Mathematics II - Matrices

21/22

Exercise

VI. Matrix calculus

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VI. Matrix calculus

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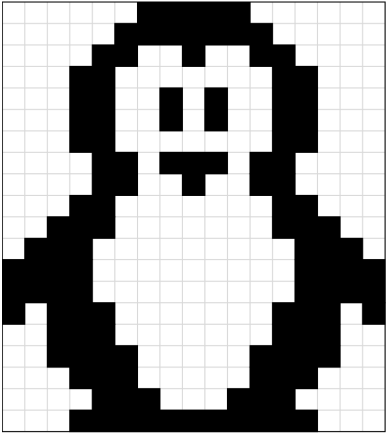

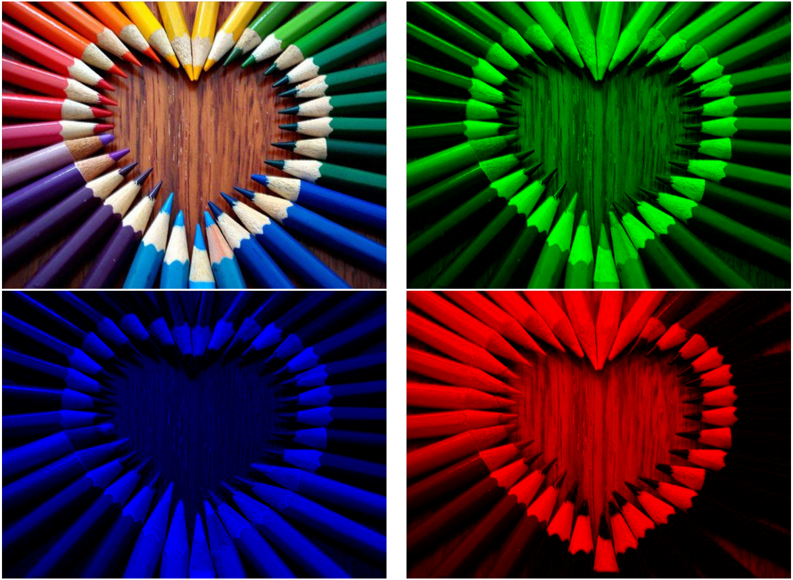

https://www.pinterest.cl/pin/527273068861820414/

Definition

A table of numbers

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix},$$

where $a_{ij} \in \mathbb{R}$, $i = 1, \dots, m$, $j = 1, \dots, n$, is called a matrix of type $m \times n$ (shortly, an m-by-n matrix). We also write $(a_{ij})_{\substack{i=1..m \\ j=1..n}}$ for short.

An n-by-n matrix is called a square matrix of order n.

VI.1. Basic operations with matrices

Definition

A table of numbers

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix},$$

where $a_{ij} \in \mathbb{R}$, $i = 1, \dots, m$, $j = 1, \dots, n$, is called a matrix of type $m \times n$ (shortly, an m-by-n matrix). We also write $(a_{ij})_{\substack{i=1..m \\ j=1..n}}$ for short.

An n-by-n matrix is called a square matrix of order n.

The set of all m-by-n matrices is denoted by $M(m \times n)$.

Example

(3) $\quad \begin{pmatrix} 2 & 2 & 8 \\ 5 & 0 & -2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & 1 \\ 4 & \pi & 3 \end{pmatrix}$

$\begin{pmatrix} 6 \\ 7 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 4 & -4 \end{pmatrix}$

Exercise

Find the type of the matrix

$$\begin{pmatrix} 6 & 11 & -2 \\ 23 & 31 & 5 \end{pmatrix}$$

A 2x3   B 3x2   C 6

Definition

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

The n-tuple $(a_{i1}, a_{i2}, \dots, a_{in})$, *where* $i \in \{1, 2, \dots, m\}$, *is called the ith row of the matrix* $\boldsymbol{A}$.

Definition

Let

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

The m-tuple $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$, *where* $j \in \{1, 2, \dots, n\}$, *is called the jth column of the matrix* $\boldsymbol{A}$.

Definition

We say that two matrices are equal, if they are of the same type and the corresponding elements are equal, i.e. if $\mathbf{A} = (a_{ij})_{\substack{i=1..m \\ j=1..n}}$ *and* $\mathbf{B} = (b_{uv})_{\substack{u=1..r, \\ v=1..s}}$ *then* $\mathbf{A} = \mathbf{B}$ *if and only if* $m = r$, $n = s$ *and* $a_{ij} = b_{ij}\ \forall i \in \{1, \ldots, m\}, \forall j \in \{1, \ldots, n\}$.

Exercise

Are **A** and **B** equal?

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 4 \\ -1 & -2 & 5 \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} 4 & 0 & 4 \\ 1 & 2 & 3 \\ -1 & -2 & 5 \end{pmatrix}$$

Definition

Let $\boldsymbol{A}, \boldsymbol{B} \in M(m \times n)$, $\boldsymbol{A} = (a_{ij})_{\substack{i=1..m \\ j=1..n}}$, $\boldsymbol{B} = (b_{ij})_{\substack{i=1..m \\ j=1..n}}$, $\lambda \in \mathbb{R}$. The sum of the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is the matrix defined by

$$\boldsymbol{A} + \boldsymbol{B} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m1} & \dots & a_{mn} + b_{mn} \end{pmatrix}.$$

Definition

Let $\boldsymbol{A}, \boldsymbol{B} \in M(m \times n)$, $\boldsymbol{A} = (a_{ij})_{\substack{i=1..m,\\ j=1..n}}$ $\boldsymbol{B} = (b_{ij})_{\substack{i=1..m,\\ j=1..n}}$ $\lambda \in \mathbb{R}$. *The sum of the matrices* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *is the matrix defined by*

$$\boldsymbol{A} + \boldsymbol{B} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m1} & \dots & a_{mn} + b_{mn} \end{pmatrix}.$$

The product of the real number λ *and the matrix* $\boldsymbol{A}$ *(or the* λ*-multiple of the matrix* $\boldsymbol{A}$*) is the matrix defined by*

$$\lambda \boldsymbol{A} = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \dots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \dots & \lambda a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda a_{m1} & \lambda a_{m2} & \dots & \lambda a_{mn} \end{pmatrix}.$$

Exercise

Let

$$A = \begin{pmatrix} 4 & 6 \\ 20 & 24 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 5 \\ 3 & 7 \end{pmatrix}.$$

Find $A + B$

A 71

B
$$\begin{pmatrix} 6 & 9 \\ 7 & 11 \end{pmatrix}$$

C
$$\begin{pmatrix} 6 & 11 \\ 23 & 31 \end{pmatrix}$$

D
$$\begin{pmatrix} 26 & 62 \\ 112 & 268 \end{pmatrix}$$

E
$$\begin{pmatrix} 4 & 6 & 2 & 5 \\ 20 & 24 & 3 & 7 \end{pmatrix}$$

Exercise

Let

$$A = \begin{pmatrix} 4 & 6 \\ 20 & 7 \end{pmatrix}$$

Find $5A$

A

$$\begin{pmatrix} 9 & 6 \\ 20 & 7 \end{pmatrix}$$

B

$$\begin{pmatrix} 9 & 11 \\ 25 & 12 \end{pmatrix}$$

C

$$\begin{pmatrix} 20 & 6 \\ 20 & 7 \end{pmatrix}$$

D

$$\begin{pmatrix} 20 & 30 \\ 100 & 35 \end{pmatrix}$$

Exercise

Let $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 2 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & 1 \\ -2 & 0 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} -4 & 0 \\ -2 & 1 \end{pmatrix}$ and $\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

1. Find $(\mathbf{A} + \mathbf{B}) + \mathbf{C}$ and $\mathbf{A} + (\mathbf{B} + \mathbf{C})$.
2. Find $\mathbf{A} + \mathbf{B}$ and $\mathbf{B} + \mathbf{A}$.
3. Find $\mathbf{A} + \mathbf{O}$ and $\mathbf{O} + \mathbf{A}$.
4. Find a matrix $\mathbf{C}_A$ such that $\mathbf{A} + \mathbf{C}_A = \mathbf{O}$.
5. Find $2 \cdot 3\mathbf{A}$ and $2(3\mathbf{A})$.
6. Find $2 \cdot 3\mathbf{A}$ and $2(3\mathbf{A})$.
7. Find $(1 + 2)\mathbf{A}$ and $1\mathbf{A} + 2\mathbf{A}$.
8. Find $2\mathbf{A} + 2\mathbf{B}$ and $2(\mathbf{A} + \mathbf{B})$.

Proposition 1 (basic properties of the sum of matrices and of a multiplication by a scalar)

The following holds:

- $\forall \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in M(m \times n)\colon \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}$, *(associativity)*
- $\forall \boldsymbol{A}, \boldsymbol{B} \in M(m \times n)\colon \boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$, *(commutativity)*
- $\exists! \boldsymbol{O} \in M(m \times n)\ \forall \boldsymbol{A} \in M(m \times n)\colon \boldsymbol{A} + \boldsymbol{O} = \boldsymbol{A}$, *(existence of a zero element)*
- $\forall \boldsymbol{A} \in M(m \times n)\ \exists \boldsymbol{C}_{\boldsymbol{A}} \in M(m \times n)\colon \boldsymbol{A} + \boldsymbol{C}_{\boldsymbol{A}} = \boldsymbol{O}$, *(existence of an opposite element)*
- $\forall \boldsymbol{A} \in M(m \times n)\ \forall \lambda, \mu \in \mathbb{R}\colon (\lambda\mu)\boldsymbol{A} = \lambda(\mu\boldsymbol{A})$,
- $\forall \boldsymbol{A} \in M(m \times n)\colon 1 \cdot \boldsymbol{A} = \boldsymbol{A}$,
- $\forall \boldsymbol{A} \in M(m \times n)\ \forall \lambda, \mu \in \mathbb{R}\colon (\lambda + \mu)\boldsymbol{A} = \lambda\boldsymbol{A} + \mu\boldsymbol{A}$,
- $\forall \boldsymbol{A}, \boldsymbol{B} \in M(m \times n)\ \forall \lambda \in \mathbb{R}\colon \lambda(\boldsymbol{A} + \boldsymbol{B}) = \lambda\boldsymbol{A} + \lambda\boldsymbol{B}$.

Remark

- The matrix $\boldsymbol{O}$ from the previous proposition is called a zero matrix and all its elements are all zeros.

Remark

- The matrix $\boldsymbol{O}$ from the previous proposition is called a zero matrix and all its elements are all zeros.
- The matrix $\boldsymbol{C_A}$ from the previous proposition is called a matrix opposite to $\boldsymbol{A}$. It is determined uniquely, it is denoted by $-\boldsymbol{A}$, and it satisfies $-\boldsymbol{A} = (-a_{ij})_{\substack{i=1..m \\ j=1..n}}$ and $-\boldsymbol{A} = -1 \cdot \boldsymbol{A}$.

Definition

Let $\boldsymbol{A} \in M(m \times n)$, $\boldsymbol{A} = (a_{is})_{\substack{i=1..m,\\ s=1..n}}$ $\boldsymbol{B} \in M(n \times k)$, $\boldsymbol{B} = (b_{sj})_{\substack{s=1..n.\\ j=1..k}}$ *Then the* *product of matrices* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *is defined as a matrix* $\boldsymbol{AB} \in M(m \times k)$, $\boldsymbol{AB} = (c_{ij})_{\substack{i=1..m,\\ j=1..k}}$ *where*

$$c_{ij} = \sum_{s=1}^{n} a_{is} b_{sj}.$$

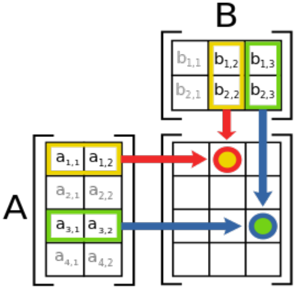

Figure: `https://en.wikipedia.org/wiki/File:Matrix_multiplication_diagram_2.svg`

Matrix multiplication

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$

Matrix multiplication

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{33} \\ a_{41}b_{11} + a_{42}b_{21} & a_{41}b_{12} + a_{42}b_{22} & a_{41}b_{13} + a_{42}b_{23} \end{pmatrix}$$

Matrix multiplication

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{33} \\ a_{41}b_{11} + a_{42}b_{21} & a_{41}b_{12} + a_{42}b_{22} & a_{41}b_{13} + a_{42}b_{23} \end{pmatrix}$$

Matrix multiplication

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{33} \\ a_{41}b_{11} + a_{42}b_{21} & a_{41}b_{12} + a_{42}b_{22} & a_{41}b_{13} + a_{42}b_{23} \end{pmatrix}$$

Matrix multiplication

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{33} \\ a_{41}b_{11} + a_{42}b_{21} & a_{41}b_{12} + a_{42}b_{22} & a_{41}b_{13} + a_{42}b_{23} \end{pmatrix}$$

Matrix multiplication

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix} \cdot \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} & a_{11}b_{13} + a_{12}b_{23} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} & a_{21}b_{13} + a_{22}b_{23} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} & a_{31}b_{13} + a_{32}b_{33} \\ a_{41}b_{11} + a_{42}b_{21} & a_{41}b_{12} + a_{42}b_{22} & a_{41}b_{13} + a_{42}b_{23} \end{pmatrix}$$

Exercise

Find **AB**, if

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ -3 & 1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 0 & -1 \\ 2 & 2 \end{pmatrix}$$

A $\begin{pmatrix} 3 & -1 \\ -2 & 2 \end{pmatrix}$

B $\begin{pmatrix} 0 & -2 \\ 2 & 5 \end{pmatrix}$

C $\begin{pmatrix} 0 & 0 \\ -6 & 2 \end{pmatrix}$

D something else

E **AB** is not well defined

Exercise

Find **AB**, if

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$$

A $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$

B $\begin{pmatrix} 10 & 7 \end{pmatrix}$

C $\begin{pmatrix} 8 & 4 \\ -3 & -2 \end{pmatrix}$

D $\begin{pmatrix} 7 \\ 10 \end{pmatrix}$

E **AB** is not well defined

Multiplication properties

Exercise

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Find

1. AI
2. IA

Multiplication properties

Exercise

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Find

1. AI
2. IA

Exercise

Let

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 3 & -3 \end{pmatrix}$$

Find

1. AB
2. BA

Theorem 2 (properties of the matrix multiplication)

Let $m, n, k, l \in \mathbb{N}$. Then:

(i) $\forall \boldsymbol{A} \in M(m \times n)\ \forall \boldsymbol{B} \in M(n \times k)\ \forall \boldsymbol{C} \in M(k \times l): \boldsymbol{A}(\boldsymbol{B}\boldsymbol{C}) = (\boldsymbol{A}\boldsymbol{B})\boldsymbol{C}$, *(associativity of multiplication)*

(ii) $\forall \boldsymbol{A} \in M(m \times n)\ \forall \boldsymbol{B}, \boldsymbol{C} \in M(n \times k): \boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A}\boldsymbol{B} + \boldsymbol{A}\boldsymbol{C}$, *(distributivity from the left)*

(iii) $\forall \boldsymbol{A}, \boldsymbol{B} \in M(m \times n)\ \forall \boldsymbol{C} \in M(n \times k): (\boldsymbol{A} + \boldsymbol{B})\boldsymbol{C} = \boldsymbol{A}\boldsymbol{C} + \boldsymbol{B}\boldsymbol{C}$, *(distributivity from the right)*

(iv) $\exists! \boldsymbol{I} \in M(n \times n)\ \forall \boldsymbol{A} \in M(n \times n): \boldsymbol{I}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{I} = \boldsymbol{A}$. *(existence and uniqueness of an identity matrix $\boldsymbol{I}$)*

Remark

Warning! The matrix multiplication is not commutative.

Definition

A transpose of a matrix

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

is the matrix

$$\boldsymbol{A}^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ a_{13} & a_{23} & \dots & a_{m3} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix},$$

i.e. if $\boldsymbol{A} = (a_{ij})_{\substack{i=1..m \\ j=1..n}}$, *then* $\boldsymbol{A}^T = (b_{uv})_{\substack{u=1..n \\ v=1..m}}$, *where* $b_{uv} = a_{vu}$ *for each* $u \in \{1, \dots, n\}$, $v \in \{1, 2, \dots, m\}$.

Definition

A transpose of a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

is the matrix

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ a_{13} & a_{23} & \dots & a_{m3} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix},$$

i.e. if $A = (a_{ij})_{\substack{i=1..m \\ j=1..n}}$, *then* $A^T = (b_{uv})_{\substack{u=1..n \\ v=1..m}}$, *where* $b_{uv} = a_{vu}$ *for each* $u \in \{1, \dots, n\}$, $v \in \{1, 2, \dots, m\}$.

Definition

A transpose of a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

is the matrix

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i.e. if $A = (a_{ij})_{\substack{i=1..m \\ j=1..n}}$, *then* $A^T = (b_{uv})_{\substack{u=1..n \\ v=1..m}}$, *where* $b_{uv} = a_{vu}$ *for each* $u \in \{1, \dots, n\}$, $v \in \{1, 2, \dots, m\}$.

Definition

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$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

is the matrix

$$\mathbf{A}^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ a_{13} & a_{23} & \dots & a_{m3} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix},$$

i.e. if $\mathbf{A} = (a_{ij})_{\substack{i=1..m \\ j=1..n}}$, *then* $\mathbf{A}^T = (b_{uv})_{\substack{u=1..n \\ v=1..m}}$, *where* $b_{uv} = a_{vu}$ *for each* $u \in \{1, \dots, n\}$, $v \in \{1, 2, \dots, m\}$.

Definition

A transpose of a matrix

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}$$

is the matrix

$$\boldsymbol{A}^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ a_{13} & a_{23} & \dots & a_{m3} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix},$$

i.e. if $\boldsymbol{A} = (a_{ij})_{\substack{i=1..m \\ j=1..n}}$, *then* $\boldsymbol{A}^T = (b_{uv})_{\substack{u=1..n \\ v=1..m}}$, *where* $b_{uv} = a_{vu}$ *for each* $u \in \{1, \dots, n\}$, $v \in \{1, 2, \dots, m\}$.

Example

$$\mathbf{D} = \begin{pmatrix} 1 & 2 \\ 4 & -4 \\ 2 & 3 \end{pmatrix}$$

$$\mathbf{D}^T = \begin{pmatrix} 1 & 4 & 2 \\ 2 & -4 & 3 \end{pmatrix}$$

$$\mathbf{F} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & 1 \\ 4 & \pi & 3 \end{pmatrix}$$

$$\mathbf{F}^T = \begin{pmatrix} 1 & 0 & 4 \\ 2 & -3 & \pi \\ 3 & 1 & 3 \end{pmatrix}$$

Exercise

$$\mathbf{A} = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 3 \\ -2 & 0 & 4 \end{pmatrix}.$$

Find $\mathbf{A}^T$?

A

$$A^T = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 3 \\ -2 & 0 & 4 \end{pmatrix}$$

B

$$A^T = \begin{pmatrix} 2 & 0 & -2 \\ 3 & -1 & 0 \\ 1 & 3 & 4 \end{pmatrix}$$

C

$$A^T = \begin{pmatrix} -2 & 0 & 4 \\ 0 & -1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

D

$$A^T = \begin{pmatrix} 1 & 3 & 4 \\ 3 & -1 & 0 \\ 2 & 0 - 2 & \end{pmatrix}$$

Exercise

Let

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 0 \\ 5 & -1 \end{pmatrix}$$

Find

1. $(AB)^T$
2. $A^T B^T$
3. $B^T A^T$

Theorem 3 (properties of the transpose of a matrix)

Platí:

(i) $\forall \boldsymbol{A} \in M(m \times n)\colon \left(\boldsymbol{A}^T\right)^T = \boldsymbol{A}$,

(ii) $\forall \boldsymbol{A}, \boldsymbol{B} \in M(m \times n)\colon (\boldsymbol{A} + \boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$,

(iii) $\forall \boldsymbol{A} \in M(m \times n)\ \forall \boldsymbol{B} \in M(n \times k)\colon (\boldsymbol{A}\boldsymbol{B})^T = \boldsymbol{B}^T \boldsymbol{A}^T$.

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Definition

We say that the matrix $\boldsymbol{A} \in M(n \times n)$ is symmetric if $\boldsymbol{A} = \boldsymbol{A}^T$.

Exercise

Let $\mathbf{A}$ and $\mathbf{B}$ are matrices of the type 2×3. Which of these operations are NOT well defined?

A $\mathbf{A} + \mathbf{B}$

B $\mathbf{A}^T\mathbf{B}$

C $\mathbf{BA}$

D $\mathbf{AB}^T$

E $\mathbf{AB}$

Exercise

Let $\mathbf{A}$ and $\mathbf{B}$ are matrices of the type 2×3. Which of these operations are NOT well defined?

- A $\mathbf{A} + \mathbf{B}$
- B $\mathbf{A}^T\mathbf{B}$
- C $\mathbf{BA}$
- D $\mathbf{AB}^T$
- E $\mathbf{AB}$

Exercise

We want to multiply matrices $\mathbf{A} \times \mathbf{B}$. We need:

- A $\mathbf{A}$ and $\mathbf{B}$ needs to have the same number of rows.
- B $\mathbf{A}$ and $\mathbf{B}$ needs to have the same number of columns.
- C the number of rows of $\mathbf{A}$ needs to be the same as the number of columns of $\mathbf{B}$
- D the number of columns of $\mathbf{A}$ needs to be the same as the number of rows of $\mathbf{B}$

Exercise

Let **A** is a matrix of the type 2×3 and **B** is of the type 3×6. Find the type of **AB**:

- A 2x6
- B 6x2
- C 3x3
- D 2x3
- E 3x6

Exercise

Let $\mathbf{A}$ is a matrix of the type 2×3 and $\mathbf{B}$ is of the type 3×6. Find the type of $\mathbf{AB}$:

A 2x6
B 6x2
C 3x3
D 2x3
E 3x6

Exercise (True or False?)

Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices of the same dimension. Then

$$(\mathbf{A}+\mathbf{B}) \times (\mathbf{A}+\mathbf{B}) = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2.$$

Exercise

Visit: `https://web.ma.utexas.edu/users/ysulyma/matrix/`
(You can change the picture: `https://www2.karlin.mff.cuni.cz/~kuncova/en/2122LS_FMat2/bilyctverec.jpg`.)
Try the following matrices:

1. 1.1 $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ 1.2 $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ 1.3 $\begin{pmatrix} 0,71 & -0,71 \\ 0,71 & 0,71 \end{pmatrix}$
2. 2.1 $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ 2.2 $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ 2.3 $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
3. 3.1 $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ 3.2 $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ 3.3 $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$
4. 4.1 $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ 4.2 $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

Exercise

And what about this matrices? What is the result of matrix multiplying?

1.
   1.1 $\begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
   1.2 $\begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

Cheat sheet for matrix transform:
`https://en.wikipedia.org/wiki/File:2D_affine_transformation_matrix.svg`

Definition

Let $\boldsymbol{A} \in M(n \times n)$. We say that $\boldsymbol{A}$ is an invertible matrix if there exist $\boldsymbol{B} \in M(n \times n)$ such that

$$\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}.$$

Definition

We say that the matrix $\boldsymbol{B} \in M(n \times n)$ is an inverse of a matrix $\boldsymbol{A} \in M(n \times n)$ if $\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$.

Remark

A matrix $\boldsymbol{A} \in M(n \times n)$ is invertible if and only if it has an inverse.

Exercise

Let

$$\mathbf{A} = \begin{pmatrix} 0 & 4 \\ 2 & 0 \end{pmatrix}$$

Find $\mathbf{A}^{-1}$

A

$$\begin{pmatrix} 0 & 4 \\ 2 & 0 \end{pmatrix}$$

B

$$\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$$

C

$$\begin{pmatrix} 0 & 1/4 \\ 1/2 & 0 \end{pmatrix}$$

D

$$\begin{pmatrix} 0 & 1/2 \\ 1/4 & 0 \end{pmatrix}$$

Remark

- If $A \in M(n \times n)$ is invertible, then it has exactly one inverse, which is denoted by A^{-1}.

Remark

- If $\boldsymbol{A} \in M(n \times n)$ is invertible, then it has exactly one inverse, which is denoted by $\boldsymbol{A}^{-1}$.
- If some matrices $\boldsymbol{A}, \boldsymbol{B} \in M(n \times n)$ satisfy $\boldsymbol{AB} = \boldsymbol{I}$, then also $\boldsymbol{BA} = \boldsymbol{I}$.

Remark

- If $\boldsymbol{A} \in M(n \times n)$ is invertible, then it has exactly one inverse, which is denoted by $\boldsymbol{A}^{-1}$.
- If some matrices $\boldsymbol{A}, \boldsymbol{B} \in M(n \times n)$ satisfy $\boldsymbol{AB} = \boldsymbol{I}$, then also $\boldsymbol{BA} = \boldsymbol{I}$.

Theorem 4 (operations with invertible matrices)

Let $\boldsymbol{A}, \boldsymbol{B} \in M(n \times n)$ be invertible matrices. Then

(i) $\boldsymbol{A}^{-1}$ *is invertible and* $\left(\boldsymbol{A}^{-1}\right)^{-1} = \boldsymbol{A}$,

(ii) $\boldsymbol{A}^{T}$ *is invertible and* $\left(\boldsymbol{A}^{T}\right)^{-1} = \left(\boldsymbol{A}^{-1}\right)^{T}$,

(iii) $\boldsymbol{AB}$ *is invertible and* $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$.

Definition 1

The Determinant of the matrix $\mathbf{A}$ *of type* $(1,1)$ *is equal to*

$$\det \mathbf{A} = a_{1,1}.$$

The Determinant of the matrix $\mathbf{A}$ *of type* $(2,2)$ *is equal to*

$$\det \mathbf{A} = a_{1,1} \cdot a_{2,2} - a_{1,2} \cdot a_{2,1}.$$

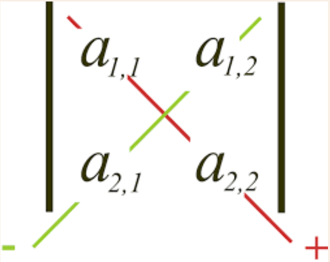

Figure:
http://umv.science.upjs.sk/madaras/MZIa/MZIa2011_4en.pdf

Exercise

Find the determinant of

$$\begin{pmatrix} 5 & 4 \\ 1 & 3 \end{pmatrix}$$

A 4

B 11

C 15

D 19

Definition 2 (Sarrus)

The Determinant of the matrix $\mathbf{A}$ *of type* $(3,3)$ *is equal to*

$$\begin{aligned}\det \mathbf{A} = {} & a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} \\ & - a_{1,3}a_{2,2}a_{3,1} - a_{1,2}a_{2,1}a_{3,3} - a_{1,1}a_{2,3}a_{3,2}.\end{aligned}$$

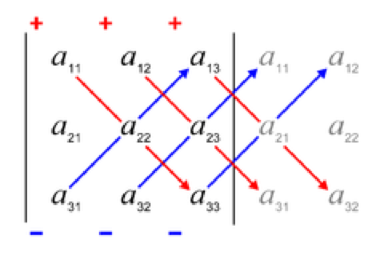

Figure: https://de.wikipedia.org/wiki/Regel_von_Sarrus

Exercise

Find the determinant of

$$\begin{pmatrix} 5 & 2 & -1 \\ 0 & 3 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

A 0

B 6

C 15

D 22

Definition 3

Let $\mathbf{A}$ be square matrix of the type $(4,4)$. Let $\mathbf{A}_{i,j}$ denote the matrix of type $(n-1,n-1)$, which is created from $\mathbf{A}$ by ommiting the ith row and jth column. Let $r \in \{1,\ldots,n\}$. Then the determinant of $\mathbf{A}$ is equal to

$$\det \mathbf{A} = a_{r,1}(-1)^{r+1}\det \mathbf{A}_{r,1} + a_{r,2}(-2)^{r+2}\det \mathbf{A}_{r,2} + \cdots + a_{r,n}(-n)^{r+n}\det \mathbf{A}_{r,n}.$$

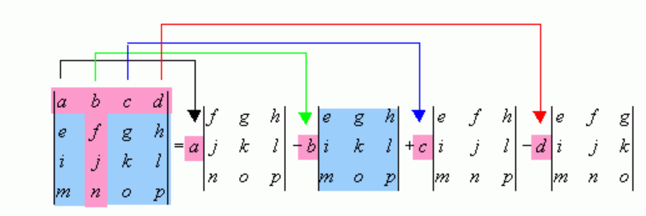

Figure: http://mathcentral.uregina.ca/QQ/database/QQ.09.06/h/suud1.html

Definition

Let $\mathbf{A} \in M(n \times n)$*. The symbol* $\mathbf{A}_{ij}$ *denotes the* $(n-1)$*-by-*$(n-1)$ *matrix which is created from* $\mathbf{A}$ *by omitting the ith row and the jth column.*

Definition

Let $\boldsymbol{A} \in M(n \times n)$. The symbol $\mathbf{A}_{ij}$ denotes the $(n-1)$-by-$(n-1)$ matrix which is created from $\mathbf{A}$ by omitting the ith row and the jth column.

$$\boldsymbol{A} = \begin{pmatrix} a_{1,1} & \dots & a_{1,j-1} & a_{1,j} & a_{1,j+1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1} & \dots & a_{i-1,j-1} & a_{i-1,j} & a_{i-1,j+1} & \dots & a_{i-1,n} \\ a_{i,1} & \dots & a_{i,j-1} & a_{i,j} & a_{i,j+1} & \dots & a_{i,n} \\ a_{i+1,1} & \dots & a_{i+1,j-1} & a_{i+1,j} & a_{i+1,j+1} & \dots & a_{i+1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,j-1} & a_{n,j} & a_{n,j+1} & \dots & a_{n,n} \end{pmatrix}$$

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$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,j-1} & & a_{1,j+1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ a_{i-1,1} & \cdots & a_{i-1,j-1} & & a_{i-1,j+1} & \cdots & a_{i-1,n} \\ & & & & & & \\ a_{i+1,1} & \cdots & a_{i+1,j-1} & & a_{i+1,j+1} & \cdots & a_{i+1,n} \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,j-1} & & a_{n,j+1} & \cdots & a_{n,n} \end{pmatrix}$$

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$$\mathbf{A}_{ij} = \begin{pmatrix} a_{1,1} & \dots & a_{1,j-1} & a_{1,j+1} & \dots & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{i-1,1} & \dots & a_{i-1,j-1} & a_{i-1,j+1} & \dots & a_{i-1,n} \\ a_{i+1,1} & \dots & a_{i+1,j-1} & a_{i+1,j+1} & \dots & a_{i+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & \dots & a_{n,j-1} & a_{n,j+1} & \dots & a_{n,n} \end{pmatrix}$$

Definition

Let $\boldsymbol{A} = (a_{ij})_{i,j=1..n}$. *The* *determinant* *of the matrix* $\boldsymbol{A}$ *is defined by*

$$\det \boldsymbol{A} = \begin{cases} a_{11} & \text{if } n = 1, \\ \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det \boldsymbol{A}_{i1} & \text{if } n > 1. \end{cases}$$

For $\det \boldsymbol{A}$ *we will also use the symbol*

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \ddots & \vdots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Theorem 5 (cofactor expansion)

Let $\boldsymbol{A} = (a_{ij})_{i,j=1..n}$, $k \in \{1, \ldots, n\}$. *Then*

$$\det \boldsymbol{A} = \sum_{i=1}^{n} (-1)^{i+k} a_{ik} \det \boldsymbol{A}_{ik} \quad \textit{(expansion along kth column),}$$

$$\det \boldsymbol{A} = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det \boldsymbol{A}_{kj} \quad \textit{(expansion along kth row).}$$

Lemma 6

Let $j, n \in \mathbb{N}$, $j \leq n$, and the matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in M(n \times n)$ coincide at each row except for the jth row. Let the jth row of $\boldsymbol{A}$ be equal to the sum of the jth rows of $\boldsymbol{B}$ and $\boldsymbol{C}$. Then $\det \boldsymbol{A} = \det \boldsymbol{B} + \det \boldsymbol{C}$.

$$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{j-1,1} & \dots & a_{j-1,n} \\ u_1+v_1 & \dots & u_n+v_n \\ a_{j+1,1} & \dots & a_{j+1,n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{j-1,1} & \dots & a_{j-1,n} \\ u_1 & \dots & u_n \\ a_{j+1,1} & \dots & a_{j+1,n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{j-1,1} & \dots & a_{j-1,n} \\ v_1 & \dots & v_n \\ a_{j+1,1} & \dots & a_{j+1,n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}$$

Theorem 7 (determinant and transformations)

Let $\boldsymbol{A}, \boldsymbol{A}' \in M(n \times n)$.

(i) *If the matrix* $\boldsymbol{A}'$ *is created from the matrix* $\boldsymbol{A}$ *by multiplying one row in* $\boldsymbol{A}$ *by a real number* μ, *then* $\det \boldsymbol{A}' = \mu \det \boldsymbol{A}$.

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(iv) *If* $\boldsymbol{A}'$ *is created from* $\boldsymbol{A}$ *by applying a transformation, then* $\det \boldsymbol{A} \neq 0$ *if and only if* $\det \boldsymbol{A}' \neq 0$.

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(iv) *If* $\boldsymbol{A}'$ *is created from* $\boldsymbol{A}$ *by applying a transformation, then* $\det \boldsymbol{A} \neq 0$ *if and only if* $\det \boldsymbol{A}' \neq 0$.

Remark

The determinant of a matrix with a zero row is equal to zero. The determinant of a matrix with two identical rows is also equal to zero.

Exercise

We have

$$\det \begin{pmatrix} -1 & 15 & 16 \\ 2 & 5 & 4 \\ 2 & 3 & 5 \end{pmatrix} = -107.$$

Find

$$\det \begin{pmatrix} 2 & 5 & 4 \\ 2 & 3 & 5 \\ -1 & 15 & 16 \end{pmatrix}?$$

A -107

B 107

C something else

Exercise

We have

$$\det \begin{pmatrix} -2 & 1 & 3 \\ 2 & 0 & 4 \\ 1 & 3 & 1 \end{pmatrix} = 44.$$

Find

$$\det \begin{pmatrix} -2 & 1 & 3 \\ 0 & 1 & 7 \\ 1 & 3 & 1 \end{pmatrix}?$$

A 44

B -44

C 88

D something else

Exercise

We have

$$\det\begin{pmatrix} -2 & 1 & 3 \\ 2 & 0 & 4 \\ 1 & 3 & 1 \end{pmatrix} = 44.$$

Find

$$\det\begin{pmatrix} -2 & 1 & 3 \\ 2 & 0 & 4 \\ 0 & 7 & 5 \end{pmatrix}?$$

A 44

B -44

C 88

D 22

E something else

Exercise

Let $\mathbf{A}$ be a matrix of type $(2x2)$. Find $\det(5\mathbf{A})$.

- A $5 \det \mathbf{A}$
- B $10 \det \mathbf{A}$
- C $25 \det \mathbf{A}$
- D something else

Definition

Let $\boldsymbol{A} = (a_{ij})_{i,j=1..n}$. *We say that* $\boldsymbol{A}$ *is an upper triangular matrix if* $a_{ij} = 0$ *for* $i > j$, $i,j \in \{1, \ldots, n\}$.

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Example

- $$\begin{pmatrix} 1 & 2 & 4 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$
- $$\begin{pmatrix} 1 & 0 & 0 \\ 2 & -4 & 0 \\ 3 & 3 & 3 \end{pmatrix}$$

Definition

Let $\boldsymbol{A} = (a_{ij})_{i,j=1..n}$. We say that $\boldsymbol{A}$ is an upper triangular matrix if $a_{ij} = 0$ for $i > j$, $i,j \in \{1,\ldots,n\}$. We say that $\boldsymbol{A}$ is a lower triangular matrix if $a_{ij} = 0$ for $i < j$, $i,j \in \{1,\ldots,n\}$.

Example

- $$\begin{pmatrix} 1 & 2 & 4 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$
- $$\begin{pmatrix} 1 & 0 & 0 \\ 2 & -4 & 0 \\ 3 & 3 & 3 \end{pmatrix}$$

Theorem 8 (determinant of a triangular matrix)

Let $\boldsymbol{A} = (a_{ij})_{i,j=1..n}$ be an upper or lower triangular matrix. Then

$$\det \boldsymbol{A} = a_{11} \cdot a_{22} \cdot \cdots \cdot a_{nn}.$$

Theorem 9 (determinant and invertibility)

Let $\boldsymbol{A} \in M(n \times n)$. *Then* $\boldsymbol{A}$ *is invertible if and only if* $\det \boldsymbol{A} \neq 0$.

Theorem 10 (determinant of a product)

Let $\boldsymbol{A}, \boldsymbol{B} \in M(n \times n)$. *Then* $\det \boldsymbol{AB} = \det \boldsymbol{A} \cdot \det \boldsymbol{B}$.

Theorem 11 (determinant of a transpose)

Let $\boldsymbol{A} \in M(n \times n)$. *Then* $\det \boldsymbol{A}^T = \det \boldsymbol{A}$.

Exercise

Let $\det \mathbf{A} = 3$. Find $\det \mathbf{A}^{-1}$.

A 1/3

B 3

C 9

D hard to say.

Exercise

We have

$$\det \begin{pmatrix} -2 & 1 & 3 \\ 2 & 0 & 4 \\ 1 & 3 & 1 \end{pmatrix} = 44.$$

Find

$$\det \begin{pmatrix} -2 & 2 & 1 \\ 1 & 0 & 3 \\ 3 & 4 & 1 \end{pmatrix} ?$$

A 44

B 1/44

C 88

D 22

E -44

Exercise

Which of the following matrices do NOT have inverse matrix?

A
$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

B
$$\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}$$

C
$$\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$

D
$$\begin{pmatrix} 0 & 4 \\ 2 & 0 \end{pmatrix}$$

E All of them have inverse matrix.

Definition

Let $k, n \in \mathbb{N}$ and $\boldsymbol{v}^1, \ldots, \boldsymbol{v}^k \in \mathbb{R}^n$. We say that a vector $\boldsymbol{u} \in \mathbb{R}^n$ is a linear combination of the vectors $\boldsymbol{v}^1, \ldots, \boldsymbol{v}^k$ with coefficients $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ if

$$\boldsymbol{u} = \lambda_1 \boldsymbol{v}^1 + \cdots + \lambda_k \boldsymbol{v}^k.$$

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$$\boldsymbol{u} = \lambda_1 \boldsymbol{v}^1 + \cdots + \lambda_k \boldsymbol{v}^k.$$

By a trivial linear combination of vectors $\boldsymbol{v}^1, \ldots, \boldsymbol{v}^k$ we mean the linear combination $0 \cdot \boldsymbol{v}^1 + \cdots + 0 \cdot \boldsymbol{v}^k$. Linear combination which is not trivial is called non-trivial.

Exercise

Let $u = (1, 2, 4)$ and $v = (-2, 0, 5)$. Then $2u - 3v$ is

A $(-4, 4, 23)$

B $(8, 4, -7)$

C $(8, 4, 23)$

D $(7, 6, 2)$

Exercise

Express $z = (-5, 3, 6)$ as the linear combination of $x = (1, -1, 4)$ and $y = (-3, 2, 6)$.

A $-5x$

B $-2x + y$

C $x + 2y$

D $2x + y$

E impossible

Exercise

Express w as the linear combination of u and v.

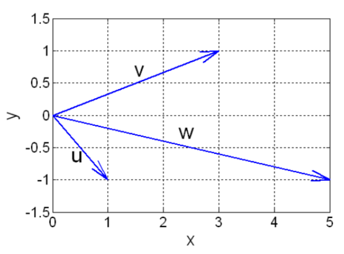

Figure: https://www.chegg.com/homework-help/questions-and-answers/write-vector-w-linear-combination-u-v-q55559120

A $w = 2u + v$

B $w = u + v$

C $w = -u + v$

D $w = u - v$

E w cannot be written like that.

Exercise

Which of the following vector can be written as the linear combination of vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$?

- A $(0, 2, 0)$
- B $(-3, 0, 1)$
- C $(0.4, 3.7, -1.5)$

Exercise

Which of the following vector can be written as the linear combination of vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$?

A $(0, 2, 0)$

B $(-3, 0, 1)$

C $(0.4, 3.7, -1.5)$

Exercise

Describe the set of all linear combinations of vectors $(2, 4, 6)$ and $(-1, -2, -3)$?

A point B line C vector D plane E space

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Describe the set of all linear combinations of vectors $(2, 4, 6)$ and $(-1, -2, -3)$?

A point B line C vector D plane E space

Exercise

Describe the set of all linear combinations of vectors $(1, 2, 0)$ and $(-1, 1, 0)$?

A point B line C vector D plane E space

Definition

We say that vectors $\boldsymbol{v}^1, \ldots, \boldsymbol{v}^k \in \mathbb{R}^n$ *are* ***linearly dependent*** *if there exists their non-trivial linear combination which is equal to the zero vector.*

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Remark

Vectors $\boldsymbol{v}^1, \ldots, \boldsymbol{v}^k$ are linearly dependent if and only if one of them can be expressed as a linear combination of the others.

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Remark

Vectors $\boldsymbol{v}^1, \dots, \boldsymbol{v}^k$ are linearly dependent if and only if one of them can be expressed as a linear combination of the others.

Exercise

The vectors $(1, 0, 0)$, $(0, 0, 2)$, $(3, 0, 4)$ are

A linearly dependent

B linearly independent

Definition

Let $\mathbf{A} \in M(m \times n)$. The rank of the matrix $\mathbf{A}$ is the maximal number of linearly independent row vectors of $\mathbf{A}$, i.e. the rank is equal to $k \in \mathbb{N}$ if

(i) *there is k linearly independent row vectors of $\mathbf{A}$ and*

(ii) *each l-tuple of row vectors of $\mathbf{A}$, where $l > k$, is linearly dependent.*

The rank of the zero matrix is zero. Rank of $\mathbf{A}$ is denoted by $\operatorname{rank}(\boldsymbol{A})$.

Exercise

Find the rank of the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

Definition

We say that a matrix $\mathbf{A} \in M(m \times n)$ is in a row echelon form if for each $i \in \{2, \dots, m\}$ the ith row of $\mathbf{A}$ is either a zero vector or it has more zeros at the beginning than the $(i-1)$th row.

Example

$$\begin{pmatrix} 1 & 4 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 4 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 4 & 1 \\ 0 & 5 & 2 \end{pmatrix}$$

Definition

We say that a matrix $\boldsymbol{A} \in M(m \times n)$ *is in a* *row echelon form* *if for each* $i \in \{2, \dots, m\}$ *the ith row of* $\boldsymbol{A}$ *is either a zero vector or it has more zeros at the beginning than the* $(i-1)$*th row.*

Example

$$\begin{pmatrix} 1 & 4 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 4 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 4 & 1 \\ 0 & 5 & 2 \end{pmatrix}$$

Remark

The rank of a row echelon matrix is equal to the number of its non-zero rows.

Definition

The elementary row operations on the matrix $\mathbf{A}$ *are:*

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The *elementary row operations* *on the matrix* $\boldsymbol{A}$ *are:*

(i) *interchange of two rows,*

(ii) *multiplication of a row by a non-zero real number,*

(iii) *addition of a multiple of a row to another row.*

Definition

A matrix *transformation* *is a finite sequence of elementary row operations. If a matrix* $\boldsymbol{B} \in M(m \times n)$ *results from the matrix* $\boldsymbol{A} \in M(m \times n)$ *by applying a transformation* T *on the matrix* $\boldsymbol{A}$*, then this fact is denoted by* $\boldsymbol{A} \overset{T}{\rightsquigarrow} \boldsymbol{B}$.

Theorem 12 (properties of matrix transformations)

(i) *Let* $\boldsymbol{A} \in M(m \times n)$*. Then there exists a transformation transforming* $\boldsymbol{A}$ *to a row echelon matrix.*

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(ii) *Let T_1 be a transformation applicable to m-by-n matrices. Then there exists a transformation T_2 applicable to m-by-n matrices such that for any two matrices $\boldsymbol{A}, \boldsymbol{B} \in M(m \times n)$ we have $\boldsymbol{A} \overset{T_1}{\rightsquigarrow} \boldsymbol{B}$ if and only if $\boldsymbol{B} \overset{T_2}{\rightsquigarrow} \boldsymbol{A}$.*

Theorem 12 (properties of matrix transformations)

(i) *Let* $\boldsymbol{A} \in M(m \times n)$. *Then there exists a transformation transforming* $\boldsymbol{A}$ *to a row echelon matrix.*

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(iii) *Let* $\boldsymbol{A}, \boldsymbol{B} \in M(m \times n)$ *and there exist a transformation* T *such that* $\boldsymbol{A} \overset{T}{\rightsquigarrow} \boldsymbol{B}$. *Then* $\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{B})$.

Exercise

Let

$$\mathbf{A} = \begin{pmatrix} 5 & 4 & -8 & 1 \\ 1 & 3 & 4 & 8 \\ 0 & 2 & 1 & 3 \\ -1 & -2 & 4 & 1 \end{pmatrix}.$$

After the transformation we get

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Find the rank of $\mathbf{A}$:

A 0
B 1
C 2
D 3
E 4

Remark

Similarly as the elementary row operations one can define also elementary column operations. It can be shown that the elementary column operations do not change the rank of the matrix.

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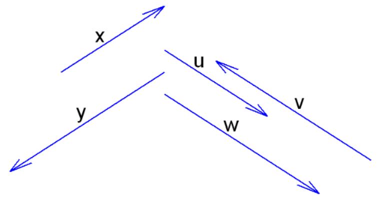
Remark

It can be shown that $\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A}^T)$ for any $\boldsymbol{A} \in M(m \times n)$.

Exercise

We made a matrix from the vectors x, y, u, v and w. Find rank of this matrix.



http://mathquest.carroll.edu/libraries/FHMW.student.edition.pdf

A 1

B 2

C 3

D 4

E 5

Theorem 13 (reprezentation of a transformation)

Let T be a transformation on $m \times n$ matrices. Then there exists an invertible matrix $\boldsymbol{C}_T \in M(m \times m)$ satisfying:
whenever we apply the transformation T to a matrix $\boldsymbol{A} \in M(m \times n)$, we obtain the matrix $\boldsymbol{C}_T\boldsymbol{A}$.

Example

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 0 & -1 & -2 \end{pmatrix}$$

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Remark

Also the converse is true: For every invertible matrix $\boldsymbol{C}$ the mapping $\boldsymbol{A} \mapsto \boldsymbol{C}\boldsymbol{A}$ is a transformation.

Lemma 14

Let $\boldsymbol{A} \in M(n \times n)$ *and* $\operatorname{rank}(\boldsymbol{A}) = n$. *Then there exists a transformation transforming* $\boldsymbol{A}$ *to* $\boldsymbol{I}$.

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Theorem 15

Let $\boldsymbol{A} \in M(n \times n)$. *Then* $\boldsymbol{A}$ *is invertible if and only if* $\operatorname{rank}(\boldsymbol{A}) = n$.

A system of m equations in n unknowns $x_1, \ldots, x_n$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned} \tag{S}$$

where $a_{ij} \in \mathbb{R}$, $b_i \in \mathbb{R}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$.

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where $a_{ij} \in \mathbb{R}$, $b_i \in \mathbb{R}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. The matrix form is

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},$$

where $\boldsymbol{A} = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix} \in M(m \times n)$, is called the coefficient matrix,

$\boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in M(m \times 1)$ is called the vector of the right-hand side and

$\boldsymbol{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in M(n \times 1)$ is the vector of unknowns.

Definition

The matrix

$$(\boldsymbol{A}|\boldsymbol{b}) = \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right)$$

is called the augmented matrix of the system (S).

$$\begin{aligned} 6x + 2y &= 100 \\ 4x + y &= 60 \end{aligned}$$

$$\left(\begin{array}{cc|c} 6 & 2 & 100 \\ 4 & 1 & 60 \end{array}\right)$$

Proposition 16 (solutions of a transformed system)

Let $\boldsymbol{A} \in M(m \times n)$, $\boldsymbol{b} \in M(m \times 1)$ and let T be a transformation of matrices with m rows. Denote $\boldsymbol{A} \overset{T}{\rightsquigarrow} \boldsymbol{A}'$, $\boldsymbol{b} \overset{T}{\rightsquigarrow} \boldsymbol{b}'$. Then for any $\boldsymbol{y} \in M(n \times 1)$ we have $\boldsymbol{Ay} = \boldsymbol{b}$ if and only if $\boldsymbol{A}'\boldsymbol{y} = \boldsymbol{b}'$, i.e. the systems $\boldsymbol{Ax} = \boldsymbol{b}$ and $\boldsymbol{A}'\boldsymbol{x} = \boldsymbol{b}'$ have the same set of solutions.

Theorem 17 (Rouché-Fontené)

The system (S) *has a solution if and only if its coefficient matrix has the same rank as its augmented matrix.*

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 4 & 1 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Systems of n equations in n variables

Theorem 18 (solvability of an $n \times n$ system)

Let $\mathbf{A} \in M(n \times n)$. Then the following statements are equivalent:

(i) *the matrix $\mathbf{A}$ is invertible,*

(ii) *for each $\mathbf{b} \in M(n \times 1)$ the system* (S) *has a unique solution,*

(iii) *for each $\mathbf{b} \in M(n \times 1)$ the system* (S) *has at least one solution,*

(iv) $\det \mathbf{A} \neq 0$.

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & -4 \\ 2 & 3 & -2 & 2 \\ -1 & 0 & 1 & 2 \end{array}\right)$$

$x = 1, y = 2, z = 3$
$\det A = -6$

$$\mathbf{A}^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{3} & -\frac{5}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{2} & \frac{1}{3} & \frac{1}{6} \end{pmatrix}$$

Theorem 19 (Cramer's rule)

Let $\boldsymbol{A} \in M(n \times n)$ be an invertible matrix, $\boldsymbol{b} \in M(n \times 1)$, $\boldsymbol{x} \in M(n \times 1)$, and $\boldsymbol{Ax} = \boldsymbol{b}$. Then

$$x_j = \frac{\begin{vmatrix} a_{11} & \dots & a_{1,j-1} & b_1 & a_{1,j+1} & \dots & a_{1n} \\ \vdots & & & \vdots & & & \vdots \\ a_{n1} & \dots & a_{n,j-1} & b_n & a_{n,j+1} & \dots & a_{nn} \end{vmatrix}}{\det \boldsymbol{A}}$$

for $j = 1, \dots, n$.

Exercise

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 3 & -2 & 1 & -5 \\ 1 & 3 & -2 & 14 \end{array}\right)$$

VI.5. Definiteness of matrices

Definition

We say that a **symmetric** *matrix* $\boldsymbol{A} \in M(n \times n)$ *is*

- *positive definite (PD), if* $\boldsymbol{u}^T\boldsymbol{A}\boldsymbol{u} > 0$ *for all* $\boldsymbol{u} \in \mathbb{R}^n$, $\boldsymbol{u} \neq \boldsymbol{o}$,
- *negative definite (ND), if* $\boldsymbol{u}^T\boldsymbol{A}\boldsymbol{u} < 0$ *for all* $\boldsymbol{u} \in \mathbb{R}^n$, $\boldsymbol{u} \neq \boldsymbol{o}$,
- *positive semidefinite (PSD), if* $\boldsymbol{u}^T\boldsymbol{A}\boldsymbol{u} \geq 0$ *for all* $\boldsymbol{u} \in \mathbb{R}^n$,
- *negative semidefinite (NSD), if* $\boldsymbol{u}^T\boldsymbol{A}\boldsymbol{u} \leq 0$ *for all* $\boldsymbol{u} \in \mathbb{R}^n$,
- *indefinite (ID), if there exist* $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ *such that* $\boldsymbol{u}^T\boldsymbol{A}\boldsymbol{u} > 0$ *and* $\boldsymbol{v}^T\boldsymbol{A}\boldsymbol{v} < 0$.

$$\begin{pmatrix} x & y \end{pmatrix} \cdot \begin{pmatrix} 7 & 1 \\ 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 7x^2+4xy+y^2 = 3x^2+4x^2+4xy+y^2 = 3x^2+(y+2x)^2$$

Proposition 20 (definiteness of diagonal matrices)

Let $\boldsymbol{A} \in M(n \times n)$ be **diagonal** *(i.e. $a_{ij} = 0$ whenever $i \neq j$). Then*

- *$\boldsymbol{A}$ is PD if and only if $a_{ii} > 0$ for all $i = 1, 2, \dots, n$,*
- *$\boldsymbol{A}$ is ND if and only if $a_{ii} < 0$ for all $i = 1, 2, \dots, n$,*
- *$\boldsymbol{A}$ is PSD if and only if $a_{ii} \geq 0$ for all $i = 1, 2, \dots, n$,*
- *$\boldsymbol{A}$ is NSD if and only if $a_{ii} \leq 0$ for all $i = 1, 2, \dots, n$,*
- *$\boldsymbol{A}$ is ID if and only if there exist $i, j \in \{1, 2, \dots, n\}$ such that $a_{ii} > 0$ and $a_{jj} < 0$.*

Exercise

Decide about definiteness of the following matrices:

$$\begin{pmatrix} -2 & 0 \\ 0 & -5 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Proposition 21 (necessary conditions for definiteness)

Let $\mathbf{A} \in M(n \times n)$ be a symmetric matrix. Then

- *If $\mathbf{A}$ is PD, then $a_{ii} > 0$ for all $i = 1, 2, \ldots, n$,*
- *If $\mathbf{A}$ is ND, then $a_{ii} < 0$ for all $i = 1, 2, \ldots, n$,*
- *If $\mathbf{A}$ is PSD, then $a_{ii} \geq 0$ for all $i = 1, 2, \ldots, n$,*
- *If $\mathbf{A}$ is NSD, then $a_{ii} \leq 0$ for all $i = 1, 2, \ldots, n$,*
- *If there exist $i, j \in \{1, 2, \ldots, n\}$ such that $a_{ii} > 0$ and $a_{jj} < 0$, then $\mathbf{A}$ is ID.*

Exercise

Which of this matrices can NOT be negative semidefinite?

A

$$\begin{pmatrix} 5 & 1 & -4 \\ 3 & 9 & 4 \\ 1 & 2 & -5 \end{pmatrix}$$

B

$$\begin{pmatrix} -1 & 0 & 8 \\ 3 & -2 & 1 \\ 1 & 0 & -2 \end{pmatrix}$$

C

$$\begin{pmatrix} -1 & 2 & 4 \\ -3 & 0 & 3 \\ -11 & 6 & -5 \end{pmatrix}$$

Theorem 22 (Sylvester's criterion)

Let $\boldsymbol{A} = (a_{ij}) \in M(n \times n)$ be a symmetric matrix. Then $\boldsymbol{A}$ is

- *positive definite if and only if*

$$\begin{vmatrix} a_{11} & \dots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{kk} \end{vmatrix} > 0 \quad \textit{for all } k = 1, \dots, n,$$

- *negative definite if and only if*

$$(-1)^k \begin{vmatrix} a_{11} & \dots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{kk} \end{vmatrix} > 0 \quad \textit{for all } k = 1, \dots, n,$$

- positive semidefinite if and only if

$$\begin{vmatrix} a_{i_1 i_1} & \dots & a_{i_1 i_k} \\ \vdots & & \vdots \\ a_{i_k i_1} & \dots & a_{i_k i_k} \end{vmatrix} \geq 0$$

for each k-tuple of integers $1 \leq i_1 < \cdots < i_k \leq n$, $k = 1, \dots, n$,

- negative semidefinite if and only if

$$(-1)^k \begin{vmatrix} a_{i_1 i_1} & \dots & a_{i_1 i_k} \\ \vdots & & \vdots \\ a_{i_k i_1} & \dots & a_{i_k i_k} \end{vmatrix} \geq 0$$

for each k-tuple of integers $1 \leq i_1 < \cdots < i_k \leq n$, $k = 1, \dots, n$.

Definition 4

Let $f \in C^2(G)$. Then the matrix

$$H_f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2^2}(x) & \dots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(x) & \frac{\partial^2 f}{\partial x_n \partial x_2}(x) & \dots & \frac{\partial^2 f}{\partial x_n^2 x_2}(x) \end{pmatrix}$$

is called Hessian matrix of f.

Exercise

Find the Hessian matrix of the function $x^3 + y^4 + 3x^2$ at the point $[-2, 0]$.

Definition 5

Let $f \in C^2(G)$. Then the matrix

$$H_f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \dots & \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \frac{\partial^2 f}{\partial x_2^2}(x) & \dots & \frac{\partial^2 f}{\partial x_2 \partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1}(x) & \frac{\partial^2 f}{\partial x_n \partial x_2}(x) & \dots & \frac{\partial^2 f}{\partial x_n^2 x_2}(x) \end{pmatrix}$$

is called Hessian matrix of f.

Theorem 23

Let $G \subset \mathbb{R}^n$ be convex and $f \in C^2(G)$. If the Hessian matrix of f is positive semidefinite for every $x \in G$, then f is convex on G. If the Hessian matrix of f is positive definite for every $x \in G$, then f is strictly convex on G.

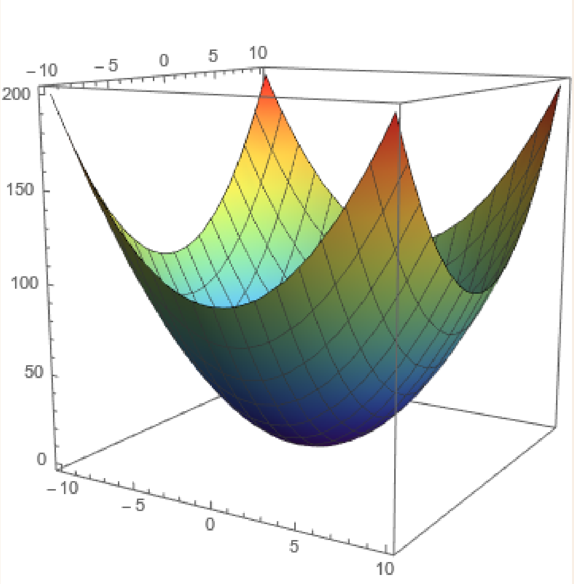
Remark (Sufficient condition of the existence of local extremum)

Let $G \subset \mathbb{R}^n$ be an open set, $a \in G$ and let $f \in C^2(G)$. Let $\nabla f(a) = 0$. Then

1. If the Hessian matrix is *positive definite*, then f has a *strict local minimum* at a.
2. If the Hessian matrix is *negative definite*, then f has a *strict local maximum* at a.
3. If the Hessian matrix is *indefinite*, then f does *not* have a local extrema at a (saddle point).

Example

$f(x,y) = x^2 + y^2$



Example

$f(x,y) = x^2 - y^2$