Limit of a composition

Theory

Theorem 1 (limit of a composition)**.** Let $c, A, B \in \mathbb{R}^*$, $\lim_{x\to c} g(x) = A$, $\lim_{y\to A} f(y) = B$ and at least one of the following conditions is satisfied:

(I) $\exists \eta \in \mathbb{R}, \eta > 0\ \forall x \in P(c,\eta)\colon g(x) \neq A$,

(C) the function f is continuous at A.

Then

$$\lim_{x\to c} f\big(g(x)\big) = B.$$

Facts

$$\lim_{y\to 0} \frac{\sin y}{y} = 1 \qquad \lim_{y\to 0} \frac{e^y - 1}{y} = 1 \qquad a^b = e^{b\ln a}$$

$$\lim_{y\to 0} \frac{1-\cos y}{y^2} = \frac{1}{2} \qquad \lim_{y\to 0} \frac{\ln(1+y)}{y} = 1$$

Exercises

1. (a) $\lim_{x\to\infty} e^{\frac{1}{3-x}}$ (b) $\lim_{x\to-\infty} e^{-x^2}$ (c) $\lim_{x\to\infty} \ln\frac{x-1}{x+1}$

2. (a) $\lim_{x\to 0} \frac{\sin 5x}{5x}$
   (b) $\lim_{x\to 0} \frac{\sin x^2}{x^2}$
   (c) $\lim_{x\to 0} \frac{\sin 3x^2}{x^2}$
   (d) $\lim_{x\to 0} \frac{\operatorname{tg} x}{x}$ $\operatorname{tg} x = \sin x/\cos x$
   (e) $\lim_{x\to 0+} \frac{\operatorname{tg}\sqrt{x}}{\sqrt{x}}$
   (f) $\lim_{x\to 0} \frac{1-\cos x^2}{x^4}$
   (g) $\lim_{x\to 0} \frac{x^2}{1-\cos x^4}$
   (h) $\lim_{x\to 0} \frac{x^2}{1-\cos 4x^4}$
   (i) $\lim_{x\to 0} x\operatorname{cotg} 3x$ $\cot x = \cos x/\sin x$

3. (a) $\lim_{x\to 0} \frac{\ln(1+3x)}{3x}$
   (b) $\lim_{x\to 0} \frac{\ln(1+3x)}{x}$
   (c) $\lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}$
   (d) $\lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)$
   (e) $\lim_{x\to 1} \frac{\ln(1+(x-1))}{x-1}$
   (f) $\lim_{x\to 0} \frac{\ln(1+x^2)}{x^2}$
   (g) $\lim_{x\to 0} \frac{-x^2}{\ln(1-x^2)}$
   (h) $\lim_{x\to 0} \frac{\ln(1+x^2)}{\ln(1-x^2)}$
   (i) $\lim_{x\to 0} \frac{e^{2x}-1}{x}$
   (j) $\lim_{x\to 0} \frac{e^{\sin x}-1}{\sin x}$
   (k) $\lim_{x\to 0} \frac{e^{\sin x}-1}{x}$

4. (a) $\lim_{x\to 0} \frac{\cos x - 1}{x}$
   (b) $\lim_{x\to 0+} \frac{\sin\sqrt{x}}{x}$
   (c) $\lim_{x\to 0} \frac{\sin 5x - \sin 3x}{\sin x}$
   (d) $\lim_{x\to 0} \frac{\sin(\sin x)}{\tan(\tan x)}$
   (e) $\lim_{x\to 0} \frac{a^x - 1}{x}$, $a > 0$ $a^b = e^{b\ln a}$