Mathematics I - Derivatives

21/22

Exercise (Motivation)

The farmer would like to enclose a rectangular place for sheep. She has 40 meters of fence and land by the river. What is the biggest possible area of the place?

Figure: https://www.cbr.com/shaun-the-sheep-best-worst-episodes-imdb/

Derivative

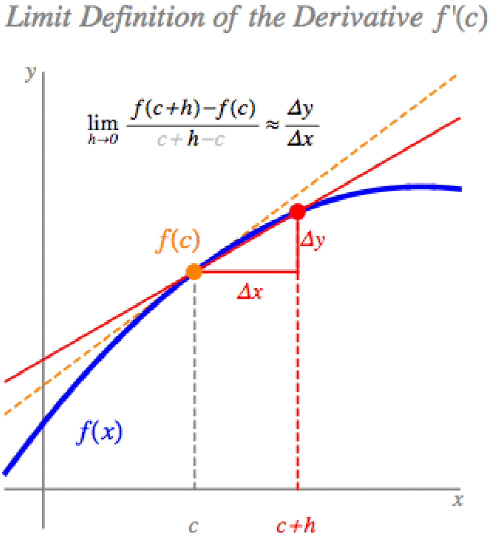

Figure: https://ginsyblog.wordpress.com/2017/02/04/how-to-solve-the-problems-of-differential-calculus/

Definition

Let f be a function and $a \in \mathbb{R}$. Then

- the derivative of the function f at the point a is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

if the respective limits exist.

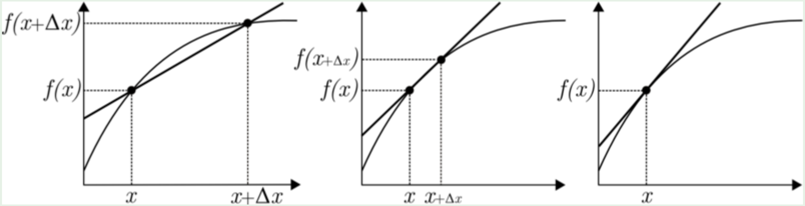

Figure: https://cs.wikipedia.org/wiki/Derivace

Definition

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- the derivative of the function f at the point a is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

- the derivative of f at a from the right is defined by

$$f'_+(a) = \lim_{h \to 0+} \frac{f(a+h) - f(a)}{h},$$

- the derivative of f at a from the left is defined by

$$f'_-(a) = \lim_{h \to 0-} \frac{f(a+h) - f(a)}{h},$$

if the respective limits exist.

Definition

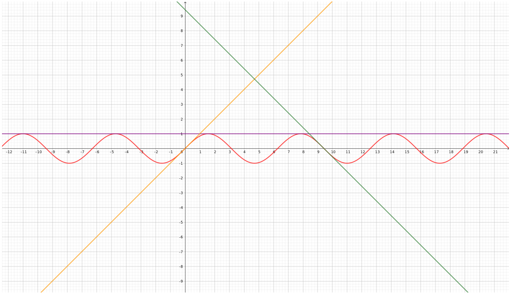
Suppose that the function f has a finite derivative at a point $a \in \mathbb{R}$. The line

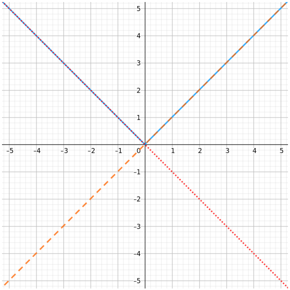
$$T_a = \left\{[x, y] \in \mathbb{R}^2;\ y = f(a) + f'(a)(x - a)\right\}.$$

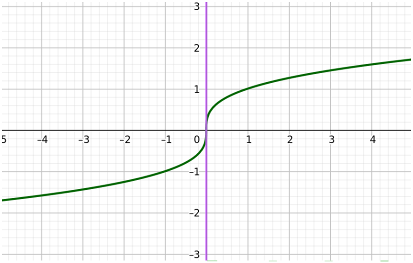
is called the tangent to the graph of f at the point $[a, f(a)]$.

https://www.desmos.com/calculator/l0puzw0zvm

Examples



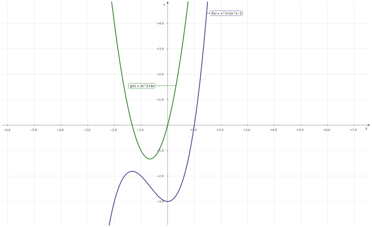




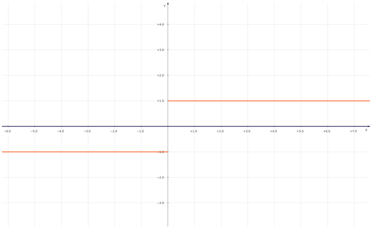
Theorem 1

Suppose that the function f has a finite derivative at a point $a \in \mathbb{R}$. Then f is continuous at a.

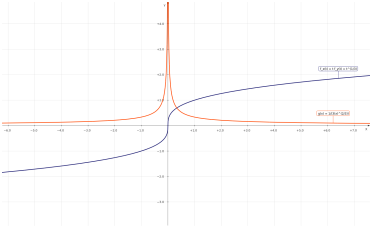
$$(x^3 + 2x^2 - 3)' = 3x^2 + 4x$$



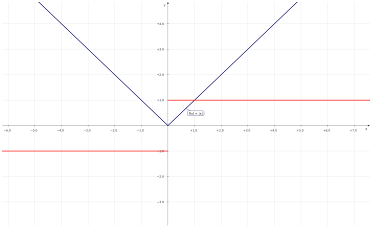
$$(\operatorname{sgn} x)'(0) = \infty$$



$$(\sqrt[3]{x})' = \frac{1}{3\sqrt[3]{x^2}}$$



$|x|'$ at 0 does not exist



Theorem 2 (arithmetics of derivatives)

Suppose that the functions f and g have finite derivatives at $a \in \mathbb{R}$ and let $\alpha \in \mathbb{R}$. Then

(i) $(f+g)'(a) = f'(a) + g'(a)$,

(ii) $(\alpha f)'(a) = \alpha \cdot f'(a)$,

(iii) $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$,

(iv) *if* $g(a) \neq 0$, *then*

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}.$$

Exercise

$f = \cos x \sin x$. Find f'.

A $\cos^2 x$

B $\sin^2 x$

C $\cos^2 x - \sin^2 x$

D $-\sin x \cos x$

Exercise

$f = \cos x \sin x$. Find f'.

A $\cos^2 x$

B $\sin^2 x$

C $\cos^2 x - \sin^2 x$

D $-\sin x \cos x$

Exercise

$f = e^7$. Find f'.

A $7e^6$

B e^7

C 0

Exercise

$f = \cos x \sin x$. Find f'.

A $\cos^2 x$

B $\sin^2 x$

C $\cos^2 x - \sin^2 x$

D $-\sin x \cos x$

Exercise

$f = e^7$. Find f'.

A $7e^6$

B e^7

C 0

Exercise

$f = \dfrac{e^x}{x^2}$ Find f'.

A $\dfrac{e^x}{2x}$

B $\dfrac{e^x(x-2)}{x^3}$

C $\dfrac{e^x x^2 - 2xe^x}{x^4}$

D $\dfrac{e^x 2x + x^2 e^x}{x^4}$

Theorem 3 (derivative of a compound function)

Suppose that the function f has a finite derivative at $y_0 \in \mathbb{R}$, the function g has a finite derivative at $x_0 \in \mathbb{R}$, and $y_0 = g(x_0)$. Then

$$(f \circ g)'(x_0) = f'(y_0) \cdot g'(x_0).$$

Exercise

$f = \sin x + e^{\sin x}$ Find f'.

- A $\cos x + e^{\cos x}$
- B $\cos x + e^{\sin x}$
- C $\cos x + \sin x \, e^{\cos x}$
- D $\cos x + \cos x \, e^{\sin x}$

Theorem 4 (derivative of an inverse function)

Let f be a function continuous and strictly monotone on an interval (a,b) and suppose that it has a finite and non-zero derivative $f'(x_0)$ at $x_0 \in (a,b)$. Then the function f^{-1} has a derivative at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}.$$

Exercise (True or false?)

1. If $f'(x) = g'(x)$, then $f(x) = g(x)$. (For every x.)
2. If $f'(a) \neq g'(a)$, then $f(a) \neq g(a)$.
   (We are talking about particular point a.)

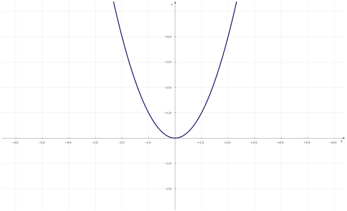
Derivatives of elementary functions

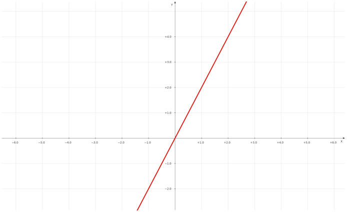
- $(\text{const.})' = 0$,
- $(x^n)' = nx^{n-1}$, $x \in \mathbb{R}$, $n \in \mathbb{N}$; $x \in \mathbb{R} \setminus \{0\}$, $n \in \mathbb{Z}$, $n < 0$,
- $(\log x)' = \frac{1}{x}$ for $x \in (0, +\infty)$,
- $(\exp x)' = \exp x$ for $x \in \mathbb{R}$,
- $(x^a)' = ax^{a-1}$ for $x \in (0, +\infty)$, $a \in \mathbb{R}$,
- $(a^x)' = a^x \log a$ for $x \in \mathbb{R}$, $a \in \mathbb{R}$, $a > 0$,
- $(\sin x)' = \cos x$ for $x \in \mathbb{R}$,
- $(\cos x)' = -\sin x$ for $x \in \mathbb{R}$,
- $(\operatorname{tg} x)' = \frac{1}{\cos^2 x}$ for $x \in D_{\operatorname{tg}}$,
- $(\operatorname{cotg} x)' = -\frac{1}{\sin^2 x}$ for $x \in D_{\operatorname{cotg}}$,
- $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ for $x \in (-1, 1)$,
- $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$ for $x \in (-1, 1)$,
- $(\operatorname{arctg} x)' = \frac{1}{1+x^2}$ for $x \in \mathbb{R}$,
- $(\operatorname{arccotg} x)' = -\frac{1}{1+x^2}$ for $x \in \mathbb{R}$.

Theorem 5 (necessary condition for a local extremum)

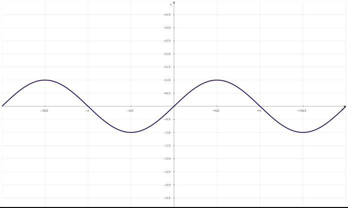
Suppose that a function f has a local extremum at $x_0 \in \mathbb{R}$. If $f'(x_0)$ exists, then $f'(x_0) = 0$.

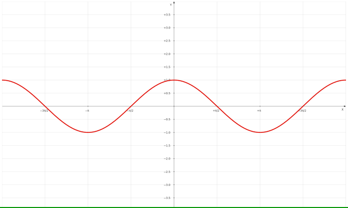
$$(x^2)' = 2x$$



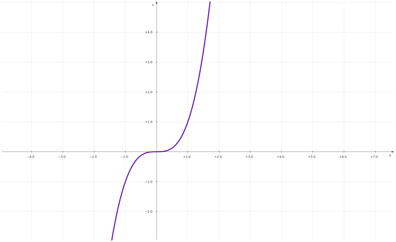


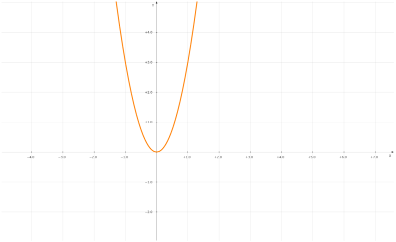
$$(\sin x)' = \cos x$$





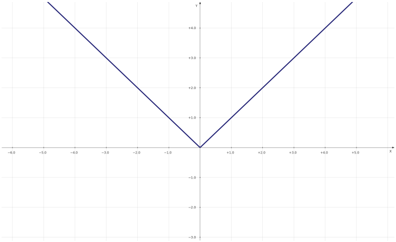
$$(x^3)' = 3x^2$$

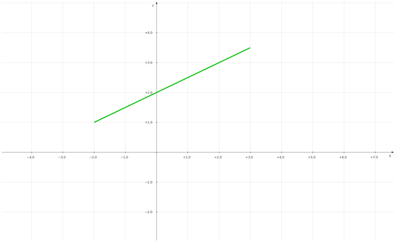




$|x|$

$x/2$





First Derivative Test for Local Extrema

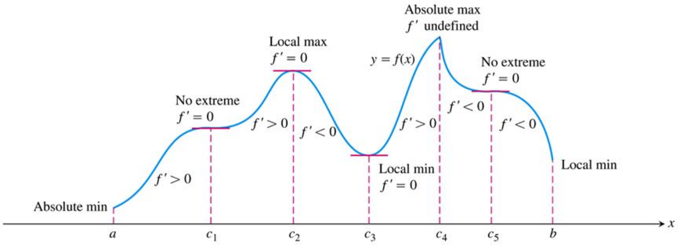


FIGURE 3.21 A function's first derivative tells how the graph rises and falls.

Figure: http://slideplayer.com/slide/7555868/

Theorem 6 (Rolle)

Suppose that $a, b \in \mathbb{R}$, $a < b$, and a function f has the following properties:

(i) *it is continuous on the interval $[a, b]$,*

(ii) *it has a derivative (finite or infinite) at every point of the open interval (a, b),*

(iii) $f(a) = f(b)$.

Then there exists $\xi \in (a, b)$ satisfying $f'(\xi) = 0$.

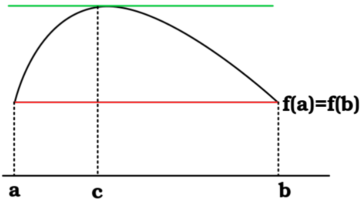


Figure: https://commons.wikimedia.org/wiki/File:Rolle%27s_theorem.svg

Theorem 7 (Lagrange, mean value theorem)

Suppose that $a, b \in \mathbb{R}$, $a < b$, a function f is continuous on an interval $[a, b]$ and has a derivative (finite or infinite) at every point of the interval (a, b). Then there is $\xi \in (a, b)$ satisfying

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

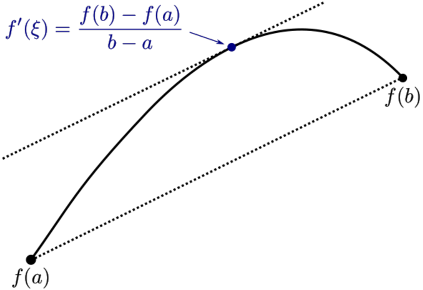


Figure: `https://en.wikipedia.org/wiki/File:Mittelwertsatz3.svg`

Theorem 8 (sign of the derivative and monotonicity)

Let $J \subset \mathbb{R}$ be a non-degenerate interval. Suppose that a function f is continuous on J and it has a derivative at every inner point of J (the set of all inner points of J is denoted by $\operatorname{Int} J$*).*

(i) *If $f'(x) > 0$ for all $x \in \operatorname{Int} J$, then f is increasing on J.*
(ii) *If $f'(x) < 0$ for all $x \in \operatorname{Int} J$, then f is decreasing on J.*
(iii) *If $f'(x) \geq 0$ for all $x \in \operatorname{Int} J$, then f in non-decreasing on J.*
(iv) *If $f'(x) \leq 0$ for all $x \in \operatorname{Int} J$, then f is non-increasing on J.*

https://mathinsight.org/applet/derivative_function

https://www.geogebra.org/m/mCTqH7u4

Theorem 9 (computation of a one-sided derivative)

Suppose that a function f is continuous from the right at $a \in \mathbb{R}$ and the limit $\lim\limits_{x \to a+} f'(x)$ exists. Then the derivative $f'_+(a)$ exists and

$$f'_+(a) = \lim_{x \to a+} f'(x).$$

Theorem 10 (l'Hospital's rule)

Suppose that functions f and g have finite derivatives on some punctured neighbourhood of $a \in \mathbb{R}^*$ and the limit $\lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$ exist. Suppose further that one of the following conditions hold:

(i) $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$,

(ii) $\lim\limits_{x\to a} |g(x)| = +\infty$.

Then the limit $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ exists and

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Exercise

$$\lim_{x\to\infty} \frac{\ln x}{x} =$$

A ∞ B 0 C 1 D $\nexists$

Convex and concave functions

Inspired by: `realisticky.cz`

Convex and concave functions

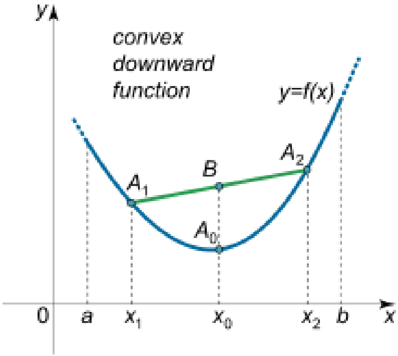


Figure: https://www.math24.net/convex-functions/

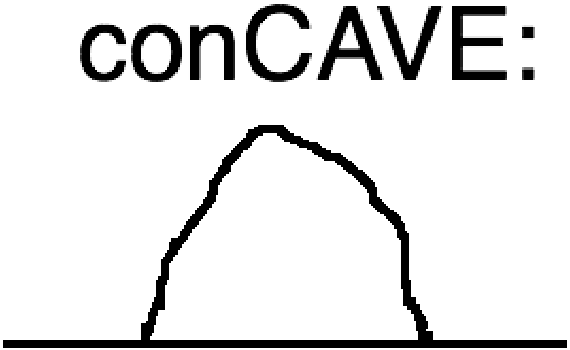


Figure: https://math.stackexchange.com/questions/3399/why-does-convex-function-mean-concave-up

Convex combination

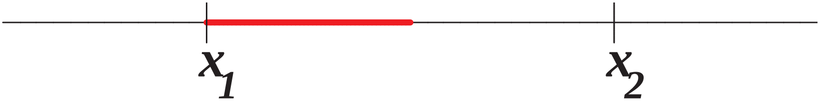
Convex combination

$$1 \cdot x_1 + 0 \cdot x_2 = x_1 + 0 \cdot (x_2 - x_1) = x_1$$

Convex combination

$$0 \cdot x_1 + 1 \cdot x_2 = x_1 + 1 \cdot (x_2 - x_1) = x_2$$

Convex combination



$$\frac{1}{2}x_1 + \frac{1}{2}x_2 = x_1 + \frac{1}{2}(x_2 - x_1)$$

Convex combination

$$\frac{3}{4}x_1 + \frac{1}{4}x_2 = x_1 + \frac{1}{4}(x_2 - x_1)$$

Convex combination

$$\frac{1}{4}x_1 + \frac{3}{4}x_2 = x_1 + \frac{3}{4}(x_2 - x_1)$$

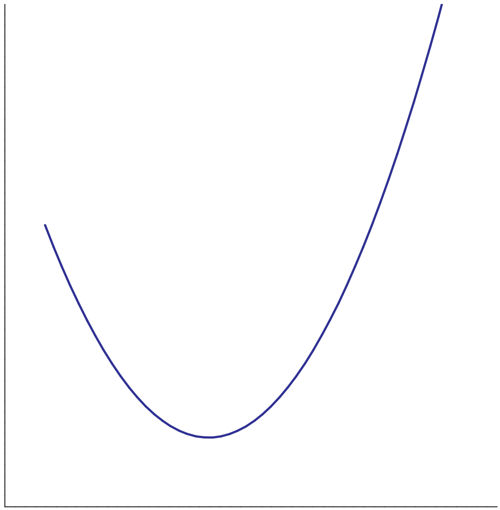
Convex combination

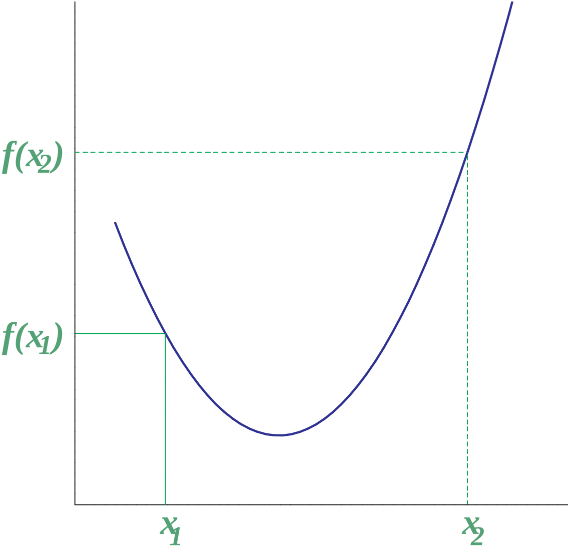
$$\lambda x_1 + (1-\lambda)x_2 = x_1 + (1-\lambda)(x_2 - x_1), \quad \lambda \in [0,1]$$

Definition

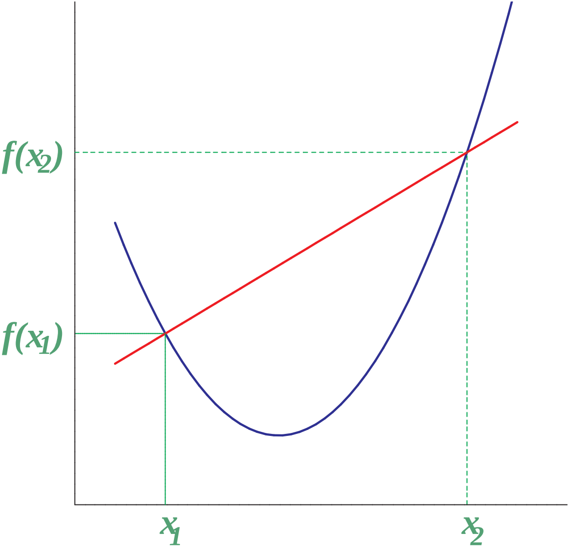
We say that a function f is

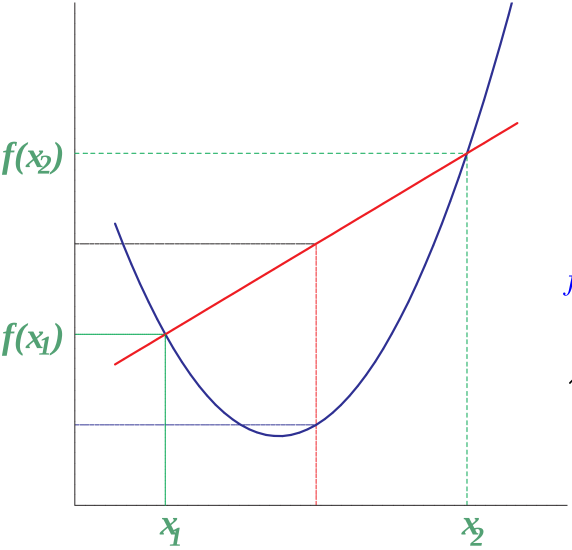
- convex on an interval I if
$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2),$$
for each $x_1, x_2 \in I$ and each $\lambda \in [0,1]$;
- concave on an interval I if
$$f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda)f(x_2),$$
for each $x_1, x_2 \in I$ and each $\lambda \in [0,1]$;
- strictly convex on an interval I if
$$f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda)f(x_2),$$
for each $x_1, x_2 \in I$, $x_1 \neq x_2$ and each $\lambda \in (0,1)$;
- strictly concave on an interval I if
$$f(\lambda x_1 + (1-\lambda)x_2) > \lambda f(x_1) + (1-\lambda)f(x_2).$$
for each $x_1, x_2 \in I$, $x_1 \neq x_2$ and each $\lambda \in (0,1)$.





f(x2)
f(x1)
x1
x2

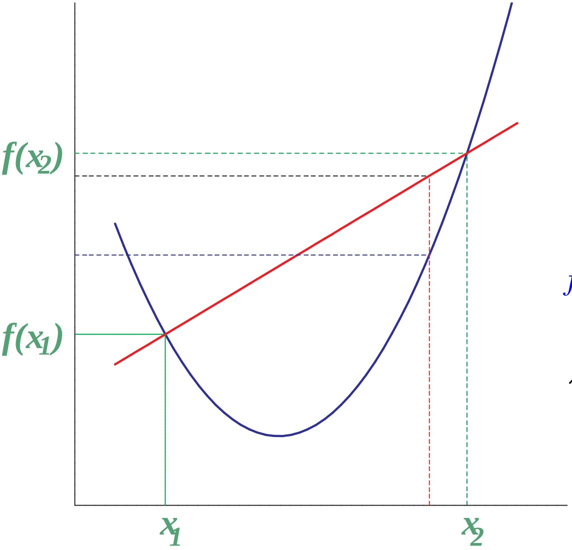

f(x2)
f(x1)
x1
x2



$$\lambda x_1 + (1-\lambda)x_2$$

$$f(\lambda x_1 + (1-\lambda)x_2)$$

$$\lambda f(x_1) + (1-\lambda) f(x_2)$$



$\lambda x_1 + (1-\lambda)x_2$

$f(\lambda x_1 + (1-\lambda)x_2)$

$\lambda f(x_1)+(1-\lambda)f(x_2)$

Lemma 11

A function f is convex on an interval I if and only if

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

for each three points $x_1, x_2, x_3 \in I$, $x_1 < x_2 < x_3$.

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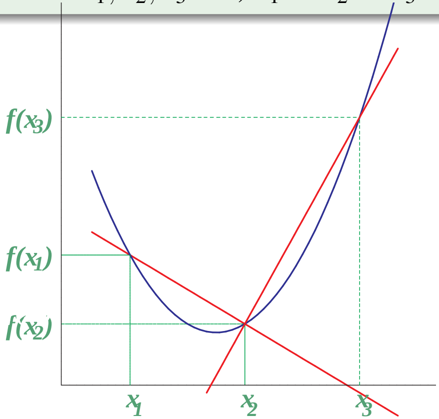
for each three points $x_1, x_2, x_3 \in I$, $x_1 < x_2 < x_3$.



Definition

Suppose that a function f has a finite derivative on some neighbourhood of $a \in \mathbb{R}$. The second derivative of f at a is defined by

$$f''(a) = \lim_{h \to 0} \frac{f'(a+h) - f'(a)}{h}$$

if the limit exists.

Definition

Suppose that a function f has a finite derivative on some neighbourhood of $a \in \mathbb{R}$. The second derivative of f at a is defined by

$$f''(a) = \lim_{h \to 0} \frac{f'(a+h) - f'(a)}{h}$$

if the limit exists.

Let $n \in \mathbb{N}$ and suppose that f has a finite nth derivative (denoted by $f^{(n)}$) on some neighbourhood of $a \in \mathbb{R}$. Then the $(n+1)$th derivative of f at a is defined by

$$f^{(n+1)}(a) = \lim_{h \to 0} \frac{f^{(n)}(a+h) - f^{(n)}(a)}{h}$$

if the limit exists.

Theorem 12 (second derivative and convexity)

Let $a, b \in \mathbb{R}^*$, $a < b$, and suppose that a function f has a finite second derivative on the interval (a, b).

(i) *If $f''(x) > 0$ for each $x \in (a, b)$, then f is strictly convex on (a, b).*

(ii) *If $f''(x) < 0$ for each $x \in (a, b)$, then f is strictly concave on (a, b).*

(iii) *If $f''(x) \geq 0$ for each $x \in (a, b)$, then f is convex on (a, b).*

(iv) *If $f''(x) \leq 0$ for each $x \in (a, b)$, then f is concave on (a, b).*

https://www.geogebra.org/m/rqebuwyw https://www.khanacademy.org/math/ap-calculus-ab/ab-diff-analytical-applications-new/ab-5-9/e/connecting-function-and-derivatives

Definition

Suppose that a function f has a finite derivative at $a \in \mathbb{R}$ and let T_a denote the tangent to the graph of f at $[a, f(a)]$. We say that the point $[x, f(x)]$ lies below the tangent T_a if

$$f(x) < f(a) + f'(a) \cdot (x - a).$$

We say that the point $[x, f(x)]$ lies above the tangent T_a if the opposite inequality holds.

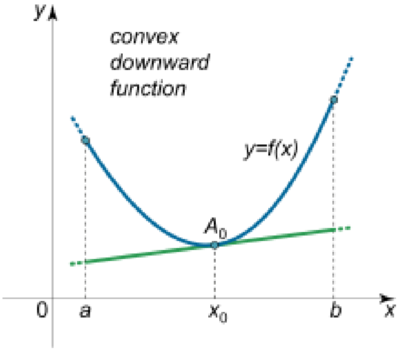


Figure: https://www.math24.net/convex-functions/

Definition

Suppose that a function f has a finite derivative at $a \in \mathbb{R}$ and let T_a denote the tangent to the graph of f at $[a, f(a)]$. We say that a is an inflection point of f if there is $\Delta > 0$ such that

(i) $\forall x \in (a - \Delta, a): [x, f(x)]$ lies below the tangent T_a,

(ii) $\forall x \in (a, a + \Delta): [x, f(x)]$ lies above the tangent T_a,

Definition

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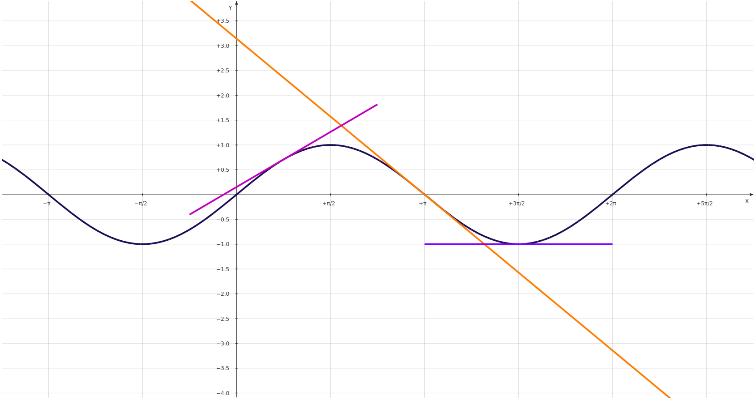
(i) $\forall x \in (a - \Delta, a)$: $[x, f(x)]$ lies below the tangent T_a,

(ii) $\forall x \in (a, a + \Delta)$: $[x, f(x)]$ lies above the tangent T_a,

or

(i) $\forall x \in (a - \Delta, a)$: $[x, f(x)]$ lies above the tangent T_a,

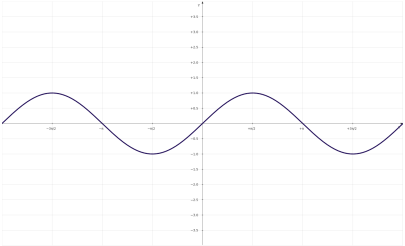
(ii) $\forall x \in (a, a + \Delta)$: $[x, f(x)]$ lies below the tangent T_a.

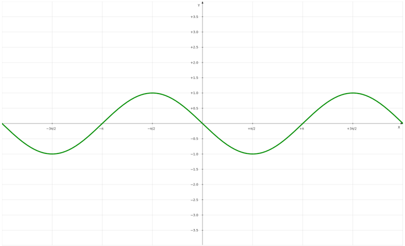


https://en.wikipedia.org/wiki/Inflection_point#/media/File:Animated_illustration_of_inflection_point.gif

Theorem 13 (necessary condition for inflection)

Let $a \in \mathbb{R}$ be an inflection point of a function f. Then $f''(a)$ either does not exist or equals zero.





Theorem 14 (necessary condition for inflection)

Let $a \in \mathbb{R}$ be an inflection point of a function f. Then $f''(a)$ either does not exist or equals zero.

$$(x^4 - x)'' = 12x^2$$

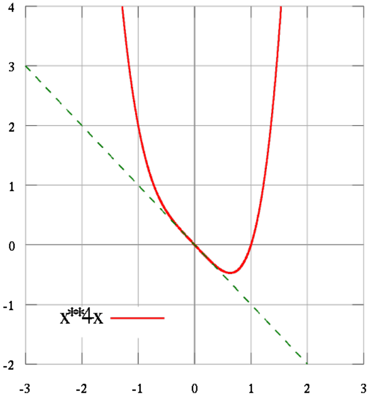


Figure:

Theorem 15 (necessary condition for inflection)

Let $a \in \mathbb{R}$ be an inflection point of a function f. Then $f''(a)$ either does not exist or equals zero.

Theorem 15 (necessary condition for inflection)

Let $a \in \mathbb{R}$ be an inflection point of a function f. Then $f''(a)$ either does not exist or equals zero.

Theorem 16 (sufficient condition for inflection)

Suppose that a function f has a continuous first derivative on an interval (a, b) and $z \in (a, b)$. Suppose further that

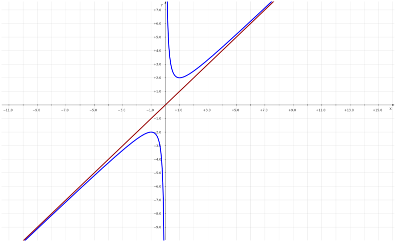
- $\forall x \in (a, z) : f''(x) > 0,$
- $\forall x \in (z, b) : f''(x) < 0.$

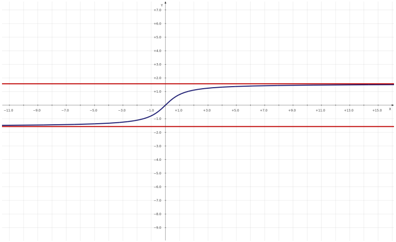
Then z is an inflection point of f.

Definition

The line which is a graph of an affine function $x \mapsto kx + q$, $k, q \in \mathbb{R}$, is called an asymptote of the function f at $+\infty$ (resp. v $-\infty$) if

$$\lim_{x \to +\infty} (f(x) - kx - q) = 0, \quad \text{(resp. } \lim_{x \to -\infty} (f(x) - kx - q) = 0\text{)}.$$





Definition

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$$\lim_{x\to+\infty} (f(x) - kx - q) = 0, \quad (\text{resp. } \lim_{x\to-\infty} (f(x) - kx - q) = 0).$$

Proposition 17

A function f has an asymptote at $+\infty$ given by the affine function $x \mapsto kx + q$ if and only if

$$\lim_{x\to+\infty} \frac{f(x)}{x} = k \in \mathbb{R} \quad \textit{and} \quad \lim_{x\to+\infty} (f(x) - kx) = q \in \mathbb{R}.$$

Exercise

Let us assume that a function $y = f(x)$ is continuous at $\mathbb{R}$. Sketch f.

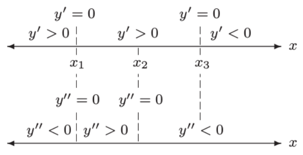

Figure: Calculus, Hughes-Hallet, Gleason, McCallum

Exercise

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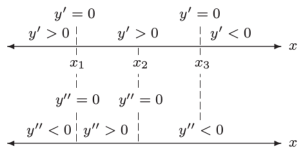
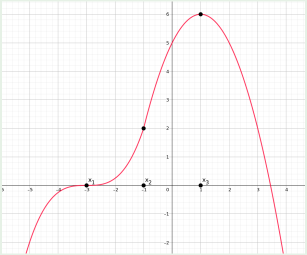


Figure: Calculus, Hughes-Hallet, Gleason, McCallum



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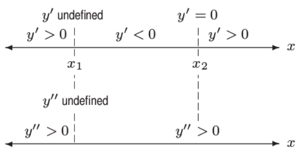

Figure: Calculus, Hughes-Hallet, Gleason, McCallum

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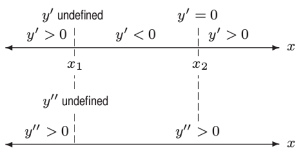
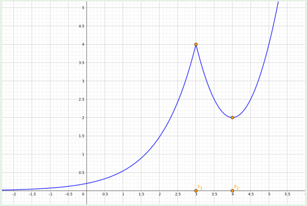


Figure: Calculus, Hughes-Hallet, Gleason, McCallum



Investigation of a function

1. Determine the domain and discuss the continuity of the function.
2. Find out symmetries: oddness, evenness, periodicity.
3. Find the limits at the "endpoints of the domain".
4. Investigate the first derivative, find the intervals of monotonicity and local and global extrema. Determine the range.
5. Find the second derivative and determine the intervals where the function is concave or convex. Find the inflection points.
6. Find the asymptotes of the function.
7. Draw the graph of the function.