Mathematics II - Integrals

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VII. Antiderivatives and Riemann integral

VII.1. Antiderivatives

Definition

Let f be a function defined on an open interval I. We say that a function $F : I \to \mathbb{R}$ is an antiderivative of f on I if for each $x \in I$ the derivative $F'(x)$ exists and $F'(x) = f(x)$.

Exercise

Connect the functions in the left column with their antiderivatives on the right.

1. 0	A $-\cos x$
2. 1	B $\sin x$
3. x	C x
4. $\cos x$	D 1
5. $\sin x$	E $\frac{x^2}{2}$

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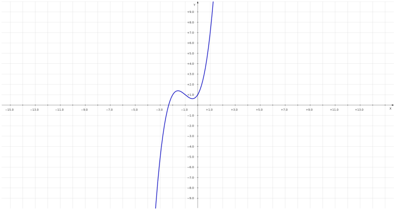
Exercise

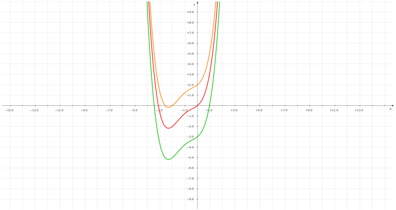
Find $\int e^x \, \mathrm{d}x$:

A e^x
B $-e^x$
C $e^x + 3$
D $e^x + e^\pi$
E $2e^x + 2$

Theorem 1 (Uniqueness of an antiderivative)

Let F and G be antiderivatives of f on an open interval I. Then there exists $c \in \mathbb{R}$ such that $F(x) = G(x) + c$ for each $x \in I$.





Remark

The set of all antiderivatives of f on an open interval I is denoted by

$$\int f(x)\,\mathrm{d}x.$$

The fact that F is an antiderivative of f on I is expressed by

$$\int f(x)\,\mathrm{d}x \stackrel{c}{=} F(x), \quad x \in I.$$

Exercise

Find $\int x \sin x$.

A $F = \sin x + x \cos x$

B $F = \sin x - x \cos x$

C $F = x \sin x + \cos x$

Exercise

Find F. You know that $F = \int 3x^2 + 2x \,\mathrm{d}x$ and $F(0) = 1$.

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Exercise (True or false?)

A If $f'(x) = g'(x)$, then $f(x) = g(x)$ (for all x).

B If $\int f(x) = \int g(x)$, then $f(x) = g(x)$ (for all x).

http://www.math.cornell.edu/~GoodQuestions/GQbysection_pdfversion.pdf

Table of basic antiderivatives

- $\int x^n \,\mathrm{d}x \overset{c}{=} \frac{x^{n+1}}{n+1}$ on $\mathbb{R}$ for $n \in \mathbb{N} \cup \{0\}$; on $(-\infty, 0)$ and on $(0, \infty)$ for $n \in \mathbb{Z}, n < -1$,
- $\int x^\alpha \,\mathrm{d}x \overset{c}{=} \frac{x^{\alpha+1}}{\alpha+1}$ on $(0, +\infty)$ for $\alpha \in \mathbb{R} \setminus \{-1\}$,
- $\int \frac{1}{x} \,\mathrm{d}x \overset{c}{=} \log|x|$ on $(0, +\infty)$ and on $(-\infty, 0)$,
- $\int e^x \,\mathrm{d}x \overset{c}{=} e^x$ on $\mathbb{R}$,
- $\int \sin x \,\mathrm{d}x \overset{c}{=} -\cos x$ on $\mathbb{R}$,
- $\int \cos x \,\mathrm{d}x \overset{c}{=} \sin x$ on $\mathbb{R}$,

- $\displaystyle\int \frac{1}{\cos^2 x}\,\mathrm{d}x \overset{c}{=} \operatorname{tg} x$ on each of the intervals $(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi)$, $k \in \mathbb{Z}$,
- $\displaystyle\int \frac{1}{\sin^2 x}\,\mathrm{d}x \overset{c}{=} -\operatorname{cotg} x$ on each of the intervals $(k\pi, \pi + k\pi)$, $k \in \mathbb{Z}$,
- $\displaystyle\int \frac{1}{1+x^2}\,\mathrm{d}x \overset{c}{=} \operatorname{arctg} x$ on $\mathbb{R}$,
- $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x \overset{c}{=} \arcsin x$ on $(-1, 1)$,
- $\displaystyle\int -\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x \overset{c}{=} \arccos x$ on $(-1, 1)$.

https://www.flippity.net/mg.php?k=
1F5i3udTbsLHBaMpm3DcWBM3sc9757rctP-S9gM8Ejgc

Theorem 2 (Existence of an antiderivative)

Let f be a continuous function on an open interval I. Then f has an antiderivative on I.

Exercise

Which of the following functions definitely have primitive function?

- A $\frac{1}{x}, x \in \mathbb{R}$
- B $\arctan x^2, x \in \mathbb{R}$
- C $\ln x, x \in (0, \infty)$
- D $\frac{x^2}{x^3+1}, x \in \mathbb{R}$
- E $\cot x, x \in (0, \pi)$

Remark

The following functions have antiderivatives, but it can not be easily expressed.

- $\int e^{x^2} \, \mathrm{d}x$
- $\int \ln(\ln x) \, \mathrm{d}x$
- $\int \sin x^2 \, \mathrm{d}x$
- $\int \frac{\sin x}{x} \, \mathrm{d}x$
- $\int \sqrt{1 - x^4} \, \mathrm{d}x$

Theorem 3 (Linearity of antiderivatives)

Suppose that f has an antiderivative F on an open interval I, g has an antiderivative G on I, and let $\alpha, \beta \in \mathbb{R}$. Then the function $\alpha F + \beta G$ is an antiderivative of $\alpha f + \beta g$ on I.

Theorem 4 (substitution)

(i) *Let F be an antiderivative of f on (a,b). Let $\varphi\colon (\alpha,\beta) \to (a,b)$ have a finite derivative at each point of (α,β). Then*

$$\int f\big(\varphi(x)\big)\varphi'(x)\,\mathrm{d}x \overset{c}{=} F\big(\varphi(x)\big) \quad \text{on } (\alpha,\beta).$$

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$$\int f\big(\varphi(x)\big)\varphi'(x)\,\mathrm{d}x \overset{c}{=} F\big(\varphi(x)\big)\quad \textit{on } (\alpha,\beta).$$

(ii) *Let φ be a function with a finite derivative in each point of (α,β) such that the derivative is either everywhere positive or everywhere negative, and such that $\varphi\big((\alpha,\beta)\big)=(a,b)$. Let f be a function defined on (a,b) and suppose that*

$$\int f\big(\varphi(t)\big)\varphi'(t)\,\mathrm{d}t \overset{c}{=} G(t)\quad \textit{on } (\alpha,\beta).$$

Then

$$\int f(x)\,\mathrm{d}x \overset{c}{=} G\big(\varphi^{-1}(x)\big)\quad \textit{on } (a,b).$$

Theorem 5 (integration by parts)

Let I be an open interval and let the functions f and g be continuous on I. Let F be an antiderivative of f on I and G an antiderivative of g on I. Then

$$\int f(x)G(x)\,\mathrm{d}x = F(x)G(x) - \int F(x)g(x)\,\mathrm{d}x \quad \textit{on } I.$$

Remark

We can write as $\int fg' = fg - \int f'g$ or $\int u'v = uv - \int uv'$.

https://cs.khanacademy.org/math/integralni-pocet/xbf9b4d9711003f1c:integracni-metody/xbf9b4d9711003f1c:integrace-per-partes/v/integral-of-ln-x

Example

$$\int x \cos x \qquad \int x \arctan x$$

Let $P(x)$ be a polynomial. The following table can help to choose u' and v.

	$v(x)$	$u'(x)$
$P(x) \cdot e^{kx}$	$P(x)$	e^{kx}
$P(x) \cdot a^{kx}$	$P(x)$	a^{kx}
$P(x) \cdot \sin(kx)$	$P(x)$	$\sin(kx)$
$P(x) \cdot \cos(kx)$	$P(x)$	$\cos(kx)$

	$v(x)$	$u'(x)$
$P(x) \cdot \ln^n x$	$\ln^n x$	$P(x)$
$P(x) \cdot \arcsin(kx)$	$\arcsin(kx)$	$P(x)$
$P(x) \cdot \arccos(kx)$	$\arccos(kx)$	$P(x)$
$P(x) \cdot \arctan(kx)$	$\arctan(kx)$	$P(x)$
$P(x) \cdot \mathrm{arccotg}(kx)$	$\mathrm{arccotg}(kx)$	$P(x)$

Exercise

Find integrals, which should be solved by integration by parts

A $\int xe^{x^2}\,\mathrm{d}x$

B $\int x\cos x\,\mathrm{d}x$

C $\int 1\ln x\,\mathrm{d}x$

D $\int \frac{x}{\ln x}\,\mathrm{d}x$

E $\int \sin x \ln x^2\,\mathrm{d}x$

Exercise

By parts or by substitution?

A $\int \arcsin x \, dx$

B $\int \frac{x}{1+x^2} \, dx$

C $\int (x^2 - 3) \ln x \, dx$

D $\int \frac{1}{x \ln x} \, dx$

E $\int x^2 \cos 2x \, dx$

https://learningapps.org/display?v=pgeigqe6j21

Exercise

True or false?

- A $\int kf = k \int f$
- B $\int f + g = \int f + \int g$
- C $\int f - g = \int f - \int g$
- D $\int f \cdot g = \int f \cdot \int g$
- E $\int f/g = \int f / \int g$

Exercise

Find a mistake.

1.

$$\int \frac{3x^2+1}{2x}\,\mathrm{d}x = \frac{x^3+x}{x^2} + c$$

2. $\forall a \in \mathbb{R}$

$$\int x^a\,\mathrm{d}x = \frac{x^{a+1}}{a+1} + c$$

Calculus: Single and Multivariable, 6th Edition, Deborah Hughes-Hallett and col.

Definition

A rational function is a ratio of two polynomials, where the polynomial in the denominator is not a zero polynomial.

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Exercise

Find rational functions.

A $\frac{3x-4+x^4}{x^2-2x+1}$

B x^6+5

C $\frac{x^5-8x+2}{3}$

D $\frac{\sqrt{2}+5}{1+\sqrt[3]{x^3-8}}$

E $\frac{(3x-4)(2x+5)}{(x-1)(x^2+2)}$

Theorem ("fundamental theorem of algebra")

Let $n \in \mathbb{N}$, $a_0, \ldots, a_n \in \mathbb{C}$, $a_n \neq 0$. *Then the equation*

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$$

has at least one solution $z \in \mathbb{C}$.

Lemma 6 (polynomial division)

Let P and Q be polynomials (with complex coefficients) such that Q is not a zero polynomial. Then there are uniquely determined polynomials S and R satisfying:

- $\deg R < \deg Q$,
- $P(x) = S(x)Q(x) + R(x)$ *for all* $x \in \mathbb{C}$.

If P and Q have real coefficients then so have S and R.

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Corollary

If P is a polynomials and $\lambda \in \mathbb{C}$ its root (i.e. $P(\lambda) = 0$), then there is a polynomial S satisfying $P(x) = (x - \lambda)S(x)$ for all $x \in \mathbb{C}$.

Example

$$x^2 - x - 6 = (x + 2)(x - 3)$$

$$x^3 + x = x(x - i)(x + i)$$

Theorem 7 (factorisation into monomials)

Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ *be a polynomial of degree* $n \in \mathbb{N}$*. Then there are numbers* $x_1, \ldots, x_n \in \mathbb{C}$ *such that*

$$P(x) = a_n(x - x_1)\cdots(x - x_n), \quad x \in \mathbb{C}.$$

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$$P(x) = a_n(x - x_1) \cdots (x - x_n), \quad x \in \mathbb{C}.$$

Definition

Let P be a polynomial that is not zero, $\lambda \in \mathbb{C}$, and $k \in \mathbb{N}$. We say that λ is a root of multiplicity k of the polynomial P if there is a polynomial S satisfying $S(\lambda) \neq 0$ and $P(x) = (x - \lambda)^k S(x)$ for all $x \in \mathbb{C}$.

Exercise

Find the multiplicity of $\lambda = -2$ of the polynomial $P(x) = (x^2 + x - 2)(x + 2)^3$.

A -2   B 1   C 2   D 3   E 4

Theorem 8 (roots of a polynomial with real coefficients)

Let P be a polynomial with real coefficients and $\lambda \in \mathbb{C}$ a root of P of multiplicity $k \in \mathbb{N}$. Then the also the conjugate number $\overline{\lambda}$ is a root of P of multiplicity k.

Theorem 9 (factorisation of a polynomial with real coefficients)

Let $P(x) = a_n x^n + \cdots + a_1 x + a_0$ *be a polynomial of degree* n *with real coefficients. Then there exist real numbers* $x_1, \ldots, x_k$, $\alpha_1, \ldots, \alpha_l$, $\beta_1, \ldots, \beta_l$ *and natural numbers* $p_1, \ldots, p_k$, $q_1, \ldots, q_l$ *such that*

- $P(x) = a_n (x - x_1)^{p_1} \cdots (x - x_k)^{p_k} (x^2 + \alpha_1 x + \beta_1)^{q_1} \cdots (x^2 + \alpha_l x + \beta_l)^{q_l}$,

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- *no two polynomials from $x - x_1, x - x_2, \ldots, x - x_k$, $x^2 + \alpha_1 x + \beta_1, \ldots, x^2 + \alpha_l x + \beta_l$ have a common root,*

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- $P(x) = a_n (x - x_1)^{p_1} \cdots (x - x_k)^{p_k} (x^2 + \alpha_1 x + \beta_1)^{q_1} \cdots (x^2 + \alpha_l x + \beta_l)^{q_l}$,
- *no two polynomials from* $x - x_1, x - x_2, \ldots, x - x_k$, $x^2 + \alpha_1 x + \beta_1, \ldots, x^2 + \alpha_l x + \beta_l$ *have a common root,*
- *the polynomials* $x^2 + \alpha_1 x + \beta_1, \ldots, x^2 + \alpha_l x + \beta_l$ *have no real root.*

Theorem 10 (decomposition to partial fractions)

Let P, Q be polynomials with real coefficients such that $\deg P < \deg Q$ and let

$$Q(x) = a_n(x - x_1)^{p_1} \cdots (x - x_k)^{p_k}(x^2 + \alpha_1 x + \beta_1)^{q_1} \cdots (x^2 + \alpha_l x + \beta_l)^{q_l}$$

be a factorisation of from Theorem 9. Then there exist unique real numbers $A_1^1, \dots, A_{p_1}^1, \dots, A_1^k, \dots, A_{p_k}^k$, $B_1^1, C_1^1, \dots, B_{q_1}^1, C_{q_1}^1, \dots, B_1^l, C_1^l, \dots, B_{q_l}^l, C_{q_l}^l$ *such that*

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$$\tfrac{P(x)}{Q(x)} = \tfrac{A_1^1}{(x-x_1)} + \cdots + \tfrac{A_{p_1}^1}{(x-x_1)^{p_1}} +$$

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$$\frac{P(x)}{Q(x)} = \frac{A_1^1}{(x-x_1)}+\cdots+\frac{A_{p_1}^1}{(x-x_1)^{p_1}}+\cdots+\frac{A_1^k}{(x-x_k)}+\cdots+\frac{A_{p_k}^k}{(x-x_k)^{p_k}}+$$

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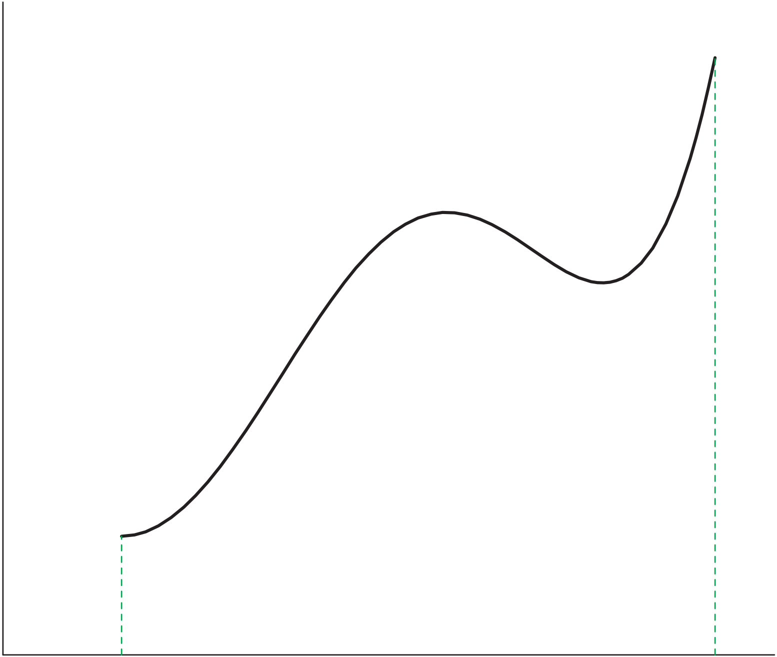
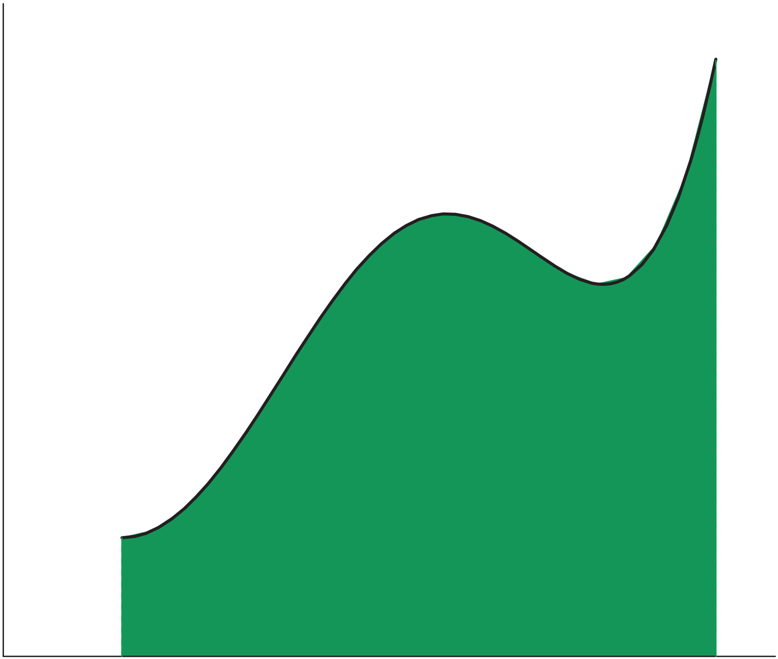
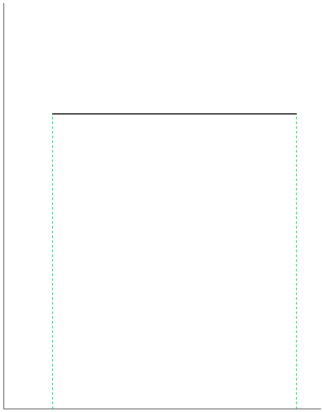
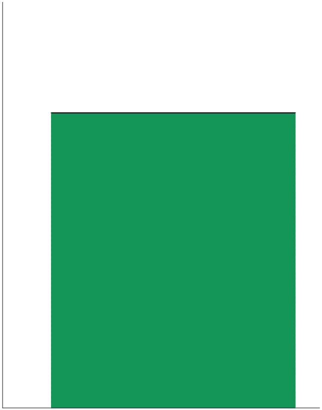
$$+ \frac{B_1^l x + C_1^l}{(x^2+\alpha_l x+\beta_l)} + \cdots + \frac{B_{q_l}^l x + C_{q_l}^l}{(x^2+\alpha_l x+\beta_l)^{q_l}}, x \in \mathbb{R} \setminus \{x_1, \ldots, x_k\}.$$

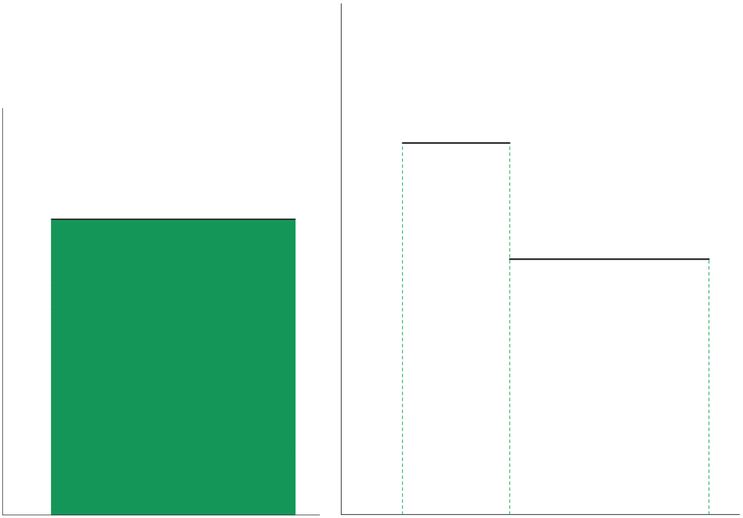
https://www.geogebra.org/calculator/aw2yjsjx

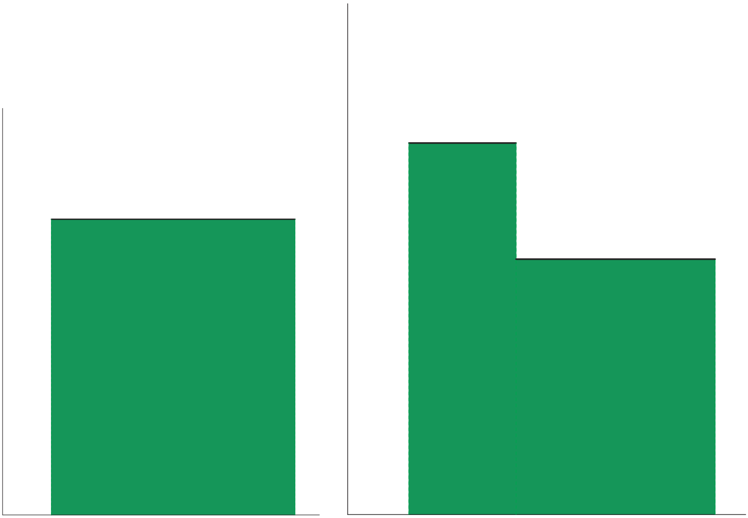


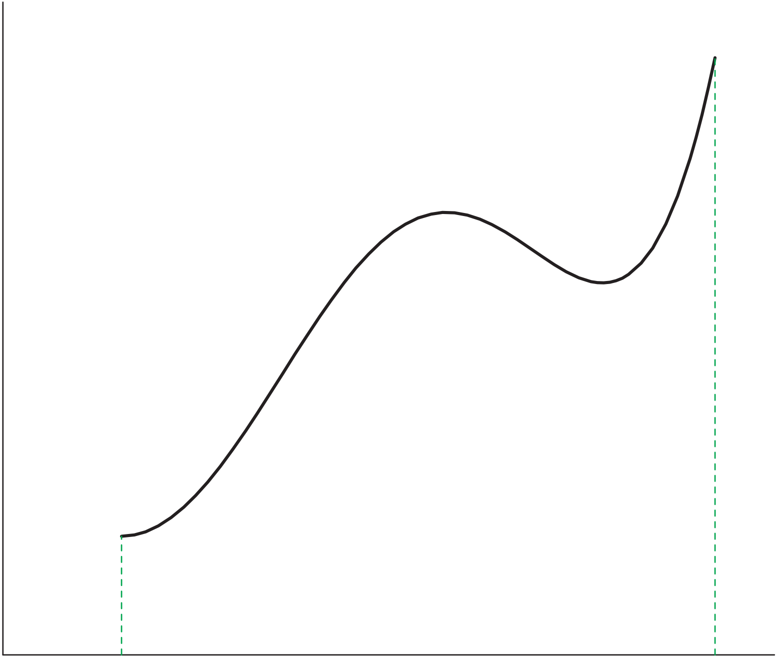


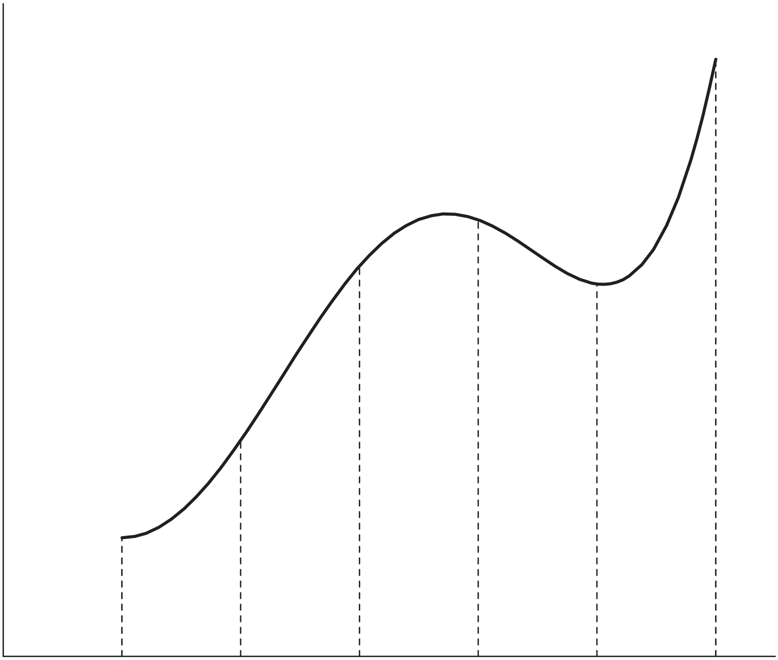


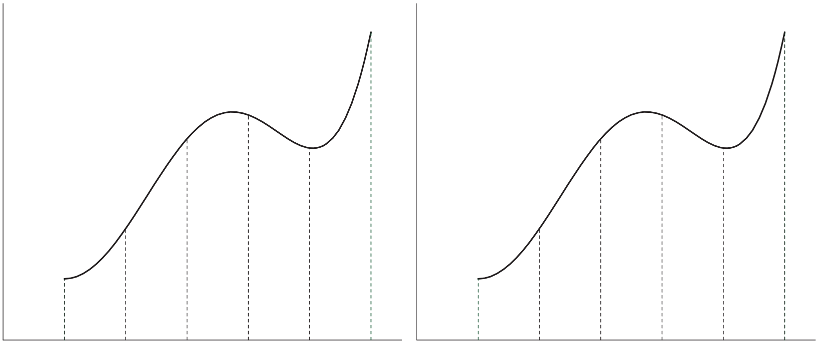


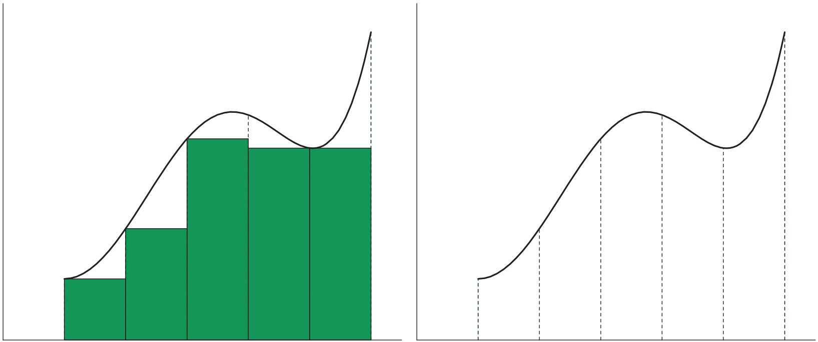


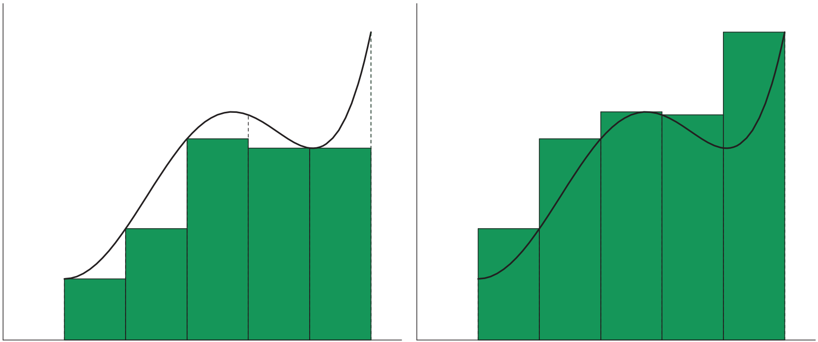


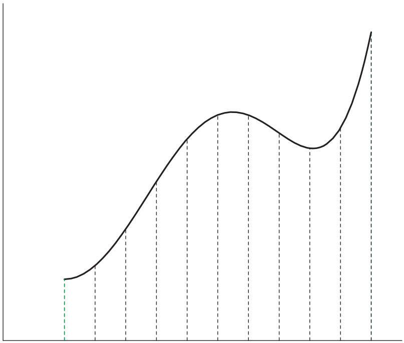


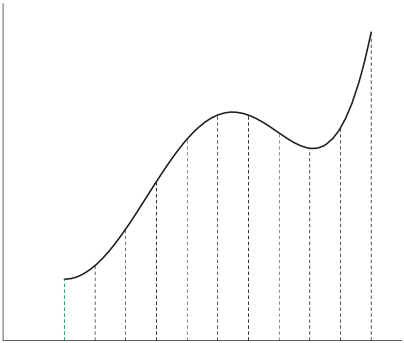


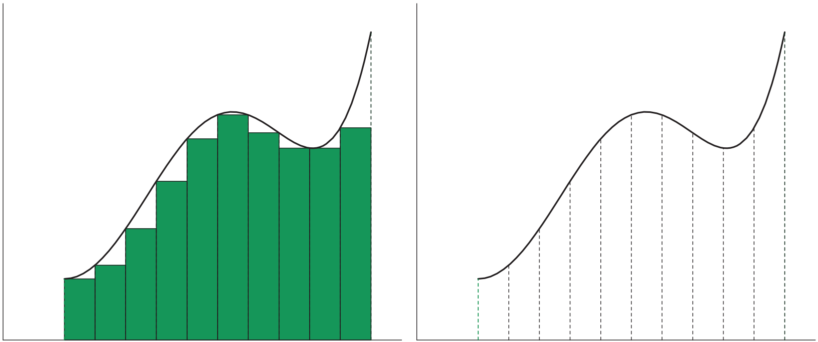


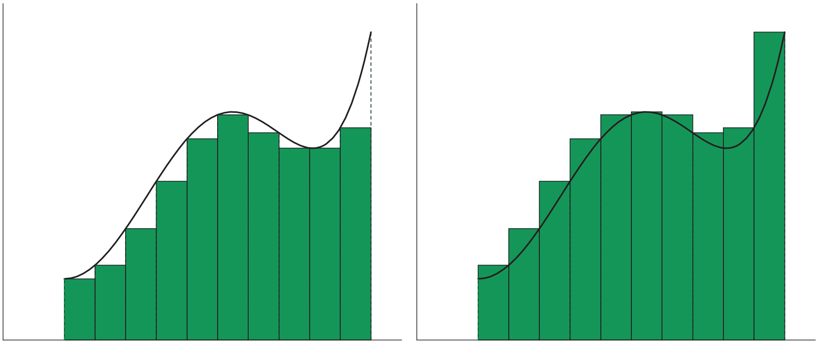












Definition

A finite sequence $\{x_j\}_{j=0}^{n}$ *is called a partition of the interval* $[a, b]$ *if*

$$a = x_0 < x_1 < \cdots < x_n = b.$$

The points $x_0, \ldots, x_n$ *are called the partition points.*

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We say that a partition D' of an interval $[a,b]$ is a refinement of the partition D of $[a,b]$ if each partition point of D is also a partition point of D'.

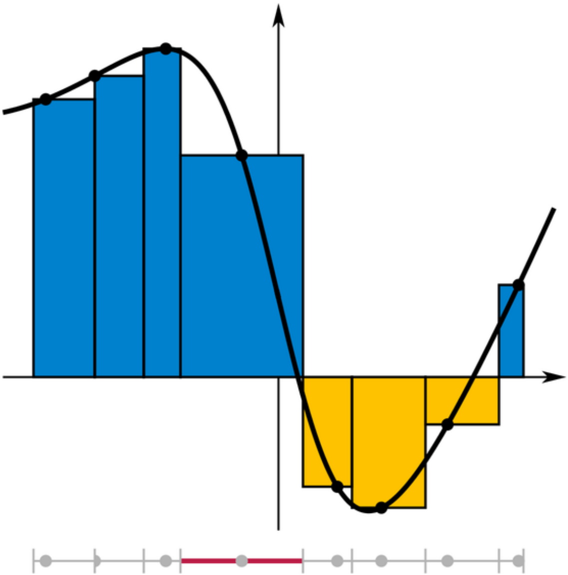

Figure: `https://en.wikipedia.org/wiki/Integral`

Definition

Suppose that $a, b \in \mathbb{R}$, $a < b$, *the function* f *is bounded on* $[a, b]$, *and* $D = \{x_j\}_{j=0}^{n}$ *is a partition of* $[a, b]$. *Denote*

$$\overline{S}(f, D) = \sum_{j=1}^{n} M_j(x_j - x_{j-1}), \text{ where } M_j = \sup\{f(x);\; x \in [x_{j-1}, x_j]\},$$

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$$\overline{\int_a^b} f = \inf\left\{\overline{S}(f, D);\ D \textit{ is a partition of } [a, b]\right\},$$

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Definition

We say that a function f *has the* *Riemann integral* *over the interval* $[a, b]$ *if* $\overline{\int_a^b} f = \underline{\int_a^b} f$.

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If $a > b$, then we define $\displaystyle\int_a^b f = -\int_b^a f$, and in case that $a = b$ we put $\displaystyle\int_a^b f = 0$.

Exercise

Use the Riemann sums and estimate the integral

$$\int_0^{15} f(x)\,\mathrm{d}x.$$

Check the table for some values of f:

x	0	3	6	9	12	15
$f(x)$	50	48	44	36	24	8

Table: Applied Calculus, 6th Edition, Deborah Hughes-Hallett and col.

Theorem 11 (Newton-Leibniz formula)

Let f be a function continuous on an interval $(a-\varepsilon, b+\varepsilon)$, $a, b \in \mathbb{R}$, $a < b$, $\varepsilon > 0$ and let F be an antiderivative of f on $(a-\varepsilon, b+\varepsilon)$. Then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a). \tag{1}$$

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$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a). \tag{1}$$

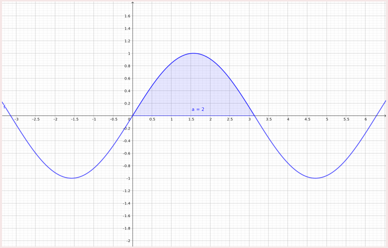
Remark

The Newton-Leibniz formula (1) holds even if $b < a$ (if $F' = f$ on $(b-\varepsilon, a+\varepsilon)$). Let us denote

$$[F]_a^b = F(b) - F(a).$$

Example

$$\int_0^\pi \sin x \, \mathrm{d}x = [-\cos x]_0^\pi = -\cos\pi - (-\cos 0) = 1 + 1 = 2$$



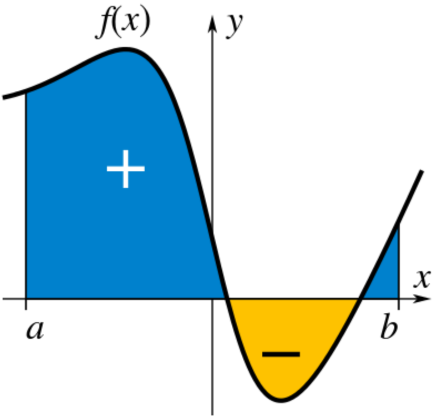
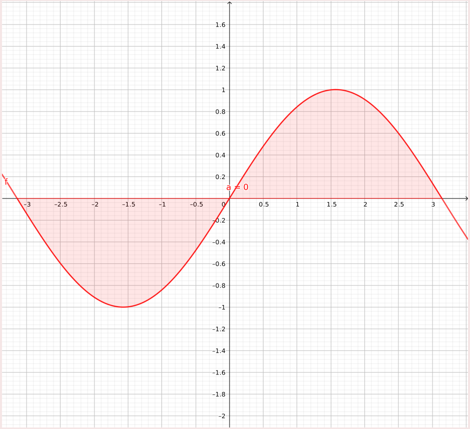


Figure: https://en.wikipedia.org/wiki/Integral

Example

$$\int_{-\pi}^{\pi} \sin x \, \mathrm{d}x = [-\cos x]_{-\pi}^{\pi} = -\cos\pi - (-\cos -\pi) = 1 - 1 = 0$$



Theorem 12 (integration by parts)

Suppose that the functions f, g, f' a g' are continuous on an interval $[a,b]$. Then

$$\int_a^b f'g = [fg]_a^b - \int_a^b fg'.$$

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Theorem 13 (substitution)

Let the function f be continuous on an interval $[a, b]$. Suppose that the function φ has a continuous derivative on $[\alpha, \beta]$ and φ maps $[\alpha, \beta]$ into the interval $[a, b]$. Then

$$\int_\alpha^\beta f\big(\varphi(x)\big)\varphi'(x)\,\mathrm{d}x = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(t)\,\mathrm{d}t.$$

https://www.geogebra.org/calculator/frvx4mtr
https://www.geogebra.org/calculator/cjuvxazd

Remark

Let D, D' be partitions of $[a, b]$, D' refines D, and let f be a bounded function on $[a, b]$. Then

$$\underline{S}(f, D) \leq \underline{S}(f, D') \leq \overline{S}(f, D') \leq \overline{S}(f, D).$$

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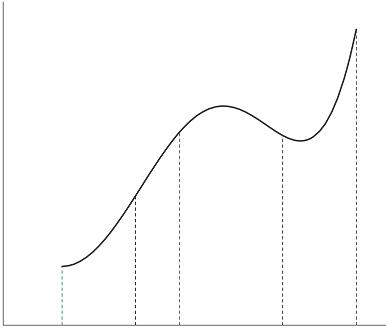

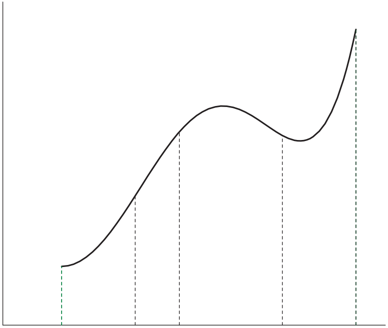

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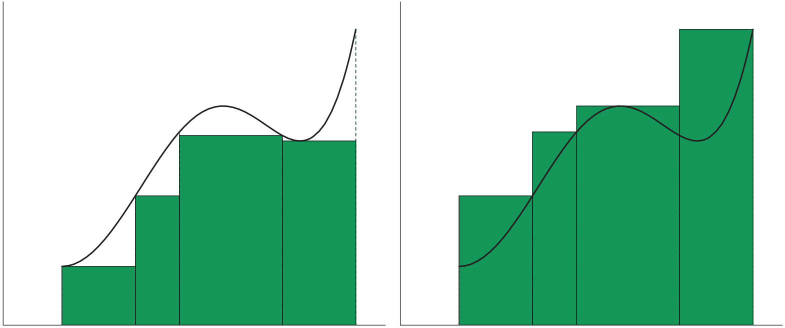

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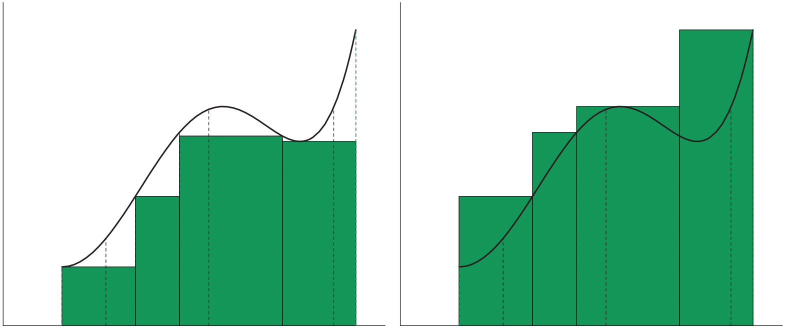

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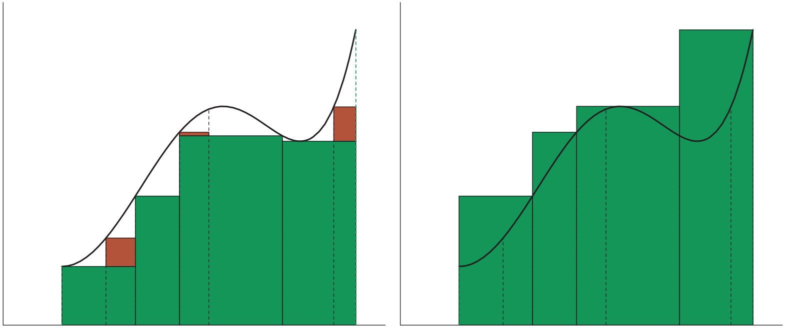

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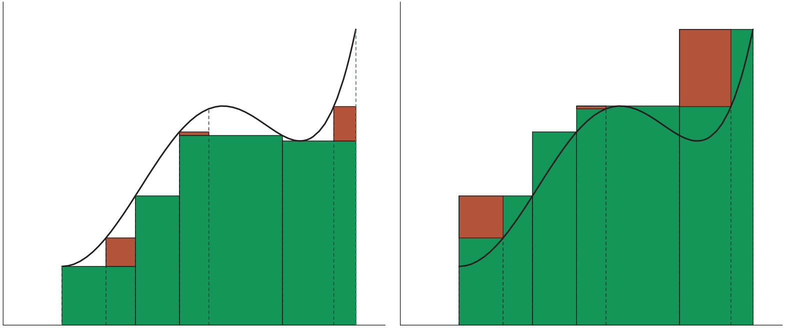

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Suppose that D_1, D_2 are partitions of $[a,b]$ and a partition D' refines both D_1 and D_2. Then

$$\underline{S}(f,D_1) \leq \underline{S}(f,D') \leq \overline{S}(f,D') \leq \overline{S}(f,D_2).$$

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$$\underline{S}(f,D_1) \leq \underline{S}(f,D') \leq \overline{S}(f,D') \leq \overline{S}(f,D_2).$$

It easily follows that $\underline{\int_a^b} f \leq \overline{\int_a^b} f$.

Theorem 14

(i) *Suppose that f has the Riemann integral over $[a, b]$ and let $[c, d] \subset [a, b]$. Then f has the Riemann integral also over $[c, d]$.*

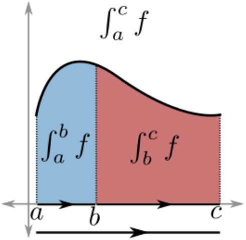


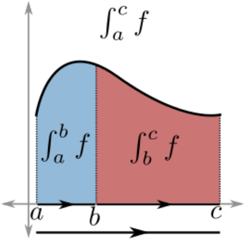
Figure: http://calculus.seas.upenn.edu/?n=Main.DefiniteIntegrals

Theorem 14

(i) *Suppose that f has the Riemann integral over $[a, b]$ and let $[c, d] \subset [a, b]$. Then f has the Riemann integral also over $[c, d]$.*

(ii) *Suppose that $c \in (a, b)$ and f has the Riemann integral over the intervals $[a, c]$ and $[c, b]$. Then f has the Riemann integral over $[a, b]$ and*

$$\int_a^b f = \int_a^c f + \int_c^b f. \tag{2}$$



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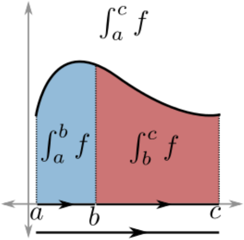

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Theorem 15

(i) *Suppose that f has the Riemann integral over $[a, b]$ and let $[c, d] \subset [a, b]$. Then f has the Riemann integral also over $[c, d]$.*

Remark

The formula (3) holds for all $a, b, c \in \mathbb{R}$ if the integral of f exists over the interval $\left[\min\{a, b, c\}, \max\{a, b, c\}\right]$.

Theorem 15

(i) *Suppose that f has the Riemann integral over $[a,b]$ and let $[c,d] \subset [a,b]$. Then f has the Riemann integral also over $[c,d]$.*

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Remark

The formula (3) holds for all $a, b, c \in \mathbb{R}$ if the integral of f exists over the interval $\big[\min\{a,b,c\}, \max\{a,b,c\}\big]$.

Theorem 16 (linearity of the Riemann integral)

Let f and g be functions with Riemann integral over $[a, b]$ and let $\alpha \in \mathbb{R}$. Then

(i) *the function αf has the Riemann integral over $[a, b]$ and*

$$\int_a^b \alpha f = \alpha \int_a^b f,$$

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(i) *the function αf has the Riemann integral over $[a, b]$ and*

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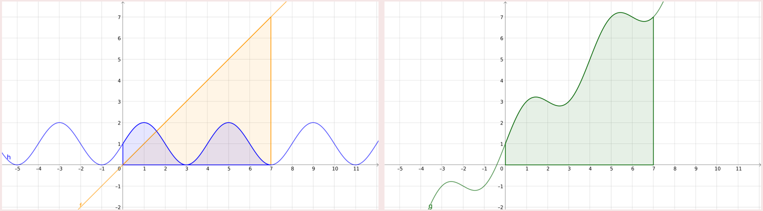
(ii) *the function $f + g$ has the Riemann integral over $[a, b]$ and*

$$\int_a^b f + g = \int_a^b f + \int_a^b g.$$

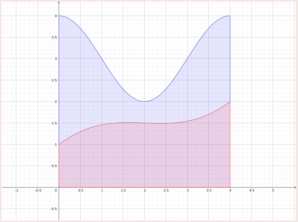


Theorem 17

Let $a, b \in \mathbb{R}$, $a < b$, and let f and g be functions with Riemann integral over $[a, b]$. Then:

(i) *If $f(x) \leq g(x)$ for each $x \in [a, b]$, then*

$$\int_a^b f \leq \int_a^b g.$$



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Let $a, b \in \mathbb{R}$, $a < b$, and let f and g be functions with Riemann integral over $[a, b]$. Then:

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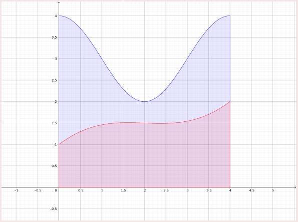


(ii) *The function $|f|$ has the Riemann integral over $[a, b]$ and*

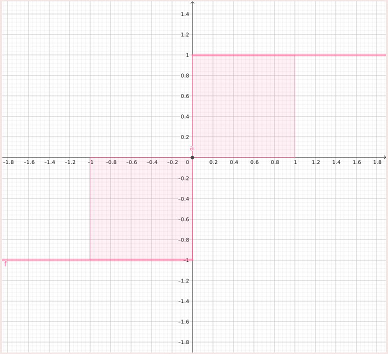
$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

Theorem 18

Let f be a function continuous on an interval $[a, b]$, $a, b \in \mathbb{R}$. Then f has the Riemann integral on $[a, b]$.

Remark

Compare with $\operatorname{sgn} x$:



Theorem 19

Let f be a function continuous on an interval (a,b) and let $c \in (a,b)$. If we denote $F(x) = \int_c^x f(t)\,\mathrm{d}t$ for $x \in (a,b)$, then $F'(x) = f(x)$ for each $x \in (a,b)$. In other words, F is an antiderivative of f on (a,b).

Exercise (True – False)

A Let f be a function. Then $\int_0^2 f(x)\,\mathrm{d}x \leq \int_0^3 f(x)\,\mathrm{d}x$.

B If $\int_2^6 g(x)\,\mathrm{d}x \leq \int_2^6 f(x)\,\mathrm{d}x$, then $g(x) \leq f(x)$ for all $2 \leq x \leq 6$.

Exercise

Let f be an odd function such that $\int_{-2}^{0} f(x)\,\mathrm{d}x = 4$. Find

1. $\int_{0}^{2} f(x)\,\mathrm{d}x$
2. $\int_{-2}^{2} f(x)\,\mathrm{d}x$

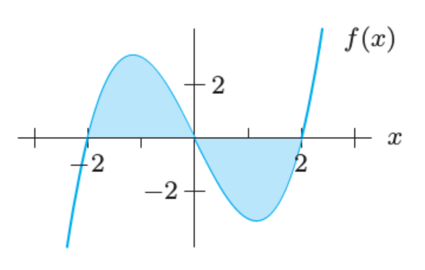


Figure: Applied Calculus, 6th Edition, Deborah Hughes-Hallett and col.

Exercise

Decide, if the integrals are

A $\int_{-\pi}^{0} \sin x \, dx$

B $\int_{0}^{\pi} \cos x \, dx$

C $\int_{-\pi}^{\pi} \sin x \, dx$

D $\int_{-\pi/2}^{\pi/2} \cos x \, dx$

E $\int_{0}^{2\pi} e^{-x} \sin x \, dx$

1. positive
2. 0
3. negative

Exercise

The half-life of phosphorous ^{32}P, which is used for biological experiments, is 14,3 days.

Suppose, that you have a sample, which emits 300 mREM/day. (1 REM=0,01 Sv)

How long can a laboratory assistant work with this sample, if according to the safety regulations she can receive only 5 000 mREM/year.

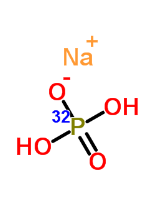


Figure: https://www.guidechem.com/cas/680178408.html

From: https://jmahaffy.sdsu.edu/courses/f14/math124/beamer_lectures/def_int.pdf