Mathematics I - Functions

21/22

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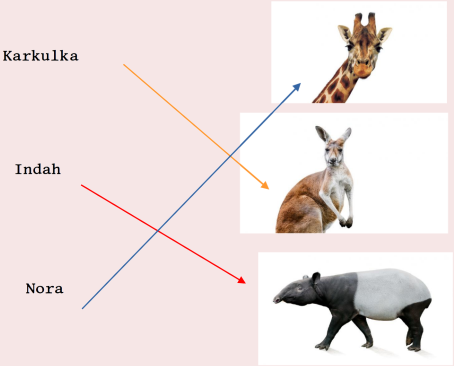

https://www.zoopraha.cz/zvirata-a-expozice/zvireci-osobnosti

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Example

- students in the classroom $\mapsto$ their date of birth
- f assigns rectangles their area
- countries $\to$ flag
- $x \mapsto \sqrt[4]{x}, f : [0, \infty) \to [0, \infty)$

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Let $f: A \to B$ be a mapping.

- The subset $G_f = \{[x, y] \in A \times B;\ x \in A, y = f(x)\}$ of the Cartesian product $A \times B$ is called the graph of the mapping f.

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$$f(M) = \{y \in B;\ \exists x \in M\colon f(x) = y\} \quad (= \{f(x);\ x \in M\}).$$

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- The set $f(A)$ is called the range of the mapping f, it is denoted by R_f.
- The pre-image of the set $W \subset B$ under the mapping f is the set

$$f_{-1}(W) = \{x \in A;\ f(x) \in W\}.$$

Exercise

Find the domain and range for the following mappings:

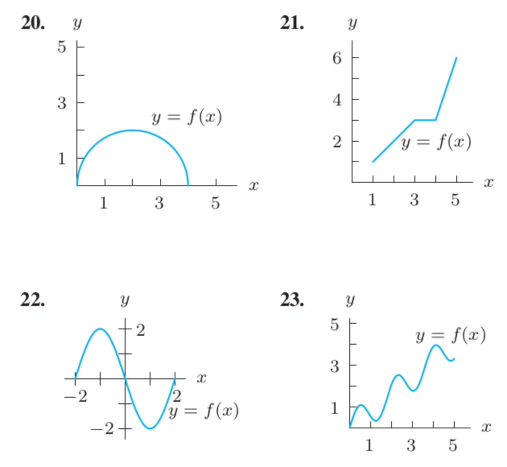


Figure: Calculus: Single and Multivariable, 6th Edition, Hughes-Hallett, col.

Exercise

Which of the following functions has its domain the same as its range?

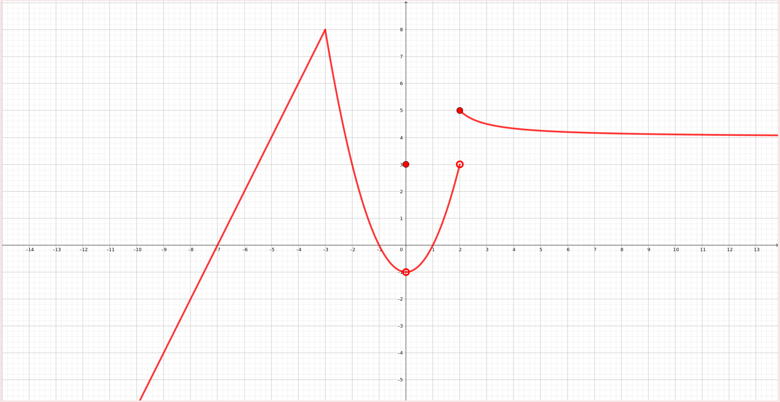
A x^2 B $\sqrt{x}$ C x^3 D $|x|$ E $2x-3$

(Inspired by: Active Calculus & Mathematical Modeling, Carroll College Mathematics Department)

Exercise

Find the image:

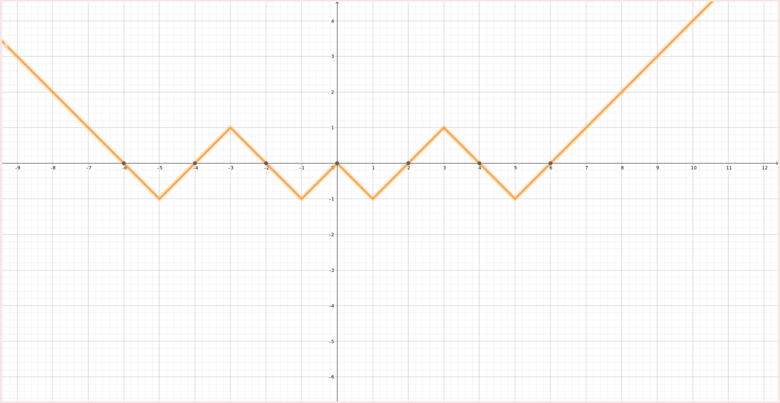
A $[-6, -2]$ **B** $[-1, 1)$ **C** $[0, 2)$ **D** $[2, \infty)$



Exercise

Find the preimage:

A $\{-1\}$ B $[2,3]$ C $[0,1]$ D $[0,1)$



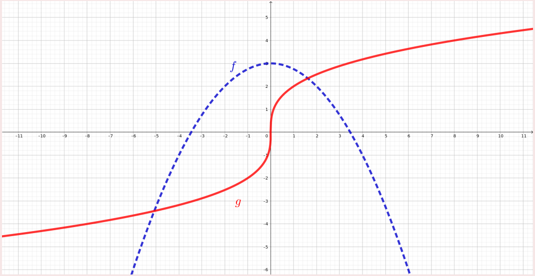
Definition

Let $f : A \to B$ and $g : B \to C$ be two mappings. The symbol $g \circ f$ denotes a mapping from A to C defined by

$$(g \circ f)(x) = g(f(x)).$$

This mapping is called a compound mapping or a composition of the mapping f and the mapping g.

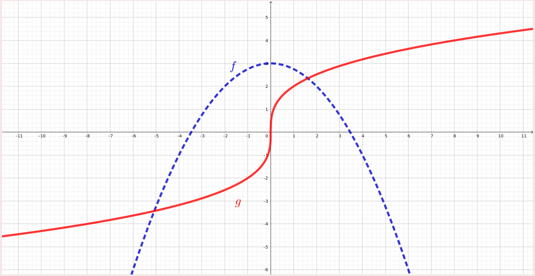
Exercise



Find $g(f(4))$.

A -2 B -1 C 0 D 1 E 2

Exercise



Find $g(f(4))$.

A -2 B -1 C 0 D 1 E 2

Find x, if $f(g(x)) = 2$.

Exercise

In the tables we can find values of functions f and g.

x	-2	-1	0	1	2
$f(x)$	1	0	-2	2	-1
$g(x)$	-1	1	2	0	-2

Find $g(f(1))$.

A -2 B -1 C 0 D 1 E 2

Find $f(f(0))$.

A -2 B -1 C 0 D 1 E 2

Exercise

In the tables we can find values of functions f and g. If $f(g(x)) = -2$, find x.

x	-2	-1	0	1	2
$f(x)$	1	0	-2	2	-1
$g(x)$	-1	1	2	0	-2

A -2 B -1 C 0 D 1 E 2

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We say that a mapping $f : A \to B$

- maps the set A onto the set B if $f(A) = B$, i.e. if to each $y \in B$ there exist $x \in A$ such that $f(x) = y$;

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$$\forall x_1, x_2 \in A: x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2),$$

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$$\forall x_1, x_2 \in A\colon x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2),$$

- is a bijection of A onto B (or a bijective mapping), if it is at the same time one-to-one and maps A onto B.

Exercise

A e^x B x^3 C $\sin x$ D $\tan x$ E $\frac{1}{x}$

Which functions are onto?

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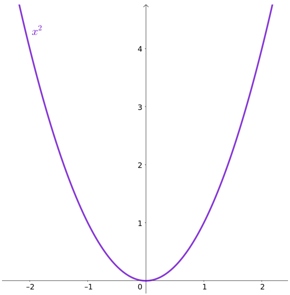
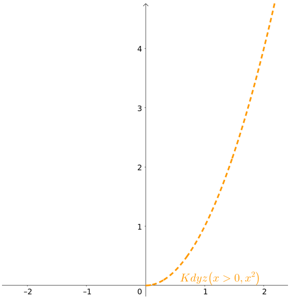
Which functions are onto? Which functions are one-to-one? Which functions are bijections?

Definition

Let A, B, C be sets, $C \subset A$ and $f\colon A \to B$. The mapping $\tilde{f}\colon C \to B$ given by the formula $\tilde{f}(x) = f(x)$ for each $x \in C$ is called the restriction of the mapping f to the set C. It is denoted by $f|_C$.





Definition

Let $f : A \to B$ be bijective (i.e. one-to-one and onto). An inverse mapping $f^{-1} : B \to A$ is a mapping that to each $y \in B$ assigns a (uniquely determined) element $x \in A$ satisfying $f(x) = y$.

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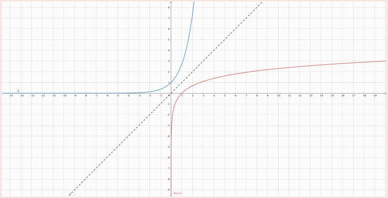
Exercise

Find inverse mappings at $\mathbb{R}$:

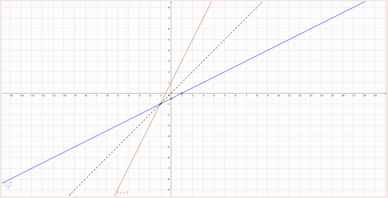
- A $2x + 1$
- B e^x
- C $\sqrt[3]{x}$
- D x^2

Exercise

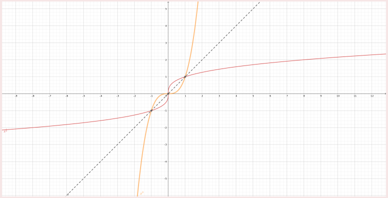
e^x vs $\ln x$



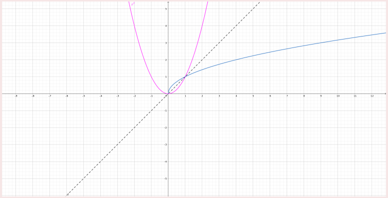
$2x+1$ vs $\frac{x-1}{2}$



x^3 vs $\sqrt[3]{x}$



x^2 vs $\sqrt{x}$



IV. Functions of one real variable

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Definition

A function f of one real variable (or a function for short) is a mapping $f : M \to \mathbb{R}$, where M is a subset of real numbers.

Definition

A function $f : J \to \mathbb{R}$ is increasing on an interval J, if for each pair $x_1, x_2 \in J$, $x_1 < x_2$ the inequality $f(x_1) < f(x_2)$ holds. Analogously we define a function decreasing (non-decreasing, non-increasing) on an interval J.

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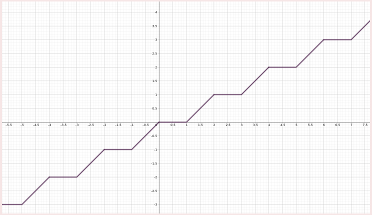
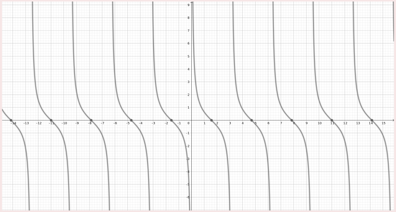
Definition

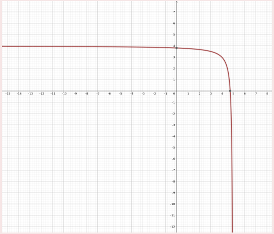
A monotone function on an interval J is a function which is non-decreasing or non-increasing on J. A strictly monotone function on an interval J is a function which is increasing or decreasing on J.

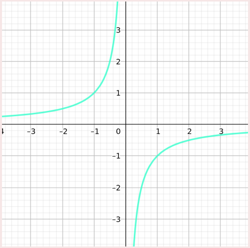
Exercise

Decide, which functions are monotone on its domain:









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- bounded from above on M if there is $K \in \mathbb{R}$ such that $f(x) \leq K$ for all $x \in M$,

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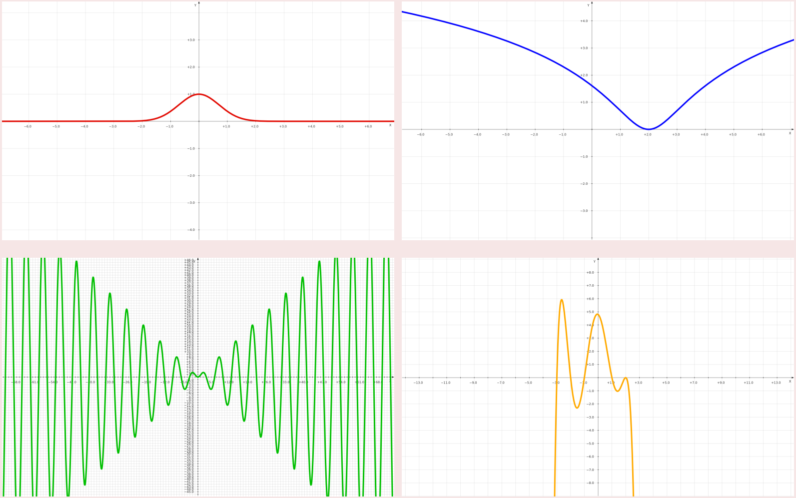
- bounded from below on M if there is $K \in \mathbb{R}$ such that $f(x) \geq K$ for all $x \in M$,
- bounded on M if there is $K \in \mathbb{R}$ such that $|f(x)| \leq K$ for all $x \in M$,

Exercise

Decide, which functions are bounded: Určete, které funkce jsou omezené, omezené shora, zdola, neomezené



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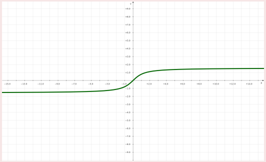
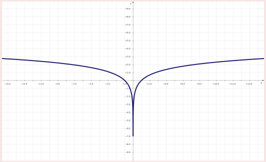
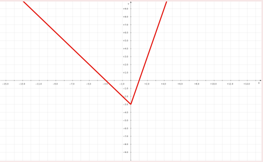
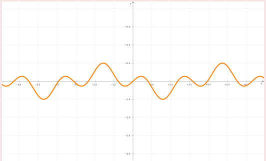
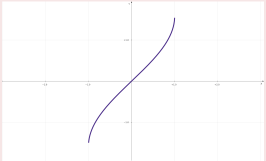
- even if for each $x \in D_f$ we have $-x \in D_f$ and $f(-x) = f(x)$,
- periodic with a period a, where $a \in \mathbb{R}$, $a > 0$, if for each $x \in D_f$ we have $x + a \in D_f$, $x - a \in D_f$ and $f(x+a) = f(x-a) = f(x)$.

Exercise

Decide, which functions are even or odd: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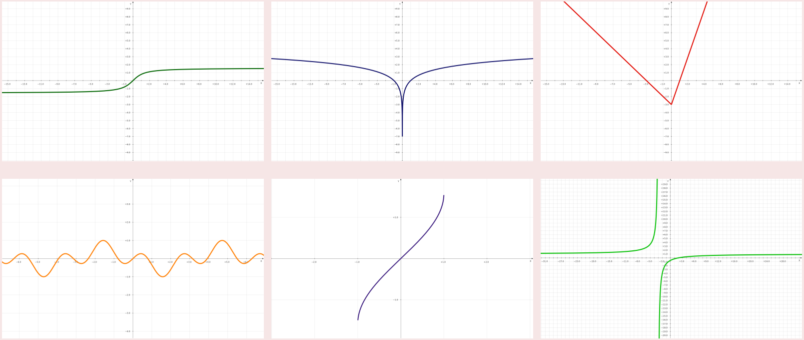


Decide, which functions are even or odd:

A $x^3 + 1$

B $x(x^2 + 1)$

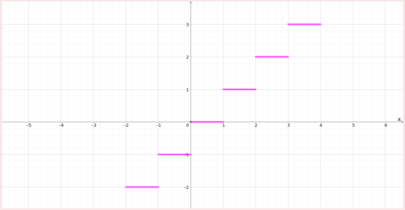
C $|x - 2|$

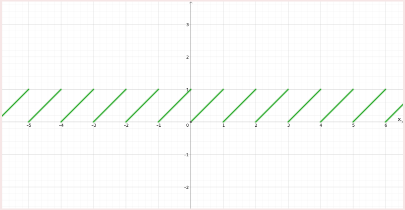
D $e^{x^2} \sin x$

E $|1 + \cos x|$

Exercise

Decide, which functions are periodic





Exercise

Sketch in the function so that it is periodic with the smallest possible period