystyle\int_a^b f = 0$.

See: https://en.wikipedia.org/wiki/Integral

Exercise

Use the Riemann sums and estimate the integral

$$\int_0^{15} f(x)\,\mathrm{d}x.$$

Check the table for some values of f:

x	0	3	6	9	12	15
$f(x)$	50	48	44	36	24	8

Table: Applied Calculus, 6th Edition, Deborah Hughes-Hallett and col.

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Table: Applied Calculus, 6th Edition, Deborah Hughes-Hallett and col.

Upper sum: 606
Lower sum: 480

Theorem 12 (Newton-Leibniz formula)

Let f be a function continuous on an interval $(a-\varepsilon, b+\varepsilon)$, $a, b \in \mathbb{R}$, $a < b$, $\varepsilon > 0$ and let F be an antiderivative of f on $(a-\varepsilon, b+\varepsilon)$. Then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a). \tag{1}$$

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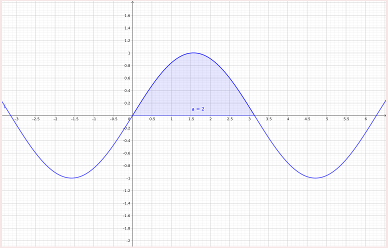
Remark

The Newton-Leibniz formula (1) holds even if $b < a$ (if $F' = f$ on $(b-\varepsilon, a+\varepsilon)$). Let us denote

$$[F]_a^b = F(b) - F(a).$$

Example

$$\int_0^{\pi} \sin x \, \mathrm{d}x = [-\cos x]_0^{\pi} = -\cos \pi - (-\cos 0) = 1 + 1 = 2$$



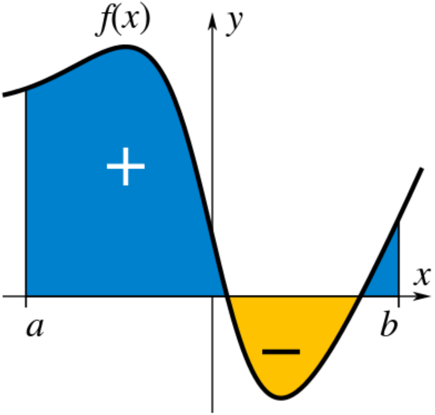
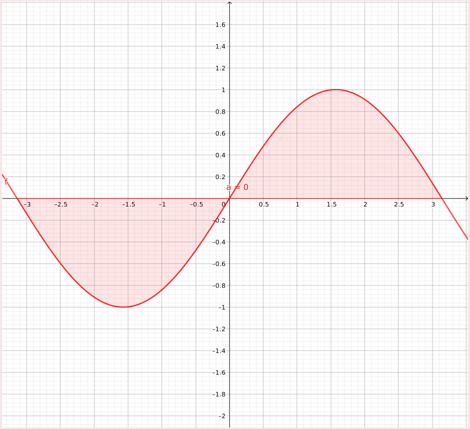


Figure: https://en.wikipedia.org/wiki/Integral

Example

$$\int_{-\pi}^{\pi} \sin x \, dx = [-\cos x]_{-\pi}^{\pi} = -\cos\pi - (-\cos -\pi) = 1 - 1 = 0$$



Theorem 13 (integration by parts)

Suppose that the functions f, g, f' a g' are continuous on an interval $[a,b]$. Then

$$\int_a^b f'g = [fg]_a^b - \int_a^b fg'.$$

Exercise

$$\int_0^\pi x\cos x\,\mathrm{d}x$$

$$\int_1^2 1\cdot\log x\,\mathrm{d}x$$

Theorem 14 (substitution)

Let the function f be continuous on an interval $[a,b]$. Suppose that the function φ has a continuous derivative on $[\alpha,\beta]$ and φ maps $[\alpha,\beta]$ into the interval $[a,b]$. Then

$$\int_{\alpha}^{\beta} f\big(\varphi(x)\big)\varphi'(x)\,\mathrm{d}x = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(t)\,\mathrm{d}t.$$

https://www.geogebra.org/calculator/frvx4mtr
https://www.geogebra.org/calculator/cjuvxazd

Exercise

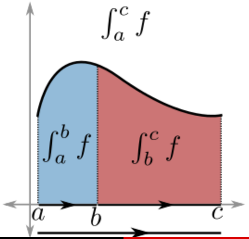
$$\int_1^{\infty} \frac{\arctan x}{1+x^2}\,\mathrm{d}x \qquad \int_0^3 e^{x^2} 2x\,\mathrm{d}x$$

Theorem 15

(i) *Suppose that f has the Riemann integral over $[a,b]$ and let $[c,d] \subset [a,b]$. Then f has the Riemann integral also over $[c,d]$.*

(ii) *Suppose that $c \in (a,b)$ and f has the Riemann integral over the intervals $[a,c]$ and $[c,b]$. Then f has the Riemann integral over $[a,b]$ and*

$$\int_a^b f = \int_a^c f + \int_c^b f. \tag{2}$$



Theorem 16 (linearity of the Riemann integral)

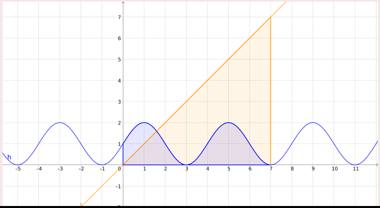
Let f and g be functions with Riemann integral over $[a, b]$ and let $\alpha \in \mathbb{R}$. Then

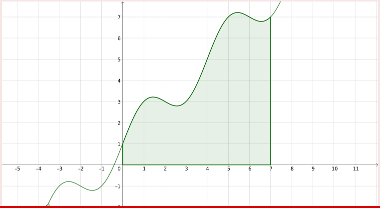
(i) *the function αf has the Riemann integral over $[a, b]$ and*

$$\int_a^b \alpha f = \alpha \int_a^b f,$$

(ii) *the function $f + g$ has the Riemann integral over $[a, b]$ and*

$$\int_a^b f + g = \int_a^b f + \int_a^b g.$$



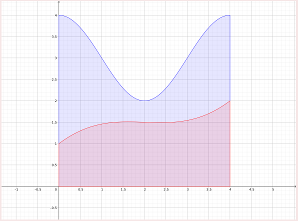


Theorem 17

Let $a, b \in \mathbb{R}$, $a < b$, and let f and g be functions with Riemann integral over $[a, b]$. Then:

(i) *If $f(x) \leq g(x)$ for each $x \in [a, b]$, then*

$$\int_a^b f \leq \int_a^b g.$$



(ii) *The function $|f|$ has the Riemann integral over $[a, b]$ and*

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

Theorem 18

Let f be a function continuous on an interval $[a,b]$, $a,b \in \mathbb{R}$. Then f has the Riemann integral on $[a,b]$.

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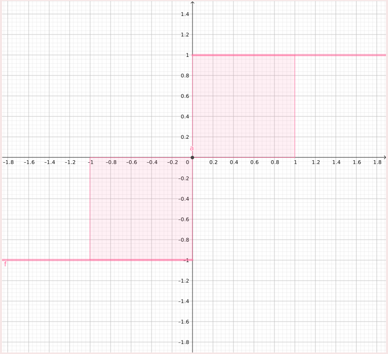
Remark

Compare with $\operatorname{sgn} x$:



Theorem 19

Let f be a function continuous on an interval (a,b) and let $c \in (a,b)$. If we denote $F(x) = \int_c^x f(t)\,\mathrm{d}t$ for $x \in (a,b)$, then $F'(x) = f(x)$ for each $x \in (a,b)$. In other words, F is an antiderivative of f on (a,b).

Exercise (True – False)

A Let f be a function. Then $\displaystyle\int_0^2 f(x)\,\mathrm{d}x \leq \int_0^3 f(x)\,\mathrm{d}x$.

B If $\displaystyle\int_2^6 g(x)\,\mathrm{d}x \leq \int_2^6 f(x)\,\mathrm{d}x$, then $g(x) \leq f(x)$ for all $2 \leq x \leq 6$.

Exercise (True – False)

A Let f be a function. Then $\int_0^2 f(x)\,\mathrm{d}x \leq \int_0^3 f(x)\,\mathrm{d}x$.

B If $\int_2^6 g(x)\,\mathrm{d}x \leq \int_2^6 f(x)\,\mathrm{d}x$, then $g(x) \leq f(x)$ for all $2 \leq x \leq 6$.

False - consider negative f.
False - consider oscillating f and g.

Exercise

Let f be an odd function such that $\int_{-2}^{0} f(x)\,\mathrm{d}x = 4$. Find

1. $\int_{0}^{2} f(x)\,\mathrm{d}x$
2. $\int_{-2}^{2} f(x)\,\mathrm{d}x$

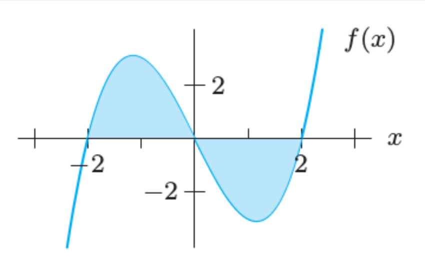


Figure: Applied Calculus, 6th Edition, Deborah Hughes-Hallett and col.

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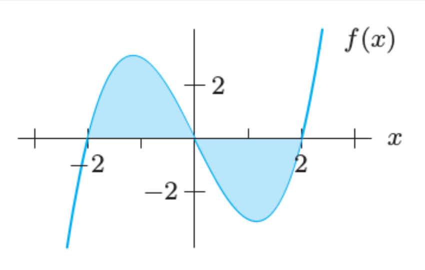

Figure: Applied Calculus, 6th Edition, Deborah Hughes-Hallett and col.

-4, 0

Exercise

Decide, if the integrals are

A $\displaystyle\int_{-\pi}^{0} \sin x \, \mathrm{d}x$

B $\displaystyle\int_{0}^{\pi} \cos x \, \mathrm{d}x$

C $\displaystyle\int_{-\pi}^{\pi} \sin x \, \mathrm{d}x$

D $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x \, \mathrm{d}x$

E $\displaystyle\int_{0}^{2\pi} e^{-x} \sin x \, \mathrm{d}x$

1. positive
2. 0
3. negative

Exercise

Decide, if the integrals are

A $\displaystyle\int_{-\pi}^{0} \sin x \, dx$

B $\displaystyle\int_{0}^{\pi} \cos x \, dx$

C $\displaystyle\int_{-\pi}^{\pi} \sin x \, dx$

D $\displaystyle\int_{-\pi/2}^{\pi/2} \cos x \, dx$

E $\displaystyle\int_{0}^{2\pi} e^{-x} \sin x \, dx$

1. positive
2. 0
3. negative

https://www.geogebra.org/calculator/ups4z7sh

positive - D, E   0 - B, C   negative - A

Exercise

The half-life of phosphorous ^{32}P, which is used for biological experiments, is 14,3 days.
Suppose, that you have a sample, which emits 300 mREM/day.
(1 REM=0,01 Sv)
How long can a laboratory assistant work with this sample, if according to the safety regulations she can receive only 5 000 mREM/year.

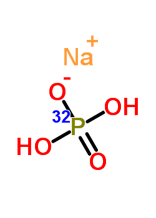

Figure:
https://www.guidechem.com/cas/680178408.html
From: https://jmahaffy.sdsu.edu/courses/f14/math124/beamer_lectures/def_int.pdf