Mathematics II - Functions of multiple variables

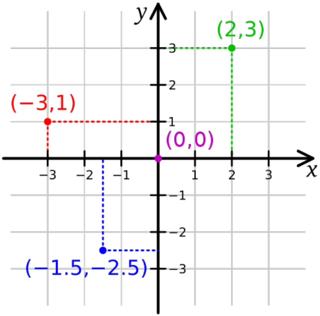
21/22

- Functions of several variables
- Matrix calculus
- Antiderivative and the Riemann Integral

Definition

The set $\mathbb{R}^n$, $n \in \mathbb{N}$, is the set of all ordered n-tuples of real numbers, i.e.

$$\mathbb{R}^n = \{[x_1, \dots, x_n] : x_1, \dots, x_n \in \mathbb{R}\}.$$



https://en.wikipedia.org/wiki/File:Cartesian-coordinate-system.svg

Exercise (2D)

Sketch the following points and connect them.

$$(14,5),(13,2),(12,0),(13,-3),(10,-1),(4,-2),(3,-4),$$

$$(1,-3),(-4,-3),(-6,-2),(-6,-7),(-8,-5),(-9,-2),$$

$$(-13,-1),(-11,0),(-14,1),(-12,2),(-9,3),(-4,3),(-2,7),$$

$$(0,3),(3,2),(9,1),(14,5).$$

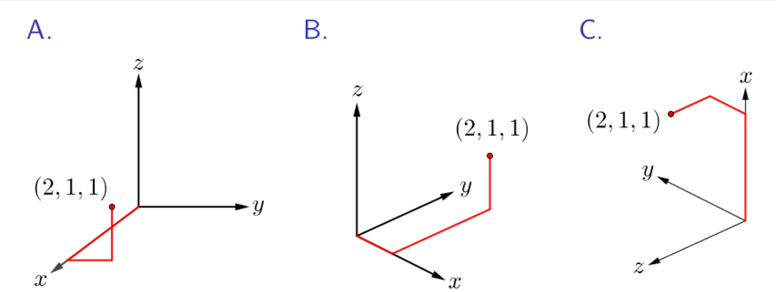
https://mathcrush.com/geometry_worksheets/

Exercise (3D)

https://www.geogebra.org/classic/ydu8a7t7

Exercise

Which picture(s) plots the point $(2, 1, 1)$ correctly?



https://www.cpp.edu/conceptests/question-library/mat214.shtml

Definition

For $\boldsymbol{x} = [x_1, \ldots, x_n] \in \mathbb{R}^n$, $\boldsymbol{y} = [y_1, \ldots, y_n] \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ we set

$$\boldsymbol{x} + \boldsymbol{y} = [x_1 + y_1, \ldots, x_n + y_n], \qquad \alpha\boldsymbol{x} = [\alpha x_1, \ldots, \alpha x_n].$$

Further, we denote $\boldsymbol{o} = [0, \ldots, 0]$ – the origin.

Exercise

Find

A $(1, 2, 3, 4) + (-2, 0, 3, -1)$

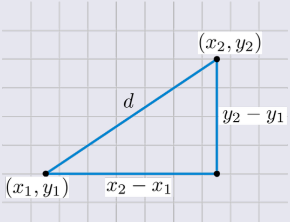
B $-2(1, 2, 3, 4)$

Definition

The Euclidean metric (distance) on $\mathbb{R}^n$ is the function $\rho\colon \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ defined by

$$\rho(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

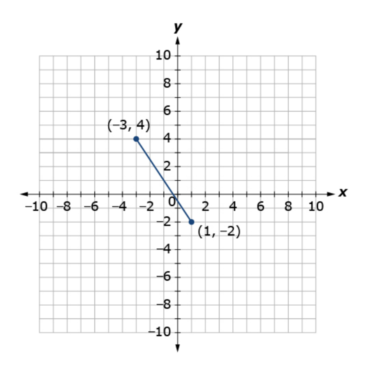
The number $\rho(\boldsymbol{x}, \boldsymbol{y})$ is called the distance of the point $\boldsymbol{x}$ from the point $\boldsymbol{y}$.



https://rosalind.info/glossary/euclidean-distance/

Exercise

Find the distance of the points

A 

https://www.summitlearning.org/guest/focusareas/862919

B $(1, -2, 3), (0, -3, -2)$

C $(-1, 0, 3, 2), (1, -1, 2, -3)$

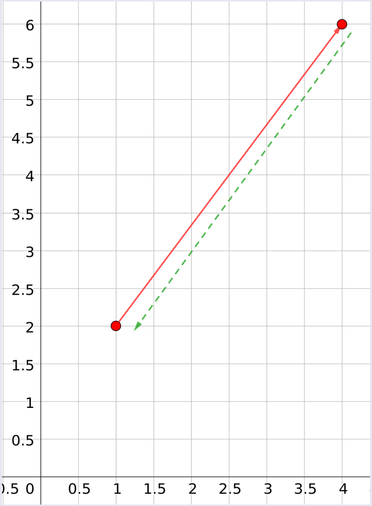
Exercise

A $\rho((1,2),(1,2))$

Exercise

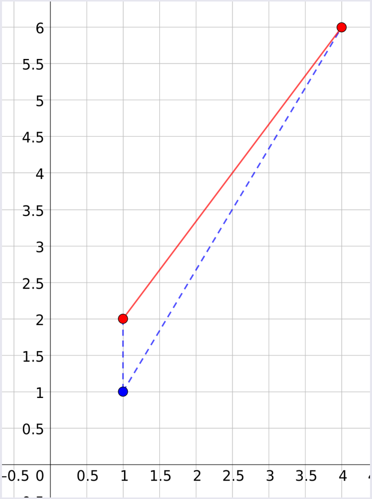
A $\rho((1,2),(1,2))$

B $\rho((1,2),(4,6)), \rho((4,6),(1,2))$



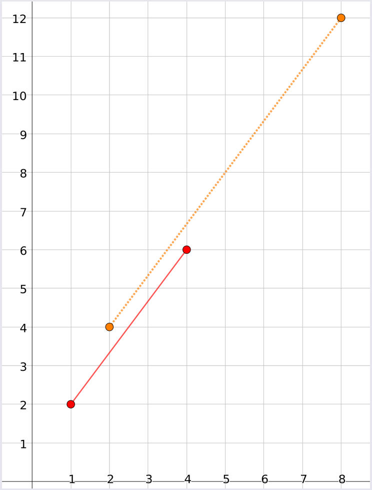
Exercise

C $\rho((1,2),(4,6))$, $\rho((1,2),(1,1)) + \rho((1,1),(4,6))$



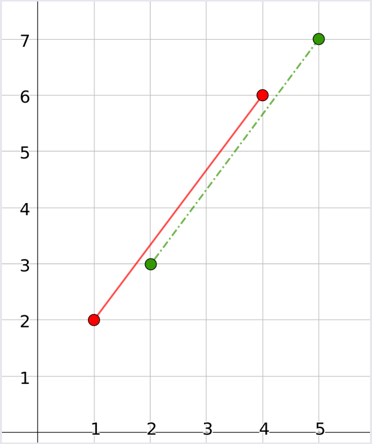
Exercise

D $2\rho((1,2),(4,6))$, $\rho((2,4),(8,12))$



Exercise

E $\rho((1,2),(4,6))$, $\rho((2,3),(5,7))$



Theorem 1 (properties of the Euclidean metric)

The Euclidean metric ρ has the following properties:

(i) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n\colon \rho(\boldsymbol{x}, \boldsymbol{y}) = 0 \Leftrightarrow \boldsymbol{x} = \boldsymbol{y}$,

(ii) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n\colon \rho(\boldsymbol{x}, \boldsymbol{y}) = \rho(\boldsymbol{y}, \boldsymbol{x})$, *(symmetry)*

(iii) $\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n\colon \rho(\boldsymbol{x}, \boldsymbol{y}) \le \rho(\boldsymbol{x}, \boldsymbol{z}) + \rho(\boldsymbol{z}, \boldsymbol{y})$, *(triangle inequality)*

(iv) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n, \forall \lambda \in \mathbb{R}\colon \rho(\lambda \boldsymbol{x}, \lambda \boldsymbol{y}) = |\lambda| \rho(\boldsymbol{x}, \boldsymbol{y})$, *(homogeneity)*

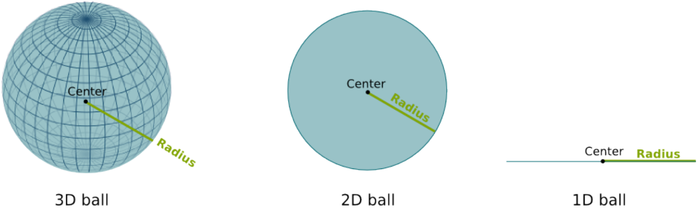
(v) $\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n\colon \rho(\boldsymbol{x} + \boldsymbol{z}, \boldsymbol{y} + \boldsymbol{z}) = \rho(\boldsymbol{x}, \boldsymbol{y})$. *(translation invariance)*

Definition

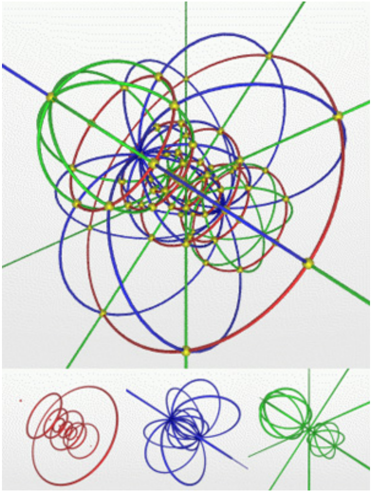
Let $\boldsymbol{x} \in \mathbb{R}^n,\ r \in \mathbb{R}, r > 0$. The set $B(\boldsymbol{x}, r)$ defined by

$$B(\boldsymbol{x}, r) = \{\boldsymbol{y} \in \mathbb{R}^n;\ \rho(\boldsymbol{x}, \boldsymbol{y}) < r\}$$

is called an open ball with radius r centred at $\boldsymbol{x}$ or the neighbourhood of $\boldsymbol{x}$.

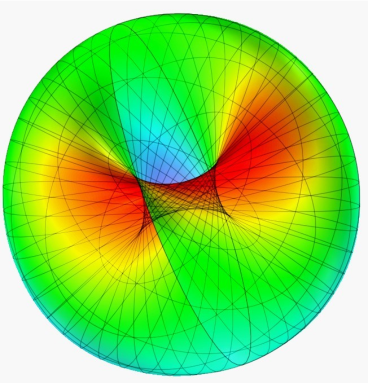


http://www.science4all.org/article/topology/

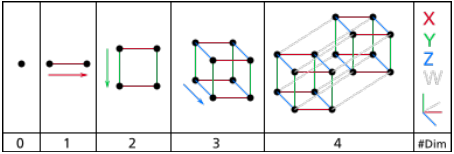


https://en.wikipedia.org/wiki/

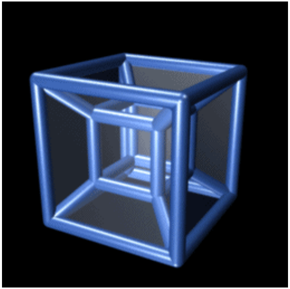
N-sphere



https://commons.wikimedia.org/
wiki/File:4dSphere.jpg



https://www.tinyepiphany.com/2011/12/visualizing-4-dimensions.html



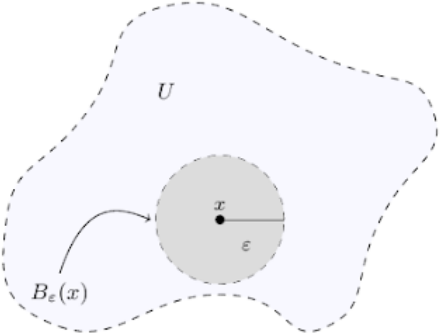
https://cs.wikipedia.org/wiki/%C4%8Ctvrt%C3%BD_rozm%C4%9Br

Definition

Let $M \subset \mathbb{R}^n$. We say that $\boldsymbol{x} \in \mathbb{R}^n$ is an interior point of M, if there exists $r > 0$ such that $B(\boldsymbol{x}, r) \subset M$.

The set of all interior points of M is called the interior of M and is denoted by $\operatorname{Int} M$.

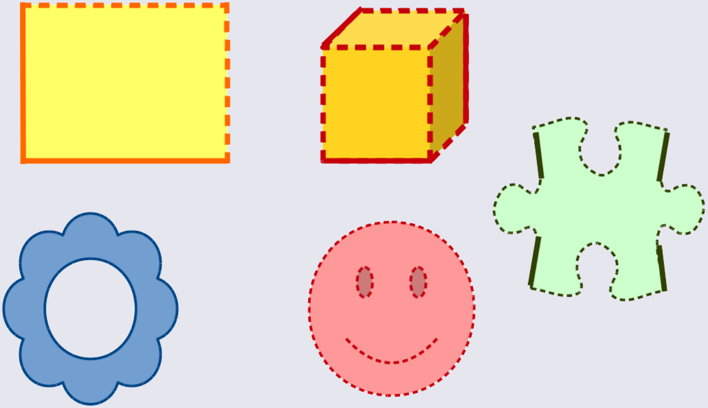
The set $M \subset \mathbb{R}^n$ is open in $\mathbb{R}^n$, if each point of M is an interior point of M, i.e. if $M = \operatorname{Int} M$.



http://www.gtmath.com/2016/07/how-close-is-close-enough-metric-spaces.html

Exercise

Find the interior



Theorem 2 (properties of open sets)

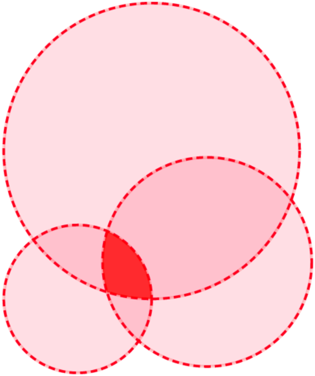
(i) *The empty set and* $\mathbb{R}^n$ *are open in* $\mathbb{R}^n$.

(ii) *Let* $G_\alpha \subset \mathbb{R}^n$, $\alpha \in A \neq \emptyset$, *be open in* $\mathbb{R}^n$. *Then* $\bigcup_{\alpha \in A} G_\alpha$ *is open in* $\mathbb{R}^n$.

(iii) *Let* $G_i \subset \mathbb{R}^n$, $i = 1, \ldots, m$, *be open in* $\mathbb{R}^n$. *Then* $\bigcap_{i=1}^{m} G_i$ *is open in* $\mathbb{R}^n$.

Remark

(ii) *A union of an arbitrary system of open sets is an open set.*
(iii) *An intersection of a finitely many open sets is an open set.*

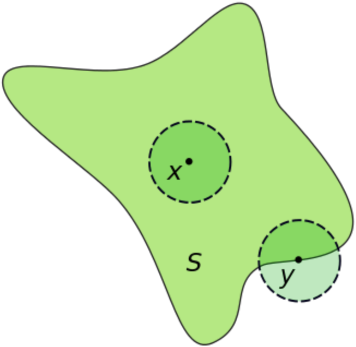


Definition

Let $M \subset \mathbb{R}^n$ and $\boldsymbol{x} \in \mathbb{R}^n$. We say that $\boldsymbol{x}$ is a boundary point of M if for each $r > 0$

$$B(\boldsymbol{x}, r) \cap M \neq \emptyset \quad \text{and} \quad B(\boldsymbol{x}, r) \cap (\mathbb{R}^n \setminus M) \neq \emptyset.$$

The boundary of M is the set of all boundary points of M (notation $\operatorname{bd} M$).



https://en.wikipedia.org/wiki/File:Interior_illustration.svg

Definition

The closure of M is the set $M \cup \operatorname{bd} M$ (notation $\overline{M}$).

A set $M \subset \mathbb{R}^n$ is said to be closed in $\mathbb{R}^n$ if it contains all its boundary points, i.e. if $\operatorname{bd} M \subset M$, or in other words if $\overline{M} = M$.

Exercise

Decide, if the set is closed or open, find the interior, the boundary, the closure.

$$M = \{[x, y] \in \mathbb{R}^2 : 1 < x \leq 2, 3 \leq y \leq 5\}.$$

Definition

Let $\boldsymbol{x}^j \in \mathbb{R}^n$ for each $j \in \mathbb{N}$ and $\boldsymbol{x} \in \mathbb{R}^n$. We say that a sequence $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$ converges to $\boldsymbol{x}$, if

$$\lim_{j\to\infty} \rho(\boldsymbol{x}, \boldsymbol{x}^j) = 0.$$

The vector $\boldsymbol{x}$ is called the limit of the sequence $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$.

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The sequence $\{\boldsymbol{y}^j\}_{j=1}^{\infty}$ of points in $\mathbb{R}^n$ is called convergent if there exists $\boldsymbol{y} \in \mathbb{R}^n$ such that $\{\boldsymbol{y}^j\}_{j=1}^{\infty}$ converges to $\boldsymbol{y}$.

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$$\lim_{j\to\infty} \rho(\boldsymbol{x}, \boldsymbol{x}^j) = 0.$$

The vector $\boldsymbol{x}$ is called the **limit of the sequence** $\{\boldsymbol{x}^j\}_{j=1}^{\infty}$.
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Exercise

$$\lim_{j\to\infty} \left(\frac{1}{j}, \frac{2j+1}{j} \right)$$

Theorem 3 (convergence is coordinatewise)

Let $\boldsymbol{x}^j \in \mathbb{R}^n$ for each $j \in \mathbb{N}$ and let $\boldsymbol{x} \in \mathbb{R}^n$. The sequence $\{\boldsymbol{x}^j\}_{j=1}^\infty$ converges to $\boldsymbol{x}$ if and only if for each $i \in \{1, \dots, n\}$ the sequence of real numbers $\{x_i^j\}_{j=1}^\infty$ converges to the real number x_i.

Remark

Theorem 3 says that the convergence in the space $\mathbb{R}^n$ is the same as the "coordinatewise" convergence. It follows that a sequence $\{\boldsymbol{x}^j\}_{j=1}^\infty$ has at most one limit. If it exists, then we denote it by $\lim_{j\to\infty} \boldsymbol{x}^j$. Sometimes we also write simply $\boldsymbol{x}^j \to \boldsymbol{x}$ instead of $\lim_{j\to\infty} \boldsymbol{x}^j = \boldsymbol{x}$.

Exercise

Find the limits of $x^j = \left(1 + \frac{1}{j}, 3 - \frac{2}{j^2}, e^{-j}\right)$ $x^j = \left((-1)^j, \arctan(j^3)\right)$

Theorem 4 (characterisation of closed sets)

Let $M \subset \mathbb{R}^n$. Then the following statements are equivalent:

(i) *M is closed in $\mathbb{R}^n$.*

(ii) *$\mathbb{R}^n \setminus M$ is open in $\mathbb{R}^n$.*

(iii) *Any $\boldsymbol{x} \in \mathbb{R}^n$ which is a limit of a sequence from M belongs to M.*

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Exercise

Decide, if the sets are closed or open (or nothing)

1. $(0, 1)$ in $\mathbb{R}$
2. $(0, \infty)$ in $\mathbb{R}$
3. $(-\infty, 2]$ in $\mathbb{R}$
4. $x^2 + y^2 < 4$ in $\mathbb{R}^2$
5. $x^2 + y^2 \geq 2$ in $\mathbb{R}^2$

Theorem 5 (properties of closed sets)

(i) *The empty set and the whole space $\mathbb{R}^n$ are closed in $\mathbb{R}^n$.*

(ii) *Let $F_\alpha \subset \mathbb{R}^n$, $\alpha \in A \neq \emptyset$, be closed in $\mathbb{R}^n$. Then $\bigcap_{\alpha \in A} F_\alpha$ is closed in $\mathbb{R}^n$.*

(iii) *Let $F_i \subset \mathbb{R}^n$, $i = 1, \ldots, m$, be closed in $\mathbb{R}^n$. Then $\bigcup_{i=1}^{m} F_i$ is closed in $\mathbb{R}^n$.*

Remark

(ii) *An intersection of an arbitrary system of closed sets is closed.*
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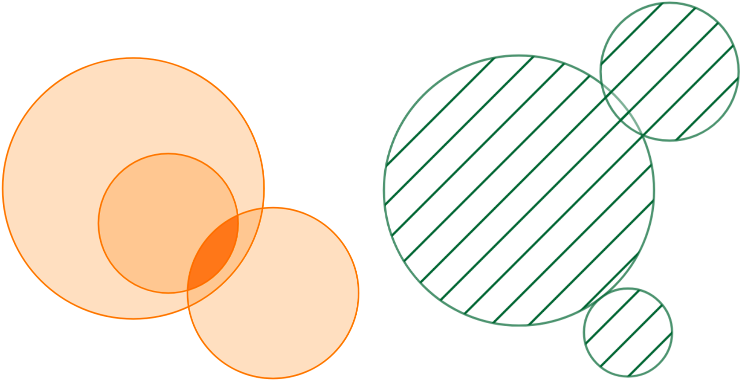

Theorem 6

Let $M \subset \mathbb{R}^n$. Then the following holds:

(i) *The set $\overline{M}$ is closed in $\mathbb{R}^n$.*

(ii) *The set $\operatorname{Int} M$ is open in $\mathbb{R}^n$.*

(iii) *The set M is open in $\mathbb{R}^n$ if and only if $M = \operatorname{Int} M$.*

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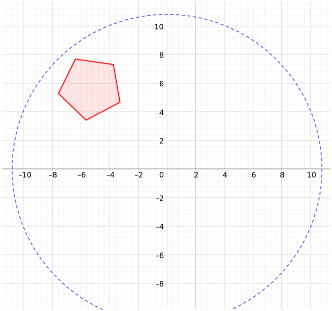
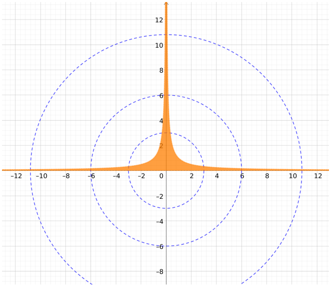
Remark

The set $\operatorname{Int} M$ is the largest open set contained in M in the following sense: If G is a set open in $\mathbb{R}^n$ and satisfying $G \subset M$, then $G \subset \operatorname{Int} M$. Similarly $\overline{M}$ is the smallest closed set containing M.

Definition

We say that the set $M \subset \mathbb{R}^n$ is bounded if there exists $r > 0$ such that $M \subset B(\boldsymbol{o}, r)$. A sequence of points in $\mathbb{R}^n$ is bounded if the set of its members is bounded.





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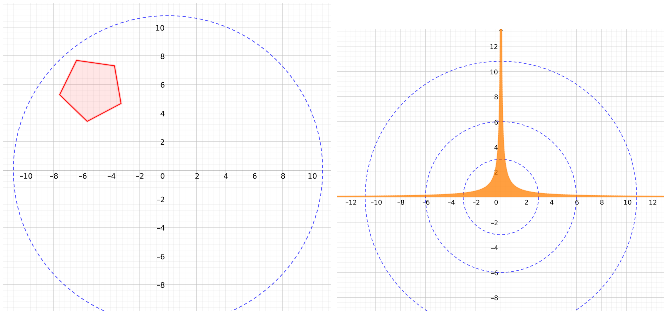

Theorem 7

A set $M \subset \mathbb{R}^n$ is bounded if and only if its closure $\overline{M}$ is bounded.

Exercise

Find bounded sets

A $x \in [-1, 3], 0 < y \leq 100$

B $x^2 + y^2 + z^2 \leq 5$

C $|x + y| < 6$

Definition

We say that a set $M \subset \mathbb{R}^n$ is compact if for each sequence of elements of M there exists a convergent subsequence with a limit in M.

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Theorem 8 (characterisation of compact subsets of $\mathbb{R}^n$)

The set $M \subset \mathbb{R}^n$ is compact if and only if M is bounded and closed.

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Theorem 8 (characterisation of compact subsets of $\mathbb{R}^n$)

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Exercise

Find compact sets

1. $(0, 1)$
2. $[1, 2] \times [-1, -3]$
3. $1 < x^2 + (y-3)^2 + z^2 \leq 4$
4. $xyz \leq 1$

Map game

Definition

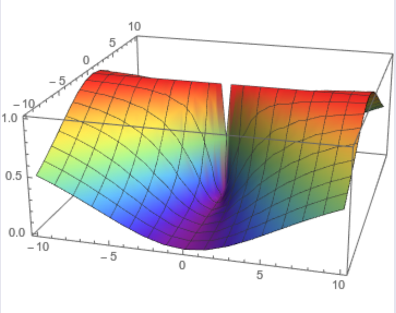
We define a function of two variables as a mapping $f : M \to \mathbb{R}$, where $M \subset \mathbb{R}^2$.

Example

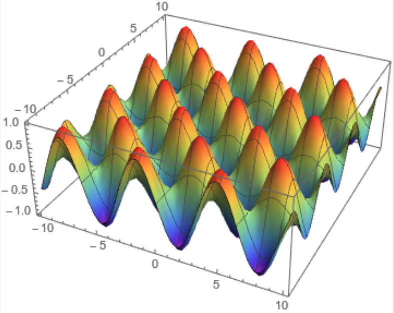
$$f(x,y) = x^2 + y^2, \qquad [x,y] \in \mathbb{R}^2$$
$$f(x,y) = \arccos y \cdot \arcsin x, \qquad D_f = [-1,1] \times [-1,1]$$
$$f(x,y) = \ln(xy), \qquad D_f = \{(x > 0 \wedge y > 0) \vee (x < 0 \wedge y < 0)\}$$
$$f(x,y) = x^3, \qquad [x,y] \in \mathbb{R}^2$$
$$f(x,y) = 5, \qquad [x,y] \in \mathbb{R}^2$$

Example

$f(x,y) = \frac{x^2}{x^2+y^2}$

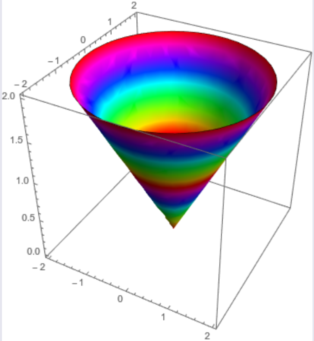


$f(x,y) = \sin x \cos y$

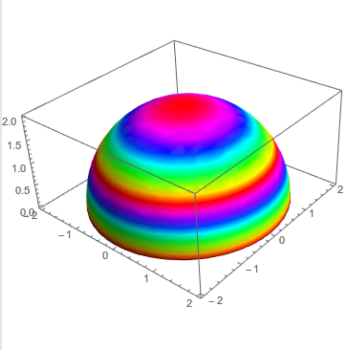


Example

$f(x, y) = \sqrt{x^2 + y^2}$

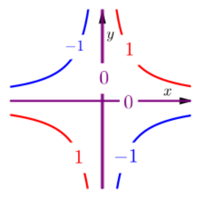


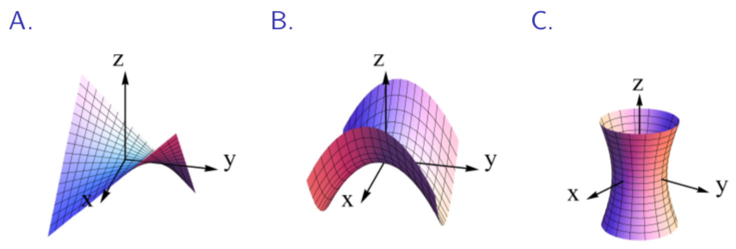
$f(x, y) = \sqrt{4 - (x^2 + y^2)}$



Exercise

Find the graph for the contourlines

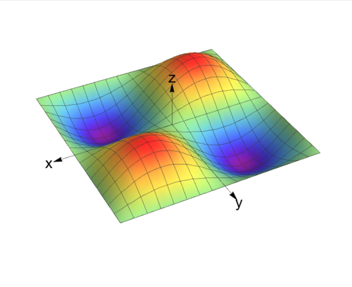


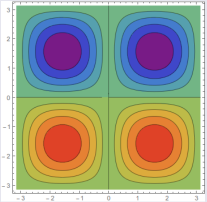


http://www.cpp.edu/~conceptests/question-library/mat214.shtml

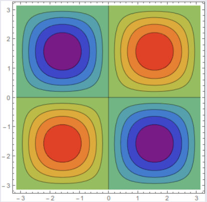
Exercise

Find the contourlines for the graph.

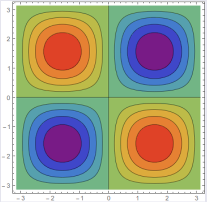




(a) *A*



(b) *B*



(c) *C*

Exercise

Connect the contourlines and the functions

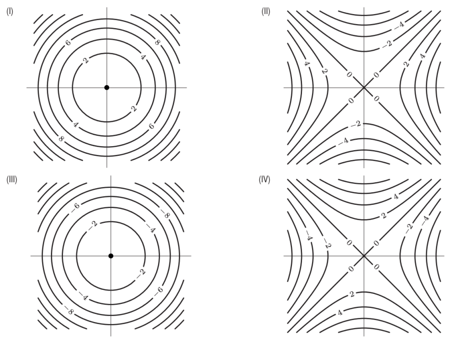


Figure: Hughes Hallett et al c 2009, John Wiley & Sons

A $-x^2 + y^2$

B $x^2 - y^2$

C $-x^2 - y^2$

D $x^2 + y^2$

Definition

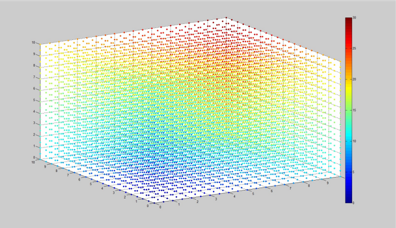
We define a function of multiple variables as a mapping $f : M \to \mathbb{R}$, where $M \subset \mathbb{R}^n$.

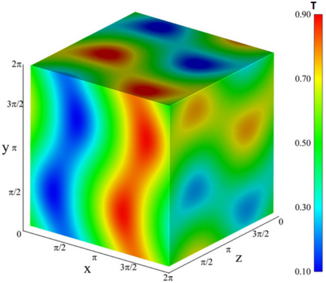
Example

$$
\begin{aligned}
f(x) &= x^3, & x &\in \mathbb{R} \\
f(x,y) &= y \sin x, & [x,y] &\in \mathbb{R}^2 \\
f(x,y,z) &= x^2 + y^2 z, & [x,y,z] &\in \mathbb{R}^3 \\
f(x,y,z) &= e^{xy} \arcsin z, & D_f &= \mathbb{R} \times \mathbb{R} \times [-1,1] \\
f(x,y,z) &= 5, & [x,y,z] &\in \mathbb{R}^3 \\
f(x,y,z,u) &= x e^{yz} \ln u, & D_f &= \{[x,y,z,u] \in \mathbb{R}^4 : u > 0\}
\end{aligned}
$$

Example

- Length of the day
- Length of your shadow.
- Compound interest.
- Storm radar.
- Drivers license tests.
- Google ads.





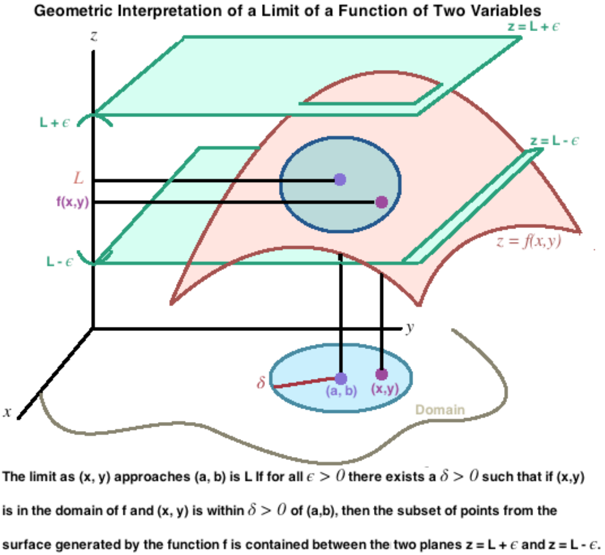
https://math.stackexchange.com/questions/703443/best-way-to-plot-a-4-dimensional-meshgrid
https://www.mathworks.com/matlabcentral/answers/224648-plotting-4d-with-3-vectors-and-1-matrix

Note: Mathematica animation

Definition

We say that a function f of n variables has a limit at a point $\boldsymbol{a} \in \mathbb{R}^n$ equal to $A \in \mathbb{R}^*$ if

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \; \exists \delta \in \mathbb{R}, \delta > 0 \; \forall \boldsymbol{x} \in B(\boldsymbol{a}, \delta) \setminus \{\boldsymbol{a}\} : f(\boldsymbol{x}) \in B(A, \varepsilon).$$



http://mathonline.wikidot.com/limits-of-functions-of-two-variables

Remark

- Each function has at a given point at most one limit. We write $\lim_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x}) = A$.
- The function f is continuous at $\boldsymbol{a}$ if and only if $\lim_{\boldsymbol{x}\to\boldsymbol{a}} f(\boldsymbol{x}) = f(\boldsymbol{a})$.
- For limits of functions of several variables one can prove similar theorems as for limits of functions of one variable (arithmetics, the sandwich theorem, ...).

Note: Mathematica animation

Exercise

1. $\lim_{(x,y)\to(2,-1)} x^2 - 2xy + 3y^2 - 4x + 3y - 6$
2. $\lim_{(x,y)\to(2,-1)} \frac{2x+3y}{4x-3y}$
3. $\lim_{(x,y)\to(0,0)} \frac{x^2+xy}{x+y}$

In the table there are values of a function $f(x,y)$. Does there exist the limit

$$\lim_{(x,y)\to(0,0)} f(x,y)?$$

$x \backslash y$	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.00	0.60	0.92	1.00	0.92	0.60	0.00
−0.5	−0.60	0.00	0.72	1.00	0.72	0.00	−0.6
−0.2	−0.92	−0.72	0.00	1.00	0.00	−0.72	−0.92
0	−1.00	−1.00	−1.00		−1.00	−1.00	−1.00
0.2	−0.92	−0.72	0.00	1.00	0.00	−0.72	−0.92
0.5	−0.60	0.00	0.72	1.00	0.72	0.00	−0.6
1.0	0.00	0.60	0.92	1.00	0.92	0.60	0.00

`https://www.cpp.edu/conceptests/question-library/mat214.shtml`

Theorem 9

Let $r, s \in \mathbb{N}$, $\boldsymbol{a} \in \mathbb{R}^s$, *and let* $\varphi_1, \ldots, \varphi_r$ *be functions of s variables such that* $\lim_{\boldsymbol{x}\to\boldsymbol{a}} \varphi_j(\boldsymbol{x}) = b_j$, $j = 1, \ldots, r$. *Set* $\boldsymbol{b} = [b_1, \ldots, b_r]$. *Let f be a function of r variables which is continuous at the point* $\boldsymbol{b}$. *If we define a compound function F of s variables by*

$$F(\boldsymbol{x}) = f(\varphi_1(\boldsymbol{x}), \varphi_2(\boldsymbol{x}), \ldots, \varphi_r(\boldsymbol{x})),$$

then $\lim_{\boldsymbol{x}\to\boldsymbol{a}} F(\boldsymbol{x}) = f(\boldsymbol{b})$.

Exercise

$$\lim_{(x,y)\to(4,1)} \sqrt{\frac{x^2 - 3xy}{x + y}}$$

Definition

Let $M \subset \mathbb{R}^n$, $\boldsymbol{x} \in M$, and $f : M \to \mathbb{R}$. We say that f is continuous at $\boldsymbol{x}$ with respect to M, if we

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \; \exists \delta \in \mathbb{R}, \delta > 0 \; \forall \boldsymbol{y} \in B(\boldsymbol{x}, \delta) \cap M : f(\boldsymbol{y}) \in B(f(\boldsymbol{x}), \varepsilon).$$

We say that f is continuous at the point $\boldsymbol{x}$ if it is continuous at $\boldsymbol{x}$ with respect to a neighbourhood of $\boldsymbol{x}$, i.e.

$$\forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \; \exists \delta \in \mathbb{R}, \delta > 0 \; \forall \boldsymbol{y} \in B(\boldsymbol{x}, \delta) : f(\boldsymbol{y}) \in B(f(\boldsymbol{x}), \varepsilon).$$

Definition

Let $M \subset \mathbb{R}^n$ and $f\colon M \to \mathbb{R}$. We say that f is continuous on M if it is continuous at each point $\boldsymbol{x} \in M$ with respect to M.

Remark

The functions $\pi_j\colon \mathbb{R}^n \to \mathbb{R}$, $\pi_j(\boldsymbol{x}) = x_j$, $1 \leq j \leq n$, are continuous on $\mathbb{R}^n$. They are called coordinate projections.

Theorem 10

Let $M \subset \mathbb{R}^n$, $\boldsymbol{x} \in M$, $f\colon M \to \mathbb{R}$, $g\colon M \to \mathbb{R}$, and $c \in \mathbb{R}$. If f and g are continuous at the point $\boldsymbol{x}$ with respect to M, then the functions cf, $f + g$ a fg are continuous at $\boldsymbol{x}$ with respect to M. If the function g is nonzero at $\boldsymbol{x}$, then also the function f/g is continuous at $\boldsymbol{x}$ with respect to M.

Theorem 11

Let $r, s \in \mathbb{N}$, $M \subset \mathbb{R}^s$, $L \subset \mathbb{R}^r$, and $\boldsymbol{y} \in M$. Let $\varphi_1, \dots, \varphi_r$ be functions defined on M, which are continuous at $\boldsymbol{y}$ with respect to M and $[\varphi_1(\boldsymbol{x}), \dots, \varphi_r(\boldsymbol{x})] \in L$ for each $\boldsymbol{x} \in M$. Let $f\colon L \to \mathbb{R}$ be continuous at the point $[\varphi_1(\boldsymbol{y}), \dots, \varphi_r(\boldsymbol{y})]$ with respect to L. Then the compound function $F\colon M \to \mathbb{R}$ defined by

$$F(\boldsymbol{x}) = f\big(\varphi_1(\boldsymbol{x}), \dots, \varphi_r(\boldsymbol{x})\big), \quad \boldsymbol{x} \in M,$$

is continuous at $\boldsymbol{y}$ with respect to M.

Exercise

Where is continuous $f(x, y) = \cos\frac{x}{y}$?

A Everywhere except at the origin

B Everywhere except along the x-axis.

C Everywhere except along the y-axis.

D Everywhere except along the line $y = x$.

Exercise

Where is continuous $f(x, y) = \operatorname{sgn} xy$?

A Everywhere except along the axes.

B Everywhere except along the x-axis.

C Everywhere except at the origin.

D Everywhere except along the line $y = x$.

Exercise

Find continuous functions (at $\mathbb{R}^2$)

- A $\ln(x^2 + y^2 + 1)$
- B $\frac{x-y}{e^{xy}}$
- C $\frac{\sqrt{y-1}}{x^2}$
- D $\sin(2x) + x\cot(x^3 + 2y)$
- E $\operatorname{sgn}(x^4 + y^4)$

Theorem 12

Let f be a continuous function on $\mathbb{R}^n$ and $c \in \mathbb{R}$. Then the following holds:

(i) *The set* $\{\boldsymbol{x} \in \mathbb{R}^n;\ f(\boldsymbol{x}) < c\}$ *is open in* $\mathbb{R}^n$.

(ii) *The set* $\{\boldsymbol{x} \in \mathbb{R}^n;\ f(\boldsymbol{x}) > c\}$ *is open in* $\mathbb{R}^n$.

(iii) *The set* $\{\boldsymbol{x} \in \mathbb{R}^n;\ f(\boldsymbol{x}) \leq c\}$ *is closed in* $\mathbb{R}^n$.

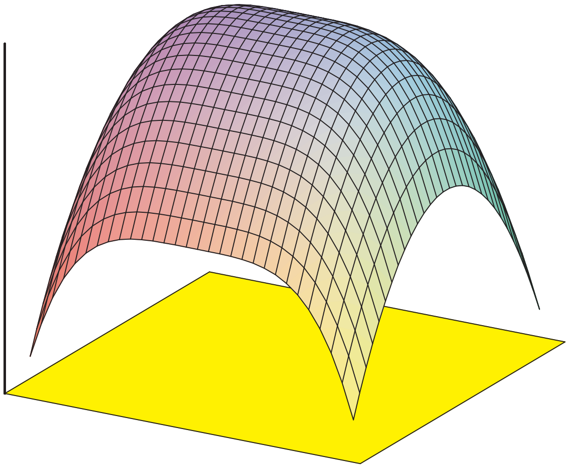
(iv) *The set* $\{\boldsymbol{x} \in \mathbb{R}^n;\ f(\boldsymbol{x}) \geq c\}$ *is closed in* $\mathbb{R}^n$.

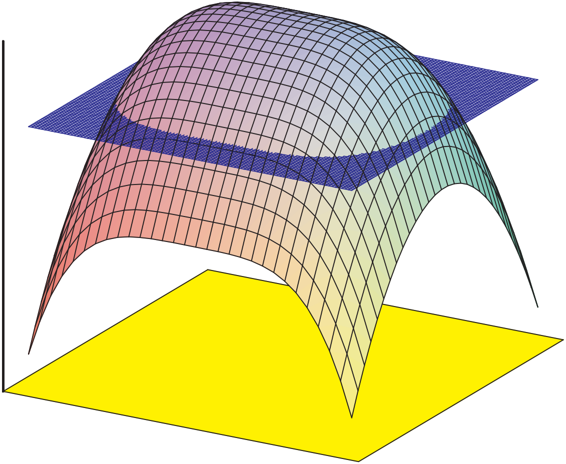
(v) *The set* $\{\boldsymbol{x} \in \mathbb{R}^n;\ f(\boldsymbol{x}) = c\}$ *is closed in* $\mathbb{R}^n$.

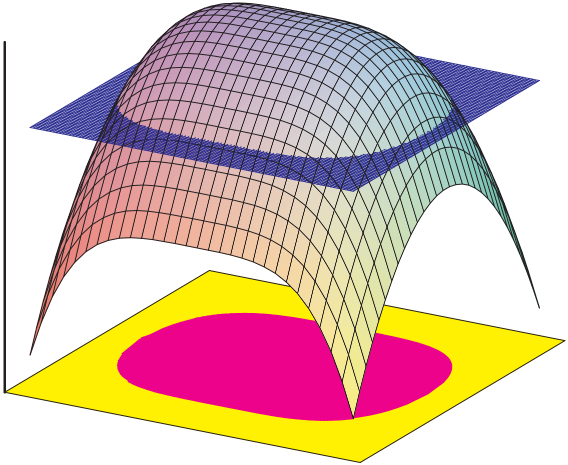
Example

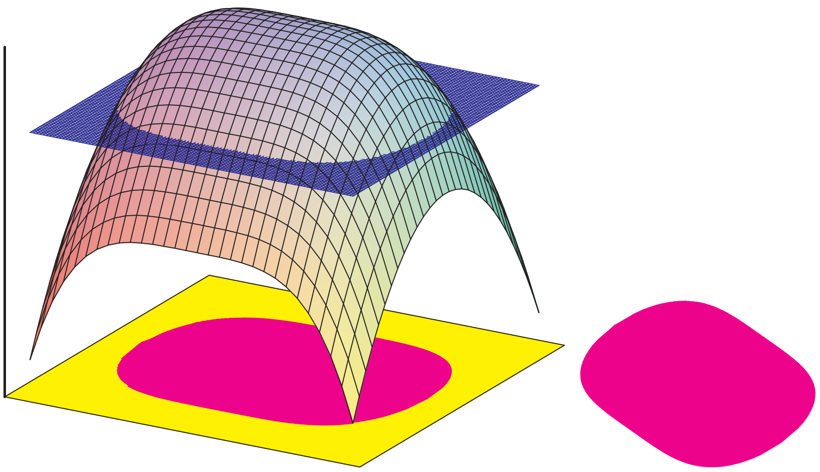
$$f(x,y) = x^2 + y^2,$$

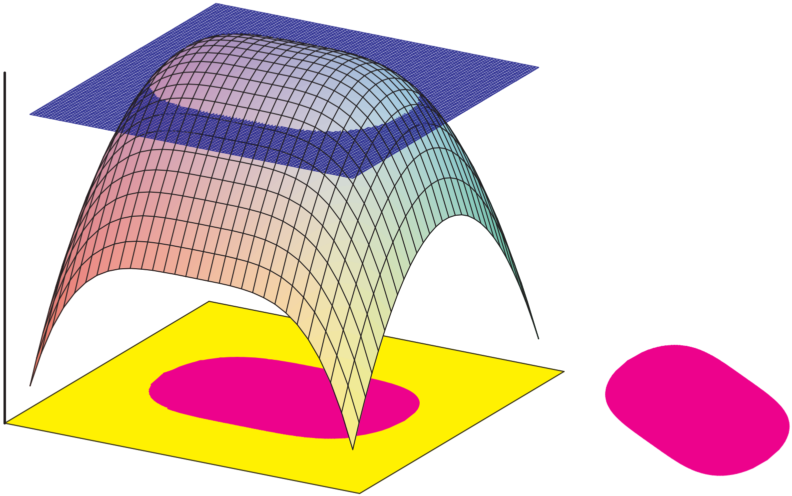
Mathematica

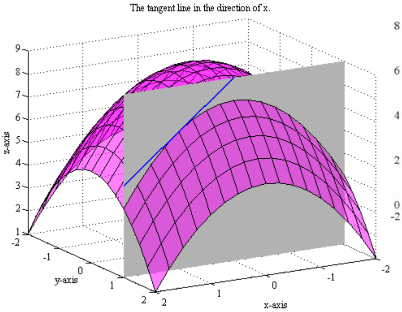


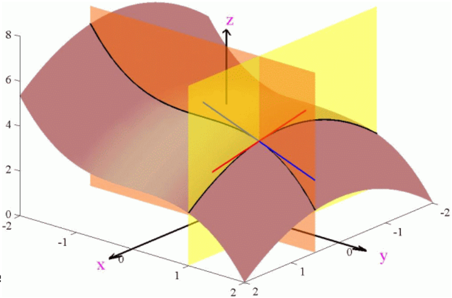












`https://www.wikihow.com/Take-Partial-Derivatives`

`http://calcnet.cst.cmich.edu/faculty/angelos/m533/lectures/pderv.htm`

Animation.

Definition

Let f be a function, $a \in \mathbb{R}$.

$$f'(a) = \lim_{t \to 0} \frac{f(a+t) - f(a)}{t}.$$

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$$f'(a) = \lim_{t \to 0} \frac{f(a+t) - f(a)}{t}.$$

Set $\boldsymbol{e}^j = [0, \ldots, 0, \underbrace{1}_{j\text{th coordinate}}, 0, \ldots, 0]$.

Definition

Let f be a function of n variables, $j \in \{1, \ldots, n\}$, $\boldsymbol{a} \in \mathbb{R}^n$. Then the number

$$\frac{\partial f}{\partial x_j}(\boldsymbol{a}) = \lim_{t \to 0} \frac{f(\boldsymbol{a} + t\boldsymbol{e}^j) - f(\boldsymbol{a})}{t}$$

$$= \lim_{t \to 0} \frac{f(a_1, \ldots, a_{j-1}, a_j + t, a_{j+1}, \ldots, a_n) - f(a_1, \ldots, a_n)}{t}$$

is called the partial derivative (of first order) of function f according to jth variable at the point $\boldsymbol{a}$ (if the limit exists).

Exercise

Find $\frac{\partial f}{\partial x}$, if $f(x,y) = x^3 + 3x^2y - 5x - 7y^3 + y - 5$

A $\dfrac{\partial f}{\partial x} = 3x^2 + 6xy - 5 - 7y^3 + y$

B $\dfrac{\partial f}{\partial x} = 3x^2 + 6xy - 5$

C $\dfrac{\partial f}{\partial x} = x^3 + 3 - 21y^2 + 1 - 5$

D $\dfrac{\partial f}{\partial x} = 3x^2 - 21y^2 + 1$

Exercise

Find $\frac{\partial f}{\partial x}$, if $f(x,y) = x^3 + 3x^2y - 5x - 7y^3 + y - 5$

A $\dfrac{\partial f}{\partial x} = 3x^2 + 6xy - 5 - 7y^3 + y$

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C $\dfrac{\partial f}{\partial x} = x^3 + 3 - 21y^2 + 1 - 5$

D $\dfrac{\partial f}{\partial x} = 3x^2 - 21y^2 + 1$

Find $\frac{\partial f}{\partial y}$, if $f(x,y) = x^2 \ln(x^2 y)$

A $\dfrac{\partial f}{\partial y} = \dfrac{2x}{y}$

B $\dfrac{\partial f}{\partial y} = \dfrac{1}{y}$

C $\dfrac{\partial f}{\partial y} = \dfrac{x^2}{y}$

D $\dfrac{\partial f}{\partial y} = \dfrac{1}{x^2 y}$

According to: https://www.wiley.com/college/hugheshallett/0470089148/conceptests/concept.pdf

Exercise

The values of a function $f(x, y)$ are in the table. Which statement is most accurate?
(In the left columnt there is x, in the the first row there is y.)

$x \backslash y$	0	1	2	3
0	3	5	7	9
1	2	4	6	8
2	1	3	5	7
3	0	2	4	6

A $\frac{\partial f}{\partial x}(1, 2) \approx -1$

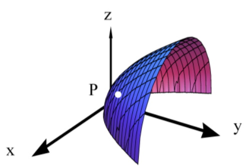
B $\frac{\partial f}{\partial y}(1, 2) \approx 2$

C $\frac{\partial f}{\partial x}(3, 2) \approx 1$

D $\frac{\partial f}{\partial y}(3, 2) \approx 4$

https://www.cpp.edu/conceptests/question-library/mat214.shtml

Exercise



A $\frac{\partial f}{\partial x} > 0, \frac{\partial f}{\partial y} > 0$

B $\frac{\partial f}{\partial x} < 0, \frac{\partial f}{\partial y} > 0$

C $\frac{\partial f}{\partial x} > 0, \frac{\partial f}{\partial y} < 0$

D $\frac{\partial f}{\partial x} < 0, \frac{\partial f}{\partial y} < 0$

https://www.cpp.edu/conceptests/question-library/mat214.shtml

Exercise (True or false?)

1. Let $f(x,y,z) = x^2 + z + 3$. Then the partial derivative $\frac{\partial f}{\partial y}$ is not defined, because there is no y in the function.
2. Is there a function $f(x,y)$ such that $\frac{\partial f}{\partial y} = 3y^2$ and $\frac{\partial f}{\partial x} = 3x^2$?

Exercise

Find a function, which is not constant, but
$\frac{\partial f}{\partial x} = 0$ for every x.

Definition

Let $G \subset \mathbb{R}^n$ be a non-empty open set. If a function $f\colon G \to \mathbb{R}$ has all partial derivatives continuous at each point of the set G (i.e. the function $\boldsymbol{x} \mapsto \frac{\partial f}{\partial x_j}(\boldsymbol{x})$ is continuous on G for each $j \in \{1, \dots, n\}$), then we say that f is of the class $\mathcal{C}^1$ on G. The set of all of these functions is denoted by $\mathcal{C}^1(G)$.

Definition

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Remark

If $G \subset \mathbb{R}^n$ is a non-empty open set and and $f, g \in \mathcal{C}^1(G)$, then $f + g \in \mathcal{C}^1(G)$, $f - g \in \mathcal{C}^1(G)$, and $fg \in \mathcal{C}^1(G)$. If moreover $g(\boldsymbol{x}) \neq 0$ for each $\boldsymbol{x} \in G$, then $f/g \in \mathcal{C}^1(G)$.

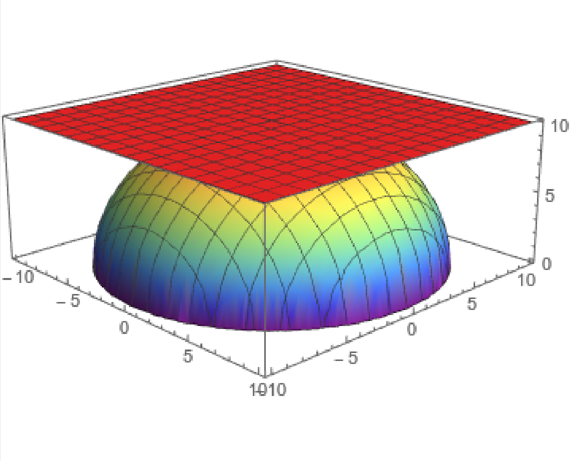
Exercise

Find functions, which are $\mathcal{C}^1(\mathbb{R}^2)$.

- A e^{xy}
- B $\sqrt[3]{x^2 + y^2}$
- C $\frac{\sin(x-2y)}{2+x^2+y^2}$
- D $\ln \frac{y}{x}$

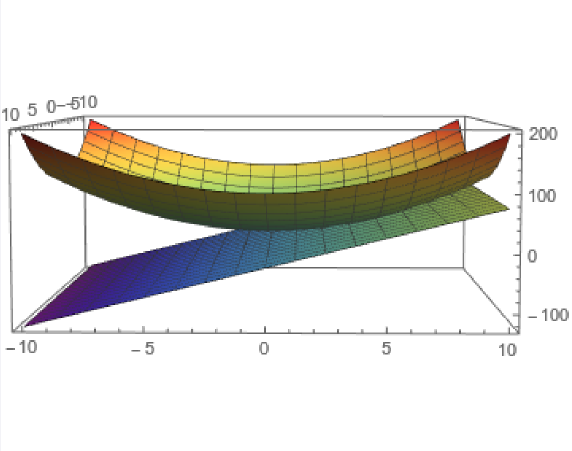
Example

$f(x, y) = \sqrt{100 - x^2 - y^2}$



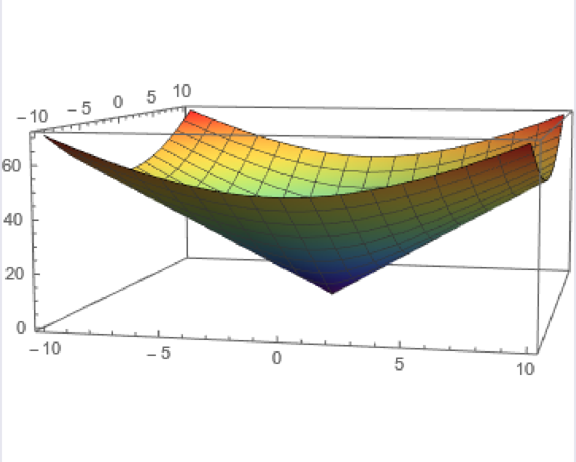
Example

$f(x,y) = x^2 + y^2$



Example

$f(x,y) = 5\sqrt{x^2 + y^2}$



Definition

Let $G \subset \mathbb{R}^n$ be an open set, $\boldsymbol{a} \in G$, and $f \in C^1(G)$. Then the graph of the function

$$T\colon \boldsymbol{x} \mapsto f(\boldsymbol{a}) + \frac{\partial f}{\partial x_1}(\boldsymbol{a})(x_1 - a_1) + \frac{\partial f}{\partial x_2}(\boldsymbol{a})(x_2 - a_2) + \dots + \frac{\partial f}{\partial x_n}(\boldsymbol{a})(x_n - a_n), \quad \boldsymbol{x} \in \mathbb{R}^n,$$

is called the tangent hyperplane to the graph of the function f at the point $[\boldsymbol{a}, f(\boldsymbol{a})]$.

Exercise

Find the tangent plane of a function $f(x,y) = xy$ at the point $(2,3)$.

A $z - 6 = x(x-2) + y(y-3)$

B $z - 6 = y(x-2) + x(y-3)$

C $z - 6 = 2(x-2) + 3(y-3)$

D $z - 6 = 3(x-2) + 2(y-3)$

Exercise

Find the tangent plane of a function $f(x,y,z,u) = \ln(xy + z^2 - u)$ at the point $a = (1,0,2,3)$.

Theorem 13 (tangent hyperplane)

Let $G \subset \mathbb{R}^n$ be an open set, $\boldsymbol{a} \in G$, $f \in C^1(G)$, and let T be a function whose graph is the tangent hyperplane of the function f at the point $[\boldsymbol{a}, f(\boldsymbol{a})]$. Then

$$\lim_{\boldsymbol{x} \to \boldsymbol{a}} \frac{f(\boldsymbol{x}) - T(\boldsymbol{x})}{\rho(\boldsymbol{x}, \boldsymbol{a})} = 0.$$

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Let $G \subset \mathbb{R}^n$ be an open set, $\boldsymbol{a} \in G$, $f \in C^1(G)$, and let T be a function whose graph is the tangent hyperplane of the function f at the point $[\boldsymbol{a}, f(\boldsymbol{a})]$. Then

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Theorem 14

Let $G \subset \mathbb{R}^n$ be an open non-empty set and $f \in C^1(G)$. Then f is continuous on G.

Theorem 13 (tangent hyperplane)

Let $G \subset \mathbb{R}^n$ be an open set, $\boldsymbol{a} \in G$, $f \in C^1(G)$, and let T be a function whose graph is the tangent hyperplane of the function f at the point $[\boldsymbol{a}, f(\boldsymbol{a})]$. Then

$$\lim_{\boldsymbol{x} \to \boldsymbol{a}} \frac{f(\boldsymbol{x}) - T(\boldsymbol{x})}{\rho(\boldsymbol{x}, \boldsymbol{a})} = 0.$$

Theorem 14

Let $G \subset \mathbb{R}^n$ be an open non-empty set and $f \in C^1(G)$. Then f is continuous on G.

Remark

Existence of partial derivatives at a **does not** imply continuity at a.

Theorem 15 (derivative of a composite function; chain rule)

Let $r, s \in \mathbb{N}$ *and let* $G \subset \mathbb{R}^s$, $H \subset \mathbb{R}^r$ *be open sets. Let* $\varphi_1, \dots, \varphi_r \in C^1(G)$, $f \in C^1(H)$ *and* $[\varphi_1(\boldsymbol{x}), \dots, \varphi_r(\boldsymbol{x})] \in H$ *for each* $\boldsymbol{x} \in G$. *Then the compound function* $F\colon G \to \mathbb{R}$ *defined by*

$$F(\boldsymbol{x}) = f\big(\varphi_1(\boldsymbol{x}), \varphi_2(\boldsymbol{x}), \dots, \varphi_r(\boldsymbol{x})\big), \quad \boldsymbol{x} \in G,$$

is of the class $\mathcal{C}^1$ *on* G. *Let* $\boldsymbol{a} \in G$ *and* $\boldsymbol{b} = [\varphi_1(\boldsymbol{a}), \dots, \varphi_r(\boldsymbol{a})]$. *Then for each* $j \in \{1, \dots, s\}$ *we have*

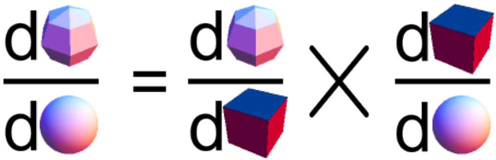
$$\frac{\partial F}{\partial x_j}(\boldsymbol{a}) = \sum_{i=1}^{r} \frac{\partial f}{\partial y_i}(\boldsymbol{b}) \frac{\partial \varphi_i}{\partial x_j}(\boldsymbol{a}).$$

Remark

Let $f(x,y,z)$ be a differentiable function, let $x = g_1(u,v)$, $y = g_2(u,v)$, $z = g_3(u,v)$, where g_1, g_2, g_3 are differentiable functions. Then for $h(u,v) = f(g_1(u,v), g_2(u,v), g_3(u,v))$ we have

$$\frac{\partial h}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial h}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}$$



http://mathinsight.org/media/image/image/chain_rule_geometric_objects.png

Exercise

Let $h(u, v) = \sin x \cos y$, where $x = (u - v)^2$ and $y = u^2 - v^2$. Find $\partial h/\partial u$ a $\partial h/\partial v$.

Exercise

Let $h(u, v) = xy$, where $x = u \cos v$ and $y = u \sin v$. Then for $\partial h/\partial v$ we have

A $\dfrac{\partial h}{\partial v} = 0$

B $\dfrac{\partial h}{\partial v} = u^2 \cos(2v)$

C $\dfrac{\partial h}{\partial v} = -u^3 \sin^2 v \cos v + u^3 \sin v \cos^2 v$

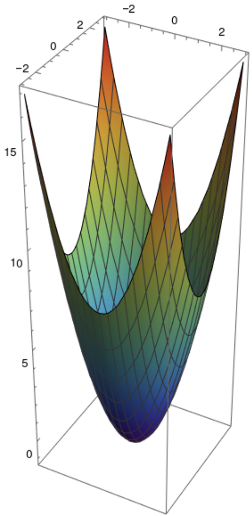
D Something else.

Exercise

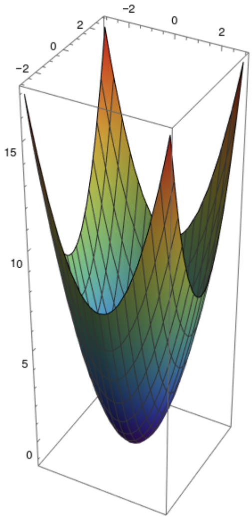
Let $f(x, y)$ satisfies the Chain rule theorem assumptions. Show, that a function $h(u, v, w) = \frac{uv}{w} \ln u + uf\left(\frac{v}{u}, \frac{w}{u}\right)$, where $x = \frac{v}{u}$,$y = \frac{w}{u}$ satisfies the following condition

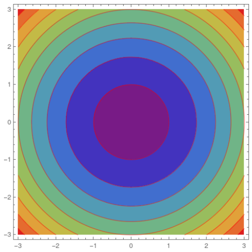
$$u\frac{\partial h}{\partial u} + v\frac{\partial h}{\partial v} + w\frac{\partial h}{\partial w} = h + \frac{uv}{w}.$$

V.4. Implicit function theorem and Lagrange multiplier theorem

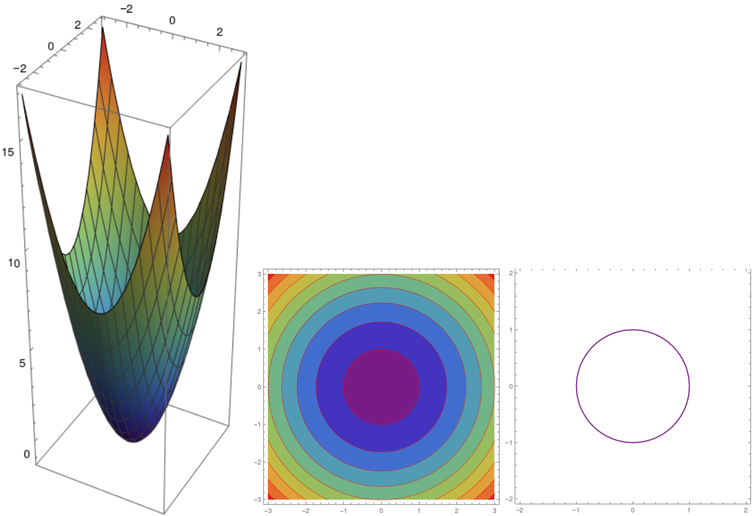


V.4. Implicit function theorem and Lagrange multiplier theorem

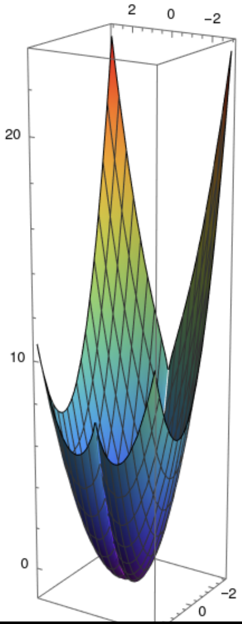




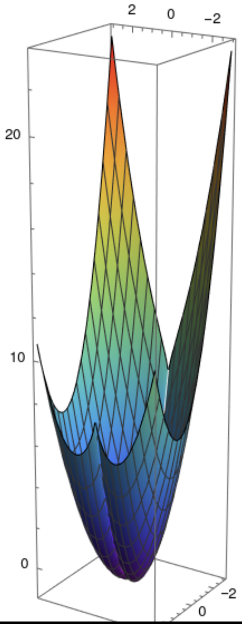
V.4. Implicit function theorem and Lagrange multiplier theorem

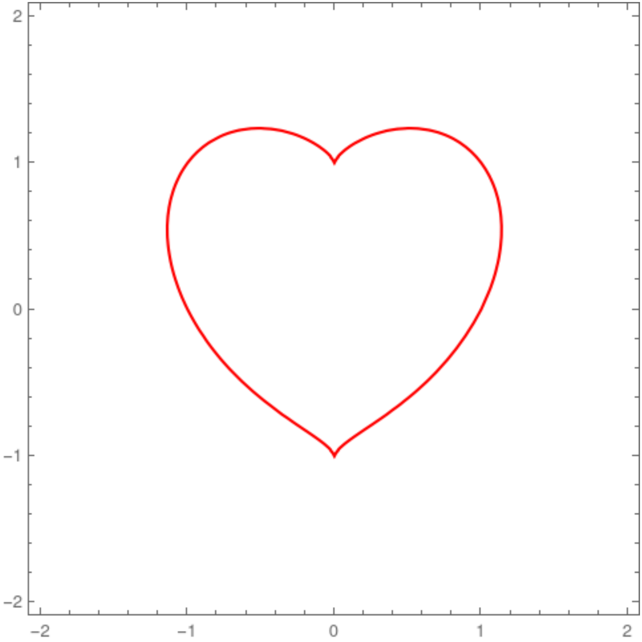


$$f(x,y) = x^2 + y^2 - 1 - y\sqrt[3]{x^2}$$



$$f(x,y) = x^2 + y^2 - 1 - y\sqrt[3]{x^2}$$




2
1
0
−1
−2
−2
−1
0
1
2

Theorem 16 (implicit function)

Let $G \subset \mathbb{R}^{n+1}$ be an open set, $F\colon G \to \mathbb{R}$, and $\tilde{\boldsymbol{x}} \in \mathbb{R}^n$, $\tilde{y} \in \mathbb{R}$ such that $[\tilde{\boldsymbol{x}}, \tilde{y}] \in G$. Suppose that

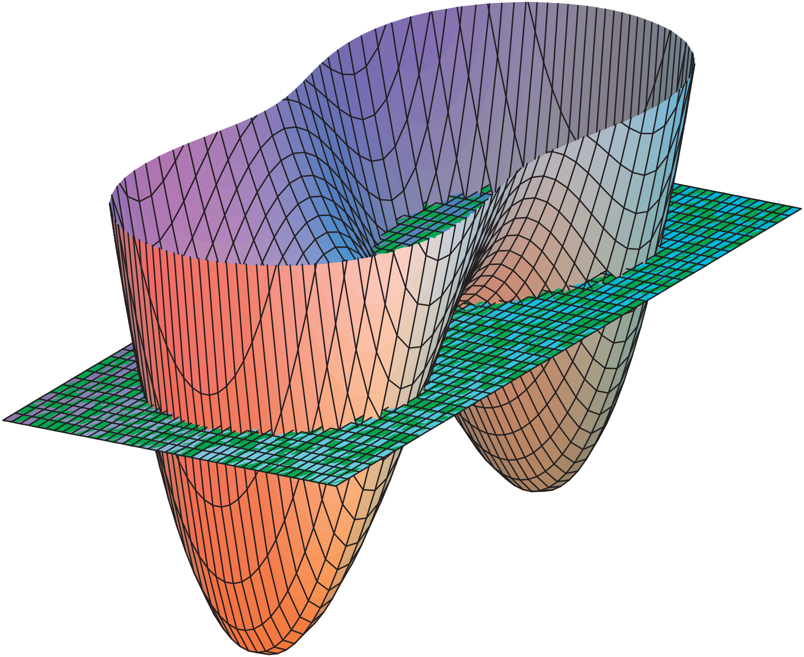
(i) $F \in C^1(G)$,

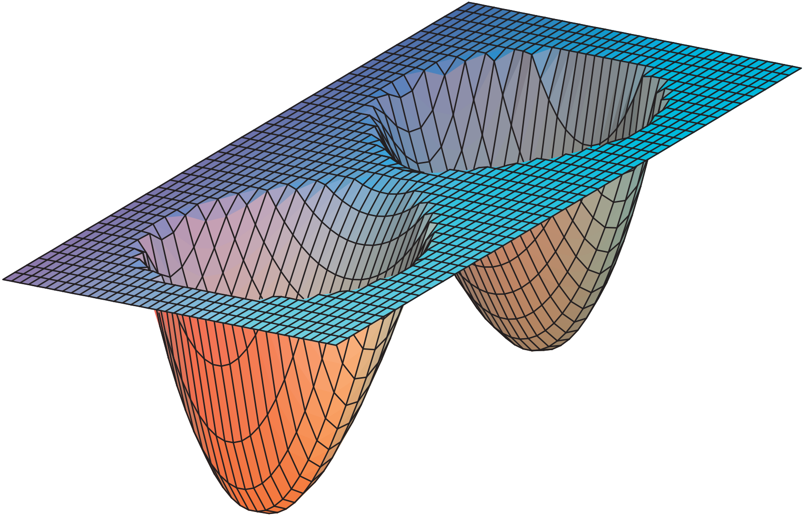
(ii) $F(\tilde{\boldsymbol{x}}, \tilde{y}) = 0$,

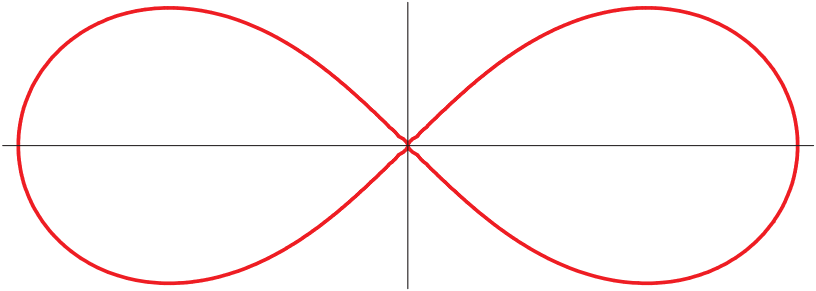
(iii) $\dfrac{\partial F}{\partial y}(\tilde{\boldsymbol{x}}, \tilde{y}) \neq 0$.

Then there exist a neighbourhood $U \subset \mathbb{R}^n$ of the point $\tilde{\boldsymbol{x}}$ and a neighbourhood $V \subset \mathbb{R}$ of the point $\tilde{y}$ such that for each $\boldsymbol{x} \in U$ there exists a unique $y \in V$ satisfying $F(\boldsymbol{x}, y) = 0$. If we denote this y by $\varphi(\boldsymbol{x})$, then the resulting function φ is in $C^1(U)$ and

$$\frac{\partial \varphi}{\partial x_j}(\boldsymbol{x}) = -\frac{\frac{\partial F}{\partial x_j}(\boldsymbol{x}, \varphi(\boldsymbol{x}))}{\frac{\partial F}{\partial y}(\boldsymbol{x}, \varphi(\boldsymbol{x}))} \quad \text{for } \boldsymbol{x} \in U,\ j \in \{1, \ldots, n\}.$$







Theorem

Let $G \subset \mathbb{R}^{n+1}$ be an open set, $F\colon G \to \mathbb{R}$, and $\tilde{\boldsymbol{x}} \in \mathbb{R}^n$, $\tilde{y} \in \mathbb{R}$ such that $[\tilde{\boldsymbol{x}}, \tilde{y}] \in G$. Suppose that

(i) $F \in C^1(G)$,

(ii) $F(\tilde{\boldsymbol{x}}, \tilde{y}) = 0$,

(iii) $\dfrac{\partial F}{\partial y}(\tilde{\boldsymbol{x}}, \tilde{y}) \neq 0$.

Then there exists a neighbourhood ...

Exercise

Which condition is NOT satisfied?

A $x^2 + y^3 = 4$ at $(2, 0)$

B $y - \frac{1}{2}\sin y = x$ at (π, π)

C $\sin(xy) + x^2 + y^2 = 1$ at $(0, 3)$

D $|x| + e^{x+y} = 1$ at $(0, 0)$

Definition

Let $G \subset \mathbb{R}^n$ be an open set, $\boldsymbol{a} \in G$, and $f \in C^1(G)$. The gradient of f at the point $\boldsymbol{a}$ is the vector

$$\nabla f(\boldsymbol{a}) = \left[\frac{\partial f}{\partial x_1}(\boldsymbol{a}), \frac{\partial f}{\partial x_2}(\boldsymbol{a}), \ldots, \frac{\partial f}{\partial x_n}(\boldsymbol{a})\right].$$

Exercise

Find the gradient of $f(x, y, z) = y\cos^3(x^2 z)$ at the point $[2, 1, 0]$:

A $(1/5, 0, 1/5)$

B $(0, 0, 1/5)$

C $(0, 1, 0)$

D $(1, 0, 1/2)$

Remark

The gradient of f at a points in the direction of steepest growth of f at a. At every point, the gradient is perpendicular to the contour of f.

Exercise

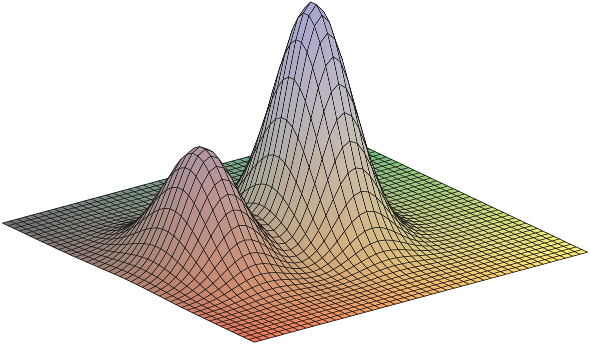
The bicyclist is on a trip up the hill, which can be described as $f(x,y) = 25 - 2x^2 - 4y^2$. When she is at the point $[1,1,19]$, it starts to rain, so she decides to go down the hill as steeply as possible (so that she is down quickly). In what direction will she start her decline?

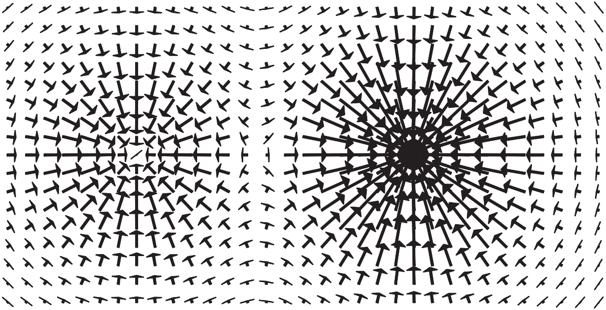
A $(-4x; -8y)$

B $(4x; 8y)$

C $(-4; -8)$

D $(4; 8)$





Definition

Let $M \subset \mathbb{R}^n$, $\boldsymbol{x} \in M$, and let f be a function defined at least on M (i.e. $M \subset D_f$). We say that f attains at the point $\boldsymbol{x}$ its

- maximum on M if $f(\boldsymbol{y}) \leq f(\boldsymbol{x})$ for every $\boldsymbol{y} \in M$,
- local maximum with respect to M if there exists $\delta > 0$ such that $f(\boldsymbol{y}) \leq f(\boldsymbol{x})$ for every $\boldsymbol{y} \in B(\boldsymbol{x}, \delta) \cap M$,
- strict local maximum with respect to M if there exists $\delta > 0$ such that $f(\boldsymbol{y}) < f(\boldsymbol{x})$ for every $\boldsymbol{y} \in \big(B(\boldsymbol{x}, \delta) \setminus \{\boldsymbol{x}\}\big) \cap M$.

The notions of a minimum, a local minimum, and a strict local minimum with respect to M are defined in analogous way.

Definition

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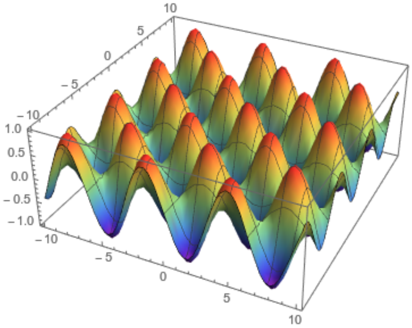
The notions of a minimum, a local minimum, and a strict local minimum with respect to M are defined in analogous way.

Definition

We say that a function f attains a local maximum at a point $\boldsymbol{x} \in \mathbb{R}^n$ if $\boldsymbol{x}$ is a local maximum with respect to some neighbourhood of $\boldsymbol{x}$.

Similarly we define local minimum, strict local maximum and strict local minimum.



Theorem 17 (attaining extrema)

Let $M \subset \mathbb{R}^n$ be a non-empty compact set and $f : M \to \mathbb{R}$ a function continuous on M. Then f attains its maximum and minimum on M.

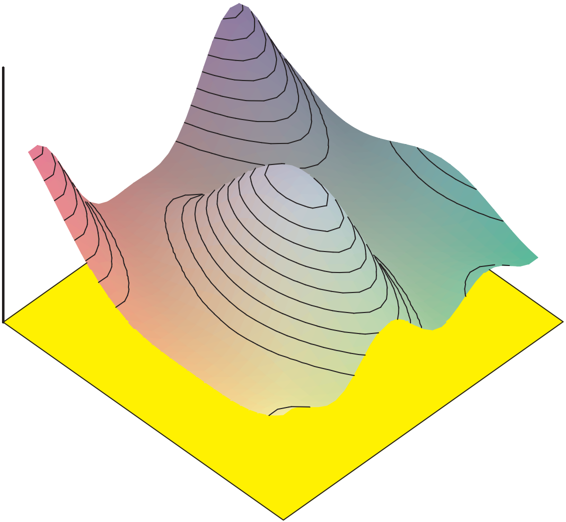
Corollary

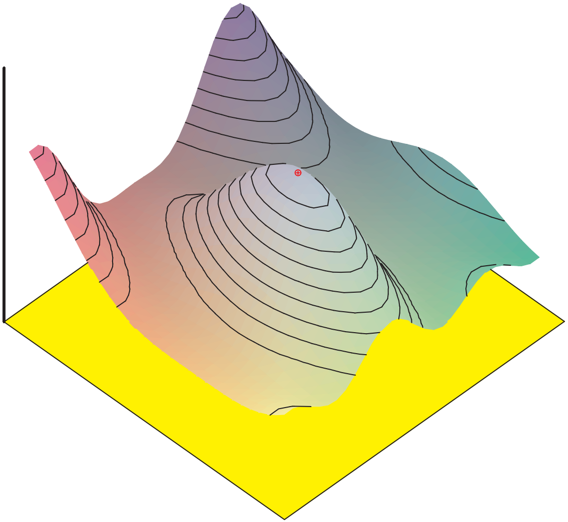
Let $M \subset \mathbb{R}^n$ be a non-empty compact set and $f : M \to \mathbb{R}$ a continuous function on M. Then f is bounded on M.

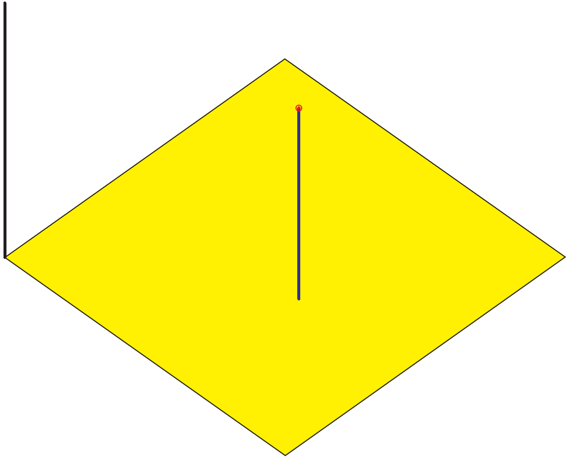
Theorem 18 (necessary condition of the existence of local extremum)

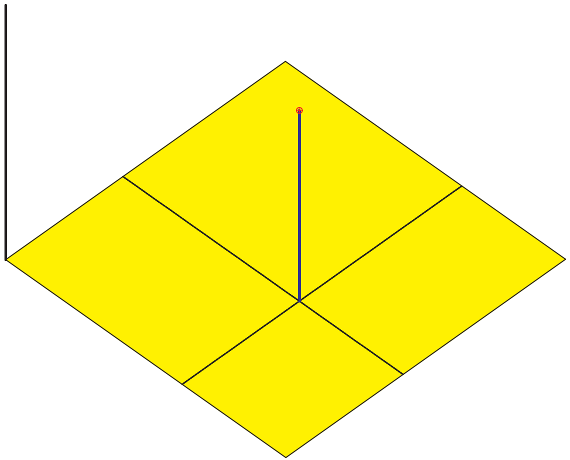
Let $G \subset \mathbb{R}^n$ be an open set, $\boldsymbol{a} \in G$, and suppose that a function $f : G \to \mathbb{R}$ has a local extremum (i.e. a local maximum or a local minimum) at the point $\boldsymbol{a}$. Then for each $j \in \{1, \dots, n\}$ the following holds:

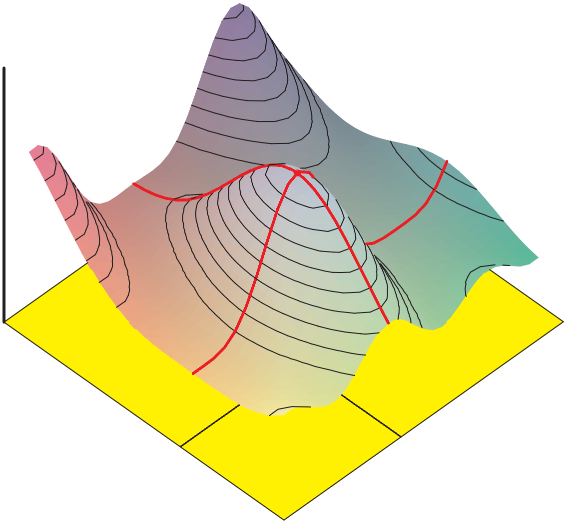
The partial derivative $\dfrac{\partial f}{\partial x_j}(\boldsymbol{a})$ either does not exist or it is equal to zero.

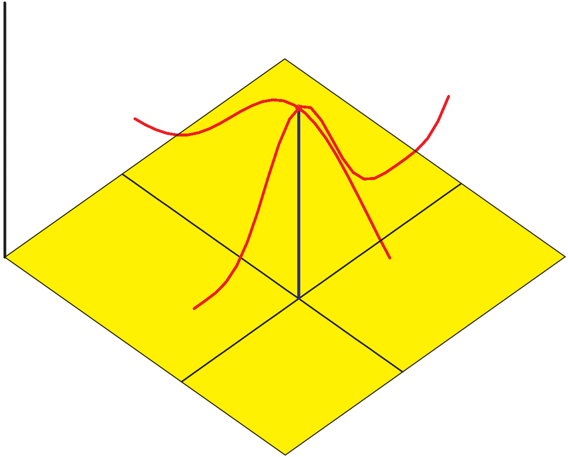


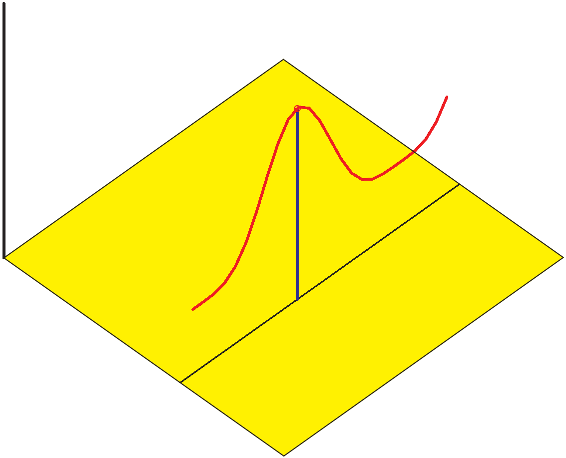


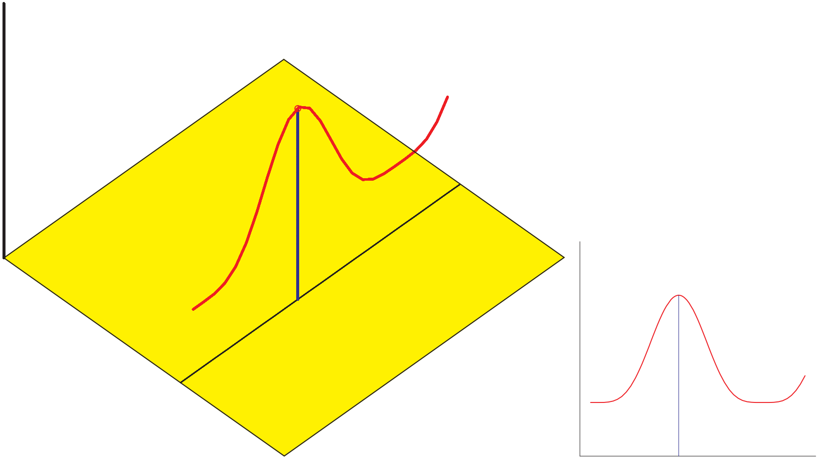










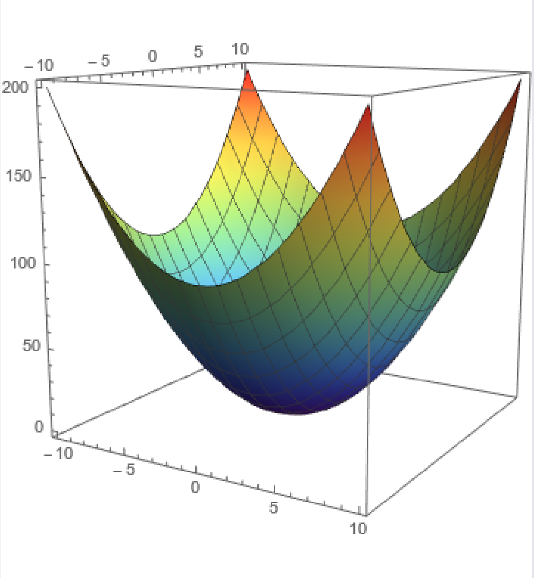


Definition

Let $G \subset \mathbb{R}^n$ be an open set, $\boldsymbol{a} \in G, f \in C^1(G)$, and $\nabla f(\boldsymbol{a}) = \boldsymbol{o}$. Then the point $\boldsymbol{a}$ is called a stationary (or critical) point of the function f.

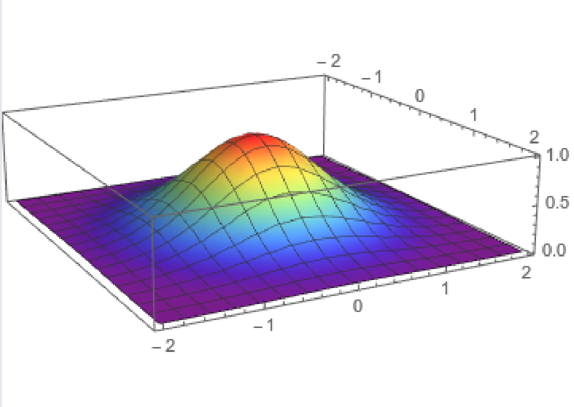
Example

$f(x,y) = x^2 + y^2$



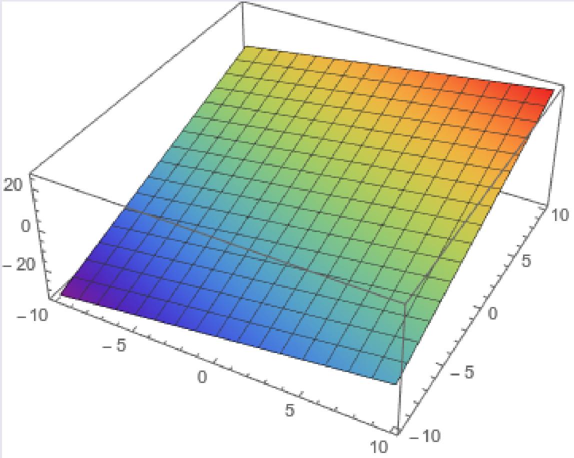
Example

$f(x,y) = e^{-x^2-y^2}$



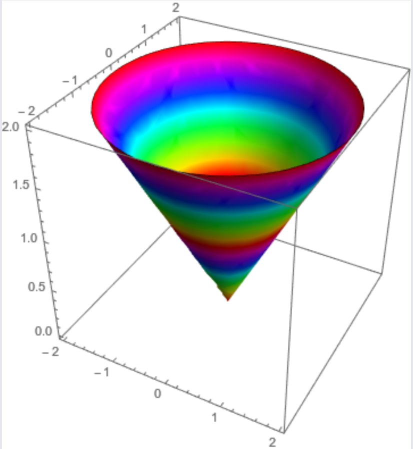
Example

$f(x,y) = x + 2y - 4$



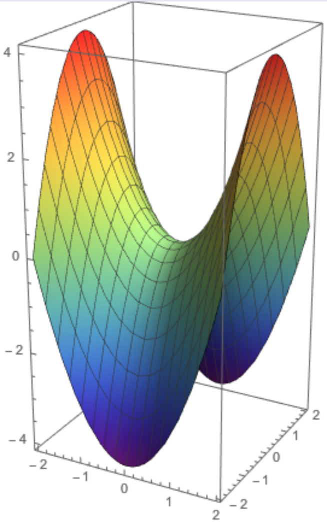
Example

$f(x,y) = \sqrt{x^2 + y^2}$



Example

$f(x, y) = x^2 - y^2$



Exercise

1. Consider the points A, B, C, D, E. Find the critical points.
2. Which of these points are probably points of
    - 2.1 local maximum,
    - 2.2 local minimum,
    - 2.3 saddle poi

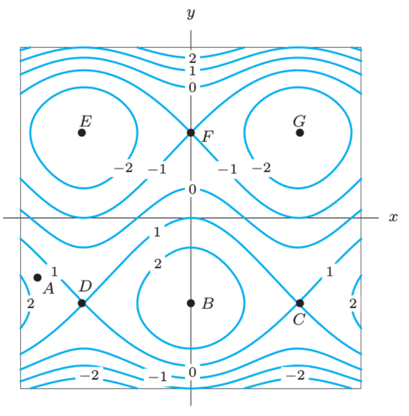


Figure: Calculus, 6th Edition; Hughes-Hallett, Gleason, McCallum et al.

Definition

Let $G \subset \mathbb{R}^n$ be an open set, $f\colon G \to \mathbb{R}$, $i, j \in \{1, \dots, n\}$, and suppose that $\frac{\partial f}{\partial x_i}(\boldsymbol{x})$ exists finite for each $\boldsymbol{x} \in G$. Then the partial derivative of the second order of the function f according to ith and jth variable at a point $\boldsymbol{a} \in G$ is defined by

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{a}) = \frac{\partial \left(\frac{\partial f}{\partial x_i} \right)}{\partial x_j}(\boldsymbol{a})$$

If $i = j$ then we use the notation $\frac{\partial^2 f}{\partial x_i^2}(\boldsymbol{a})$.

Similarly we define higher order partial derivatives.

Exercise

Find the second partial derivatives of the function $f(x, y) = x^2 + xy + y^2$.

Exercise

Find $\frac{\partial^2 f}{\partial x \partial y}$, if $f(x,y) = e^{xy}$

A e^{xy}

B ye^{xy}

C x^2e^{xy}

D $e^{xy}(xy+1)$

Exercise

Find $\frac{\partial^2 f}{\partial x \partial y}$, if $f(x,y) = e^{xy}$

A e^{xy}

B ye^{xy}

C $x^2 e^{xy}$

D $e^{xy}(xy+1)$

Exercise

Find $\frac{\partial^2 f}{\partial y \partial x}$, if $f(x,y) = e^{xy}$

A e^{xy}

B ye^{xy}

C $x^2 e^{xy}$

D $e^{xy}(xy+1)$

Remark

In general it is not true that $\frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{a}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\boldsymbol{a})$.

Remark

In general it is not true that $\frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{a}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\boldsymbol{a})$.

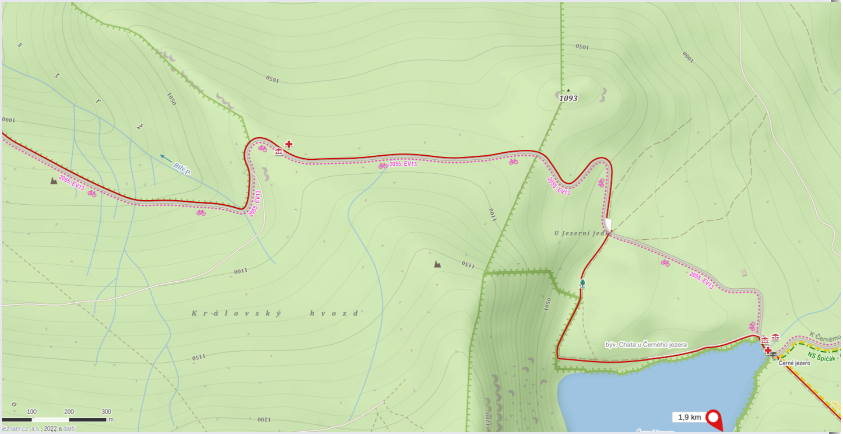
Theorem 19 (interchanging of partial derivatives)

Let $i, j \in \{1, \ldots, n\}$ and suppose that a function f has both partial derivatives $\frac{\partial^2 f}{\partial x_i \partial x_j}$ and $\frac{\partial^2 f}{\partial x_j \partial x_i}$ on a neighbourhood of a point $\boldsymbol{a} \in \mathbb{R}^n$ and that these functions are continuous at $\boldsymbol{a}$. Then

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\boldsymbol{a}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\boldsymbol{a}).$$

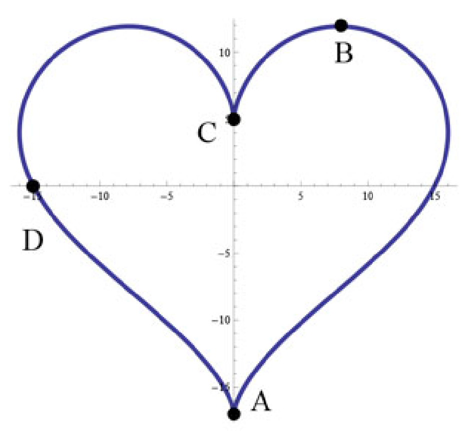
Exercise

You follow the red route. Where is the highest point of your trip?



Exercise

Where is the minimum and maximum of the function $f(x, y) = y$ along the



curve?

https://www.cpp.edu/conceptests/question-library/mat214.shtml

Theorem 20 (Lagrange multiplier theorem)

Let $G \subset \mathbb{R}^2$ be an open set, $f, g \in C^1(G)$, $M = \{[x,y] \in G;\ g(x,y) = 0\}$ and let $[\tilde{x}, \tilde{y}] \in M$ be a point of local extremum of f with respect to M. Then at least one of the following conditions holds:

(I) $\nabla g(\tilde{x}, \tilde{y}) = \boldsymbol{o}$,

(II) *there exists $\lambda \in \mathbb{R}$ satisfying*

$$\frac{\partial f}{\partial x}(\tilde{x}, \tilde{y}) + \lambda \frac{\partial g}{\partial x}(\tilde{x}, \tilde{y}) = 0,$$
$$\frac{\partial f}{\partial y}(\tilde{x}, \tilde{y}) + \lambda \frac{\partial g}{\partial y}(\tilde{x}, \tilde{y}) = 0.$$

Theorem 21 (Lagrange multipliers theorem)

Let $m, n \in \mathbb{N}$, $m < n$, $G \subset \mathbb{R}^n$ an open set, $f, g_1, \dots, g_m \in C^1(G)$,

$$M = \{z \in G;\ g_1(z) = 0, g_2(z) = 0, \dots, g_m(z) = 0\}$$

and let $\tilde{z} \in M$ be a point of local extremum of f with respect to the set M. Then at least one of the following conditions holds:

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and let $\tilde{z} \in M$ be a point of local extremum of f with respect to the set M. Then at least one of the following conditions holds:

(I) *the vectors*

$$\nabla g_1(\tilde{z}), \nabla g_2(\tilde{z}), \dots, \nabla g_m(\tilde{z})$$

are linearly dependent,

Theorem 21 (Lagrange multipliers theorem)

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$$M = \{z \in G;\ g_1(z) = 0, g_2(z) = 0, \dots, g_m(z) = 0\}$$

and let $\tilde{z} \in M$ be a point of local extremum of f with respect to the set M. Then at least one of the following conditions holds:

(I) *the vectors*

$$\nabla g_1(\tilde{z}), \nabla g_2(\tilde{z}), \dots, \nabla g_m(\tilde{z})$$

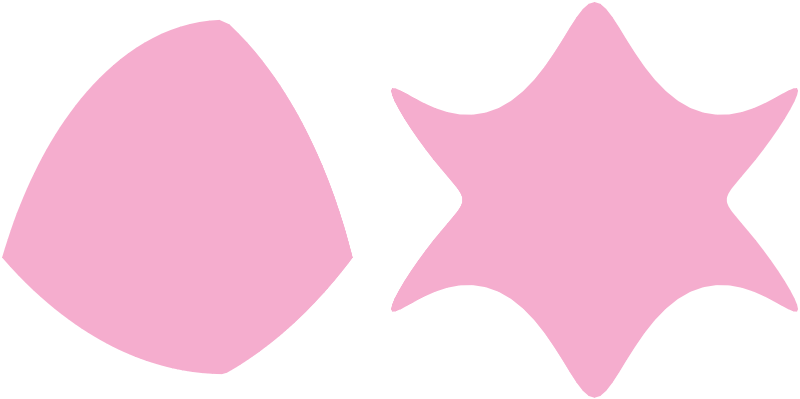
are linearly dependent,

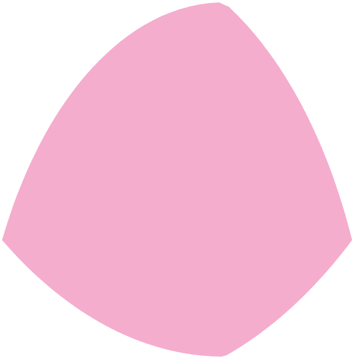
(II) *there exist numbers $\lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{R}$ satisfying*

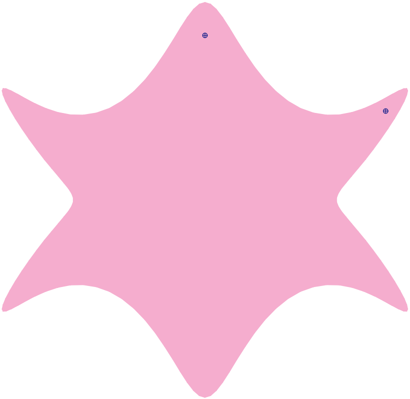
$$\nabla f(\tilde{z}) + \lambda_1 \nabla g_1(\tilde{z}) + \lambda_2 \nabla g_2(\tilde{z}) + \dots + \lambda_m \nabla g_m(\tilde{z}) = \boldsymbol{o}.$$

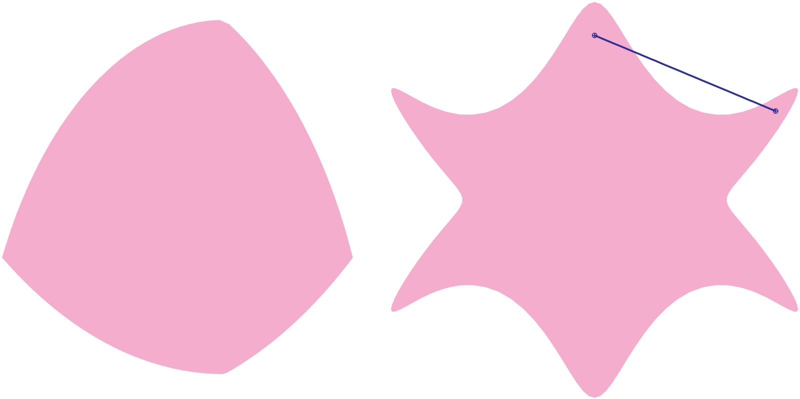
Remark

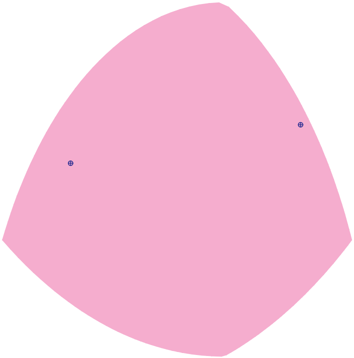
- The notion of linearly dependent vectors will be defined later.
  For $m = 1$: One vector is linearly dependent if it is the zero vector.
  For $m = 2$: Two vectors are linearly dependent if one of them is a multiple of the other one.
- The numbers $\lambda_1, \ldots, \lambda_m$ are called the Lagrange multipliers.

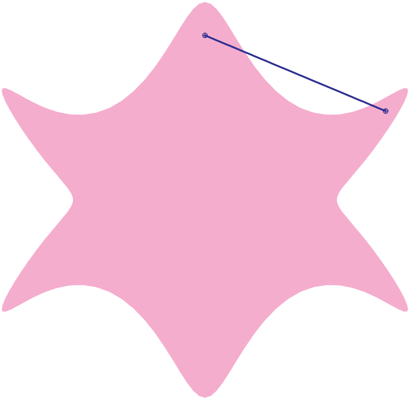


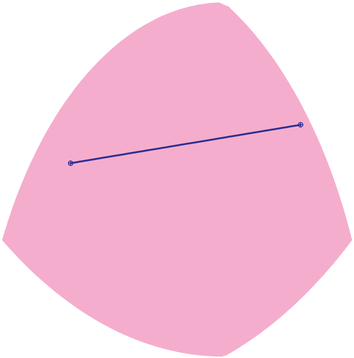


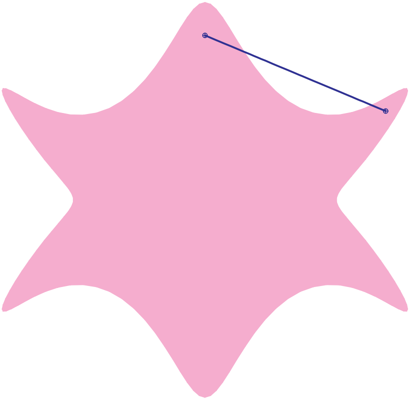


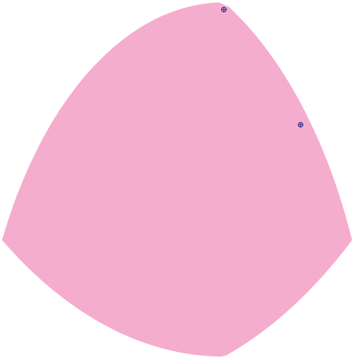


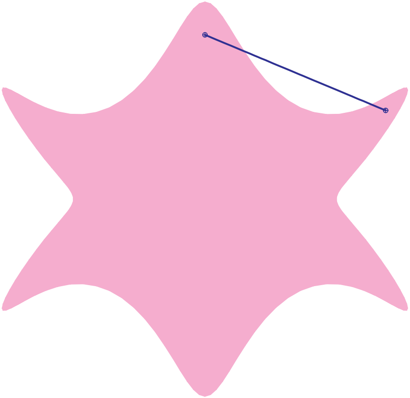


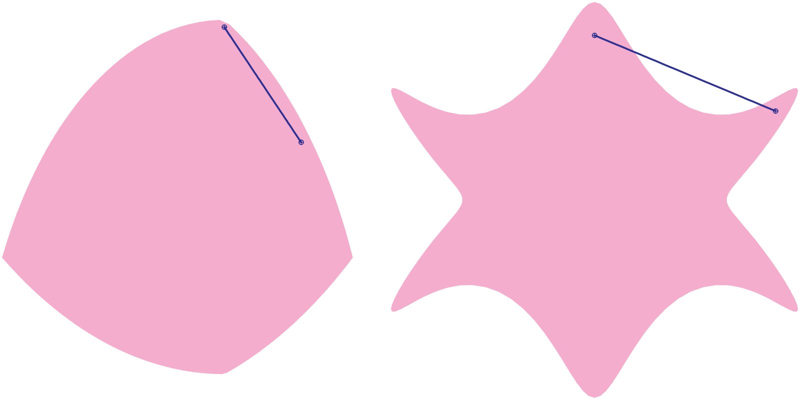


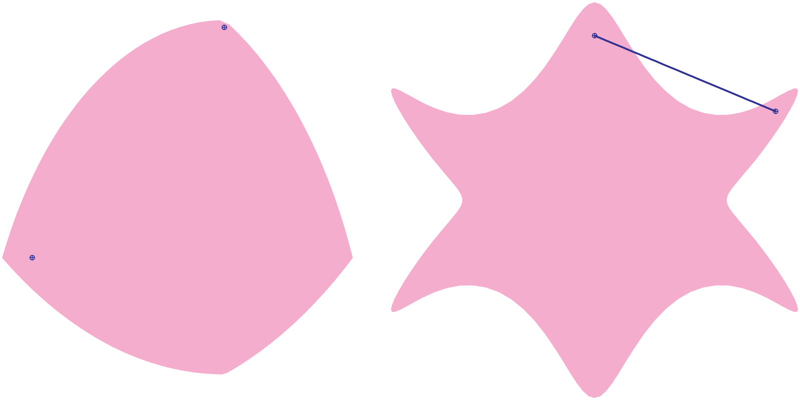


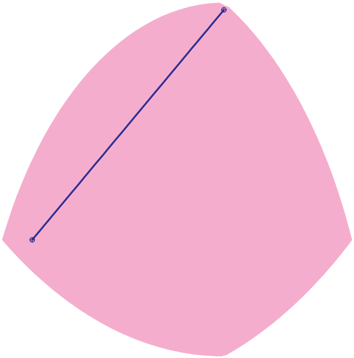


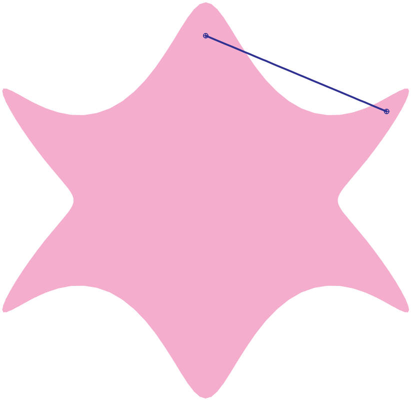






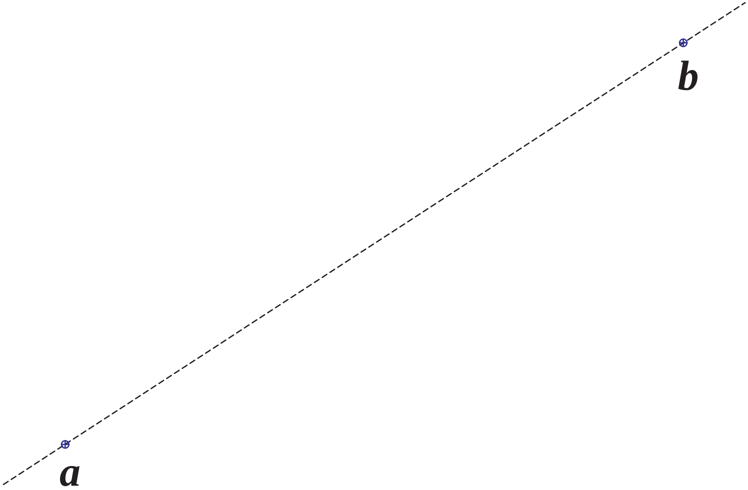




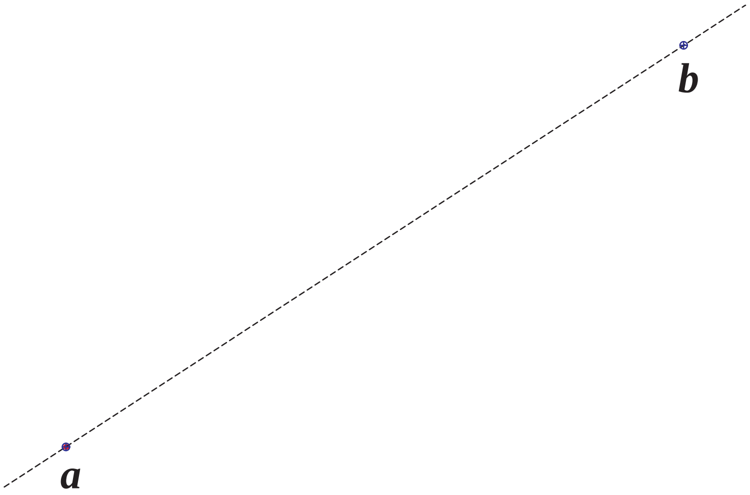


$\oplus$
b

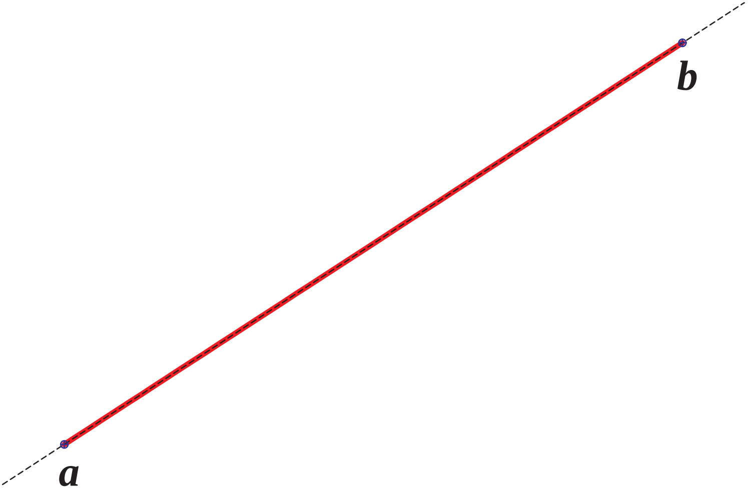
$\oplus$
a



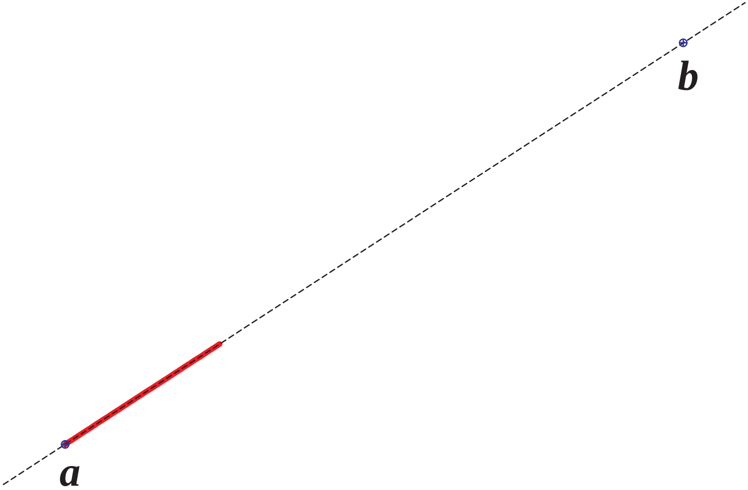
b
a



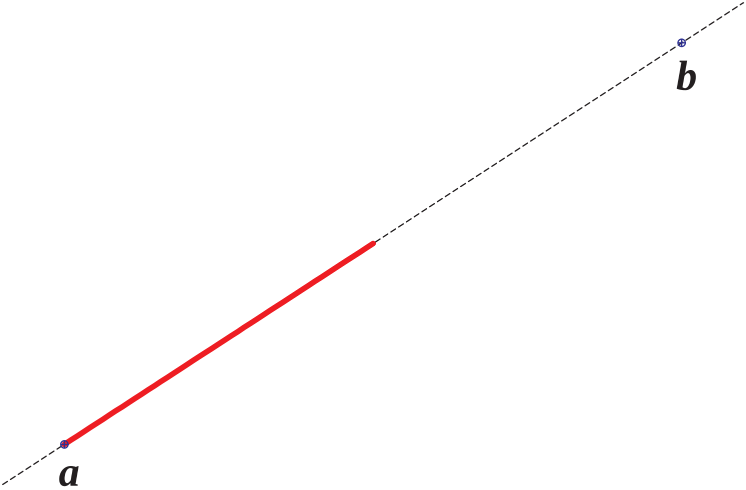
$$\boldsymbol{a} = 1 \cdot \boldsymbol{a} + 0 \cdot \boldsymbol{b} = \boldsymbol{a} + 0 \cdot (\boldsymbol{b} - \boldsymbol{a})$$



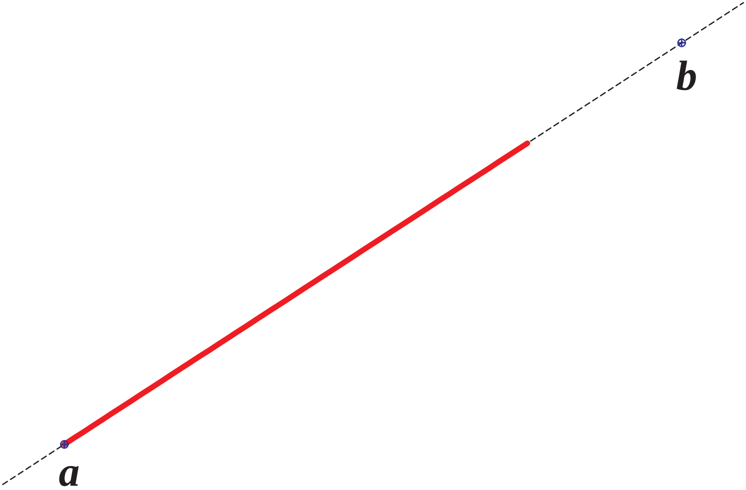
$$\boldsymbol{b} = 0 \cdot \boldsymbol{a} + 1 \cdot \boldsymbol{b} = \boldsymbol{a} + 1 \cdot (\boldsymbol{b} - \boldsymbol{a})$$



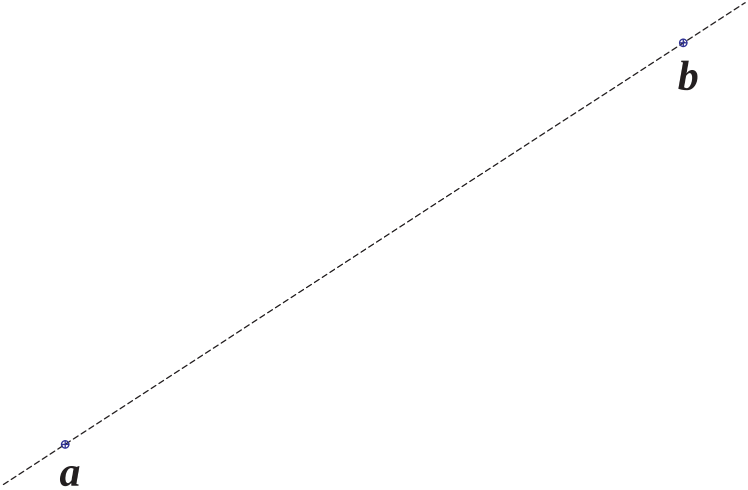
$$\frac{3}{4}\cdot \boldsymbol{a}+\frac{1}{4}\cdot \boldsymbol{b}=\boldsymbol{a}+\frac{1}{4}\cdot (\boldsymbol{b}-\boldsymbol{a})$$



$$\frac{1}{2} \cdot \boldsymbol{a} + \frac{1}{2} \cdot \boldsymbol{b} = \boldsymbol{a} + \frac{1}{2} \cdot (\boldsymbol{b} - \boldsymbol{a})$$



$$\frac{1}{4} \cdot \boldsymbol{a} + \frac{3}{4} \cdot \boldsymbol{b} = \boldsymbol{a} + \frac{3}{4} \cdot (\boldsymbol{b} - \boldsymbol{a})$$



$$t \cdot \boldsymbol{a} + (1-t) \cdot \boldsymbol{b} = \boldsymbol{a} + (1-t) \cdot (\boldsymbol{b} - \boldsymbol{a})$$

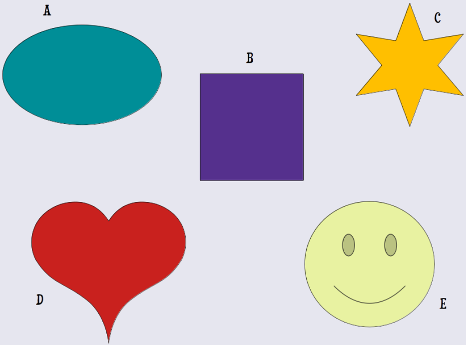
Definition

Let $M \subset \mathbb{R}^n$. We say that M is convex if

$$\forall \boldsymbol{x}, \boldsymbol{y} \in M \ \forall t \in [0,1] \colon t\boldsymbol{x} + (1-t)\boldsymbol{y} \in M.$$

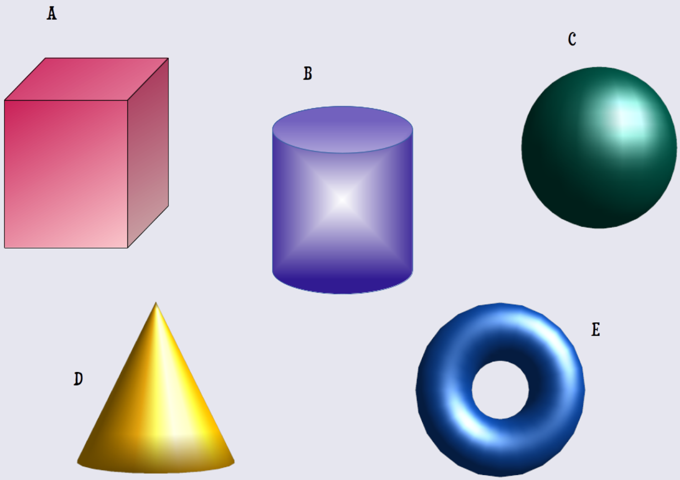
Exercise

Find convex sets



Exercise

Find convex sets



Definition

Let $M \subset \mathbb{R}^n$ be a convex set and f a function defined on M. We say that f is

- concave on M if

$$\forall \boldsymbol{a}, \boldsymbol{b} \in M \; \forall t \in [0,1] : f(t\boldsymbol{a} + (1-t)\boldsymbol{b}) \geq tf(\boldsymbol{a}) + (1-t)f(\boldsymbol{b}),$$

- strictly concave on M if

$$\forall \boldsymbol{a}, \boldsymbol{b} \in M, \boldsymbol{a} \neq \boldsymbol{b} \; \forall t \in (0,1) : \quad f(t\boldsymbol{a} + (1-t)\boldsymbol{b}) > tf(\boldsymbol{a}) + (1-t)f(\boldsymbol{b}).$$

Remark

By changing the inequalities to the opposite we obtain a definition of a *convex* and a *strictly convex* function.

Remark

A function f is convex (strictly convex) if and only if the function $-f$ is concave (strictly concave).
All the theorems in this section are formulated for concave and strictly concave functions. They have obvious analogies that hold for convex and strictly convex functions.

Remark

- If a function f is strictly concave on M, then it is concave on M.
- Let f be a concave function on M. Then f is strictly concave on M if and only if the graph of f "does not contain a segment", i.e.

$$\neg\Big(\exists \boldsymbol{a}, \boldsymbol{b} \in M, \boldsymbol{a} \neq \boldsymbol{b},\ \forall t \in [0,1]: \quad f(t\boldsymbol{a} + (1-t)\boldsymbol{b}) = tf(\boldsymbol{a}) + (1-t)f(\boldsymbol{b})\Big)$$

Theorem 22

Let f be a function concave on an open convex set $G \subset \mathbb{R}^n$. Then f is continuous on G.

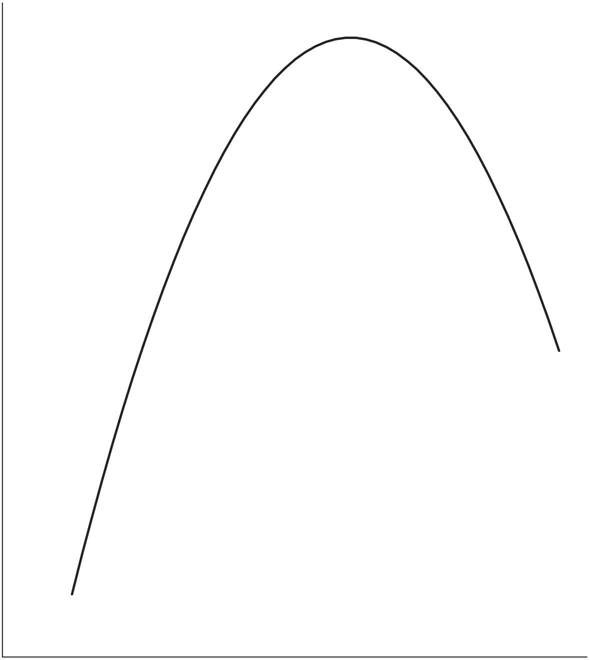
Theorem 22

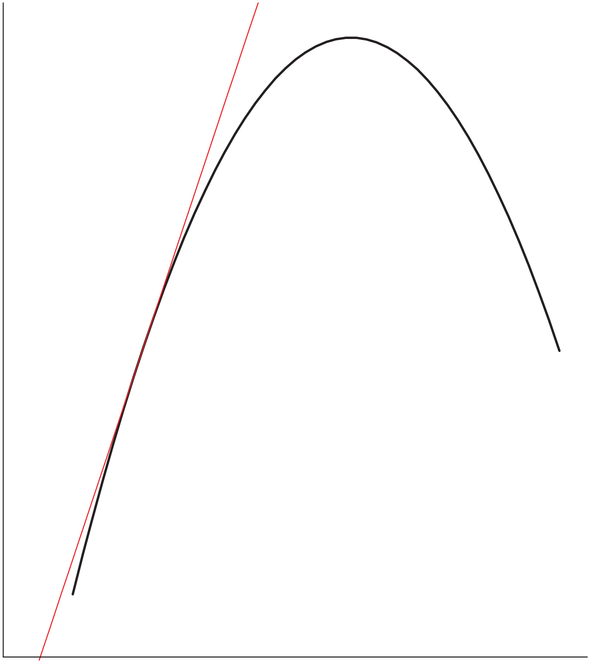
Let f be a function concave on an open convex set $G \subset \mathbb{R}^n$. Then f is continuous on G.

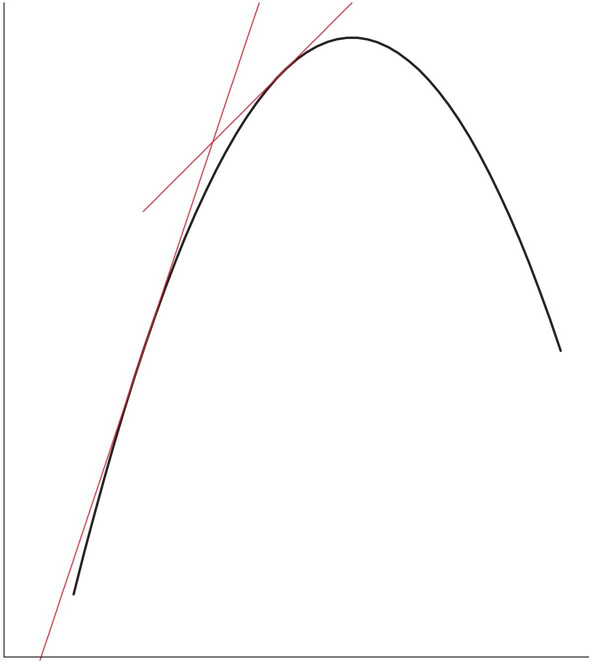
Theorem 23 (characterisation of strictly concave functions of the class $\mathcal{C}^1$)

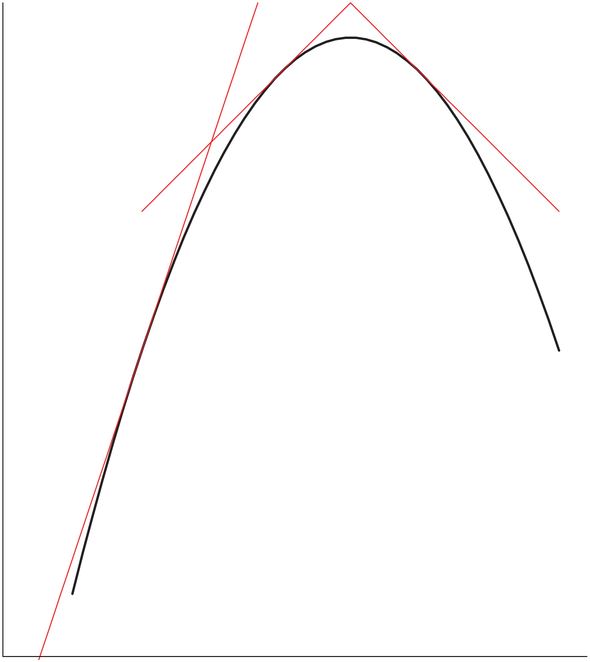
Let $G \subset \mathbb{R}^n$ be a convex open set and $f \in C^1(G)$. Then the function f is strictly concave on G if and only if

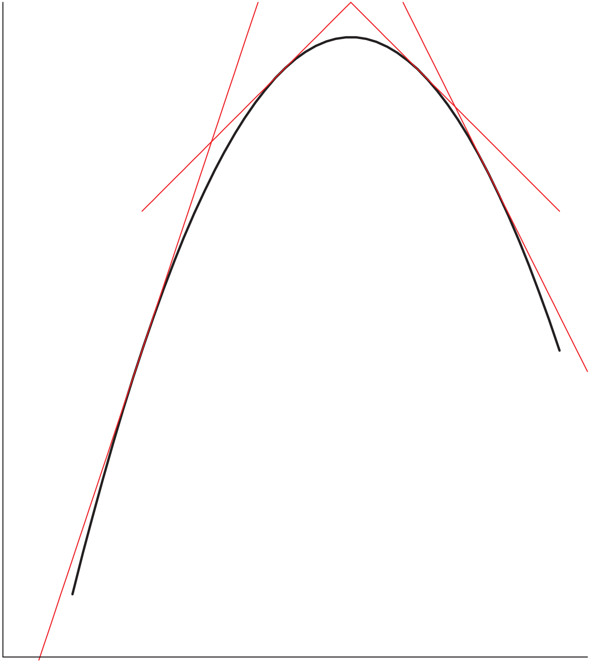
$$\forall \boldsymbol{x}, \boldsymbol{y} \in G, \boldsymbol{x} \neq \boldsymbol{y} : f(\boldsymbol{y}) < f(\boldsymbol{x}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\boldsymbol{x})(y_i - x_i).$$











Theorem 24 (characterisation of concave functions of the class $\mathcal{C}^1$)

Let $G \subset \mathbb{R}^n$ be a convex open set and $f \in C^1(G)$. Then the function f is concave on G if and only if

$$\forall \boldsymbol{x}, \boldsymbol{y} \in G \colon f(\boldsymbol{y}) \leq f(\boldsymbol{x}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\boldsymbol{x})(y_i - x_i).$$

Theorem 24 (characterisation of concave functions of the class $\mathcal{C}^1$)

Let $G \subset \mathbb{R}^n$ be a convex open set and $f \in C^1(G)$. Then the function f is concave on G if and only if

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Corollary 25

Let $G \subset \mathbb{R}^n$ be a convex open set, $f \in C^1(G)$, and let $\boldsymbol{a} \in G$ be a critical point of f (i.e. $\nabla f(\boldsymbol{a}) = \boldsymbol{o}$). If f is concave on G, then $\boldsymbol{a}$ is a maximum point of f on G.

Theorem 24 (characterisation of concave functions of the class $\mathcal{C}^1$)

Let $G \subset \mathbb{R}^n$ be a convex open set and $f \in C^1(G)$. Then the function f is concave on G if and only if

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Corollary 25

Let $G \subset \mathbb{R}^n$ be a convex open set, $f \in C^1(G)$, and let $\boldsymbol{a} \in G$ be a critical point of f (i.e. $\nabla f(\boldsymbol{a}) = \boldsymbol{o}$). If f is concave on G, then $\boldsymbol{a}$ is a maximum point of f on G. If f is strictly concave on G, then $\boldsymbol{a}$ is a strict maximum point of f on G.

Theorem 26 (level sets of concave functions)

Let f be a function concave on a convex set $M \subset \mathbb{R}^n$. Then for each $\alpha \in \mathbb{R}$ the set $Q_\alpha = \{\boldsymbol{x} \in M;\ f(\boldsymbol{x}) \geq \alpha\}$ is convex.

Definition

Let $M \subset \mathbb{R}^n$ be a convex set and let f be a function defined on M. We say that f is

- quasiconcave on M if

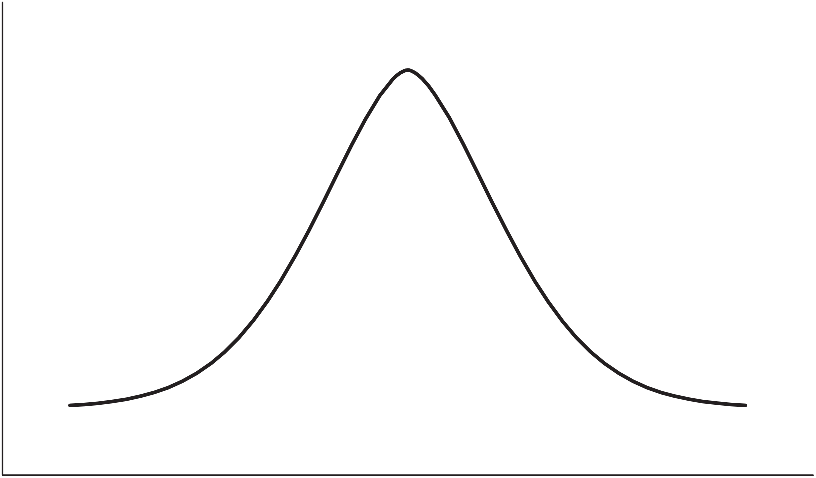
$$\forall \boldsymbol{a}, \boldsymbol{b} \in M \; \forall t \in [0,1] : f(t\boldsymbol{a} + (1-t)\boldsymbol{b}) \geq \min\{f(\boldsymbol{a}), f(\boldsymbol{b})\},$$

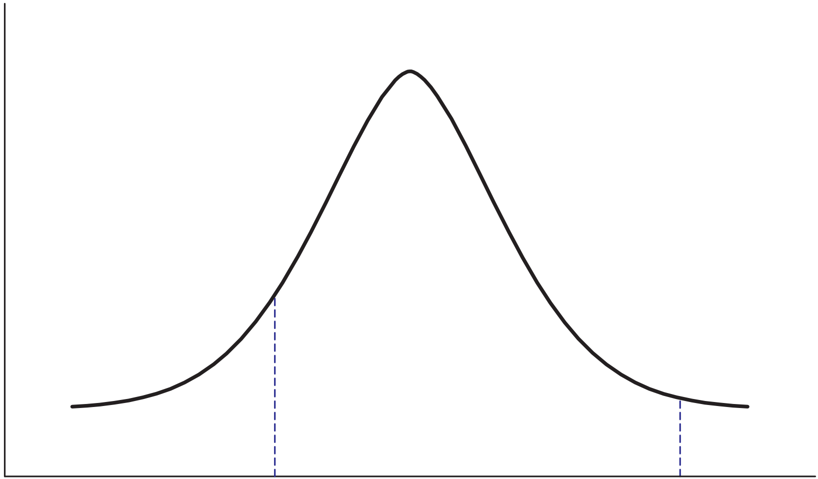
- strictly quasiconcave on M if

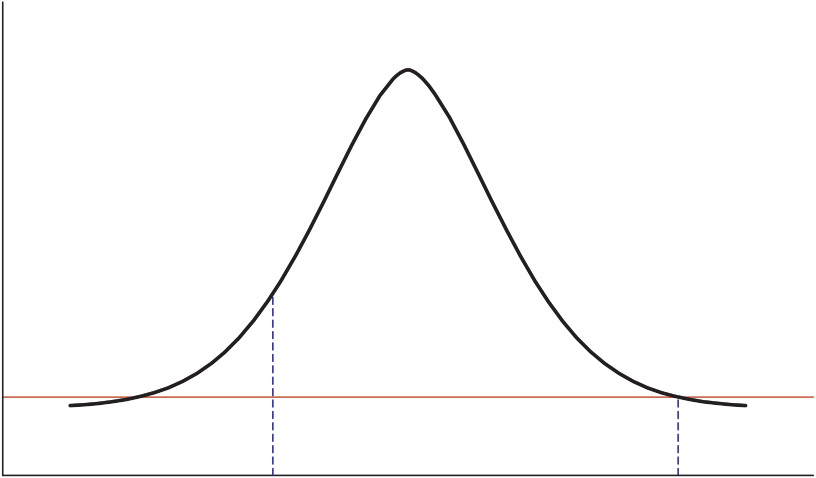
$$\forall \boldsymbol{a}, \boldsymbol{b} \in M, \boldsymbol{a} \neq \boldsymbol{b}, \; \forall t \in (0,1) : \quad f(t\boldsymbol{a} + (1-t)\boldsymbol{b}) > \min\{f(\boldsymbol{a}), f(\boldsymbol{b})\}.$$

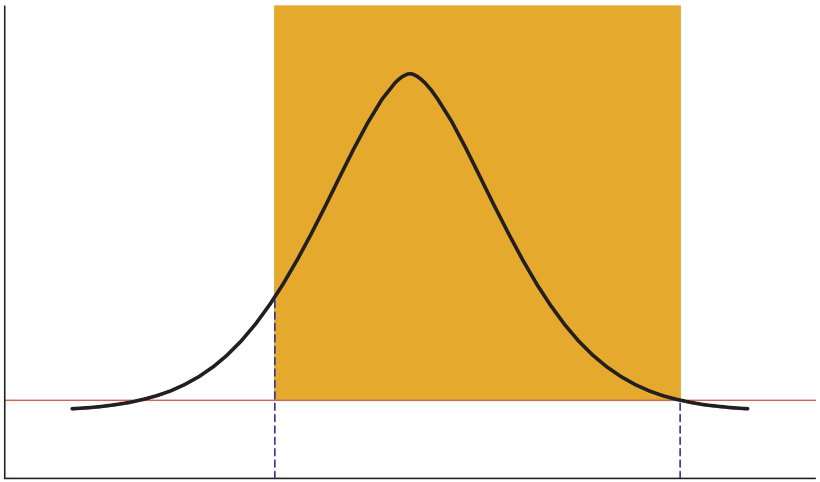
Remark

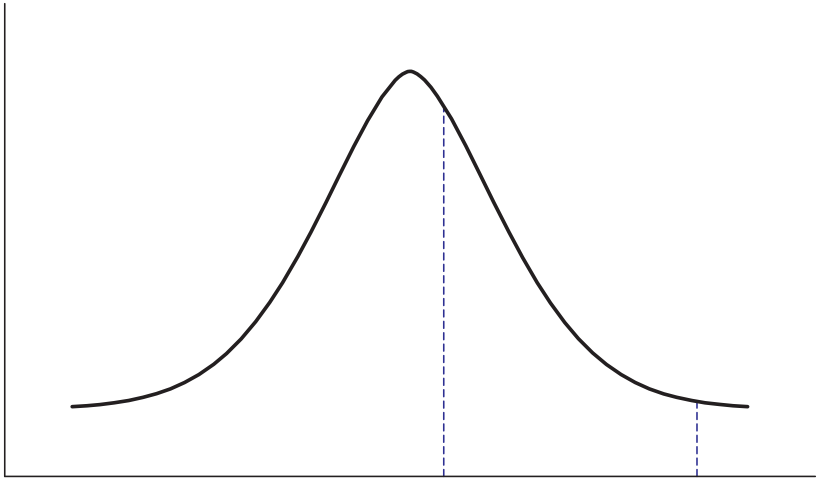
By changing the inequalities to the opposite and changing the minimum to a maximum we obtain a definition of a *quasiconvex* and a *strictly quasiconvex* function.

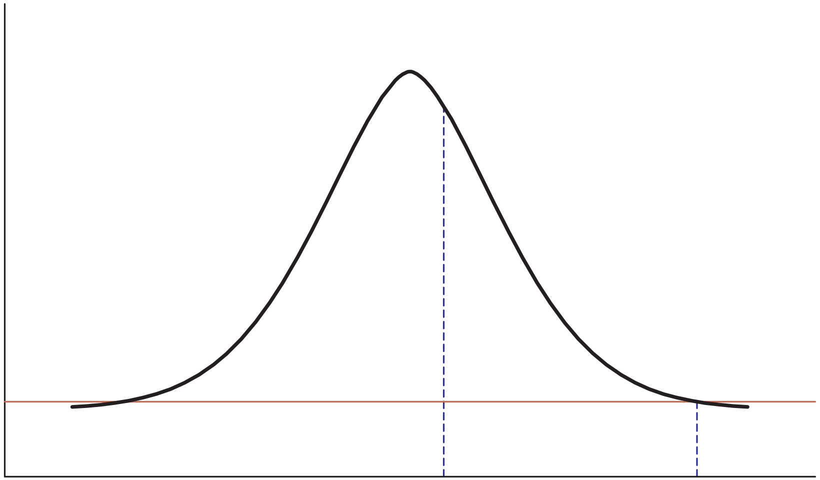


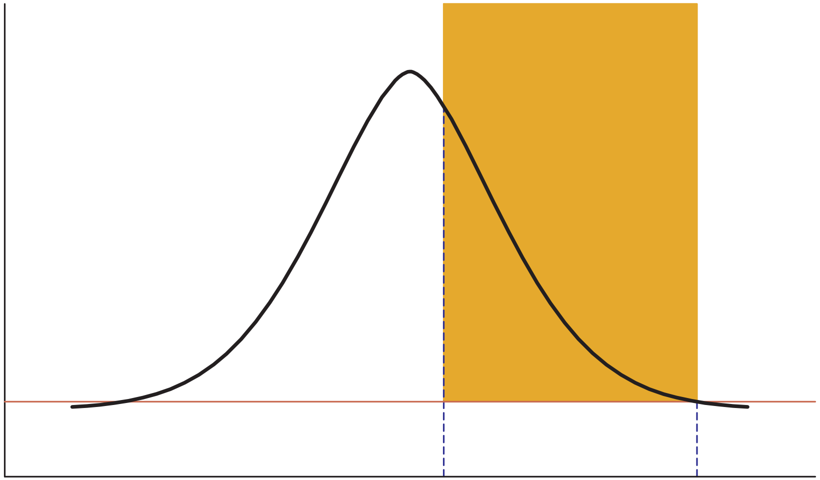


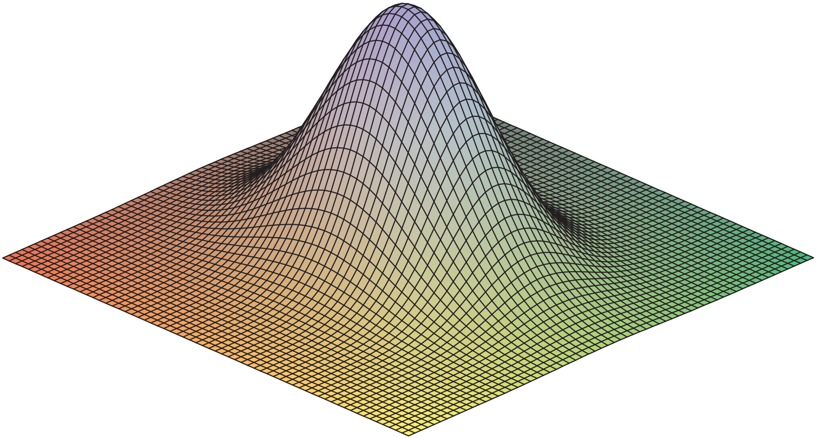




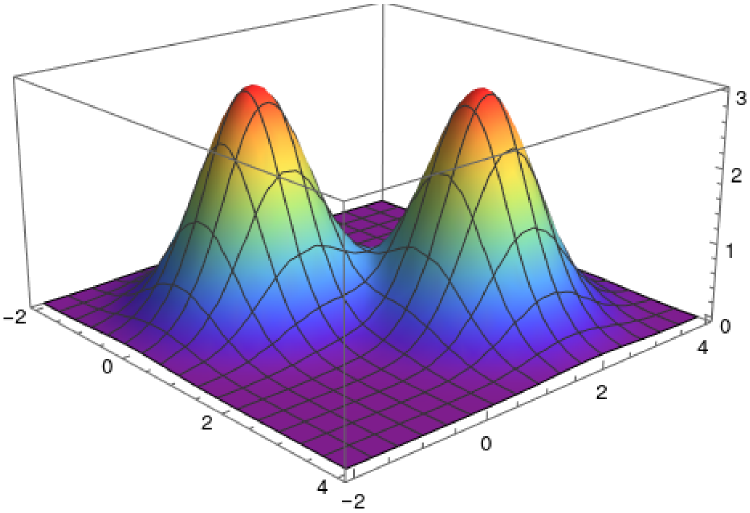








Not quasiconcave



Remark

A function f is quasiconvex (strictly quasiconvex) if and only if the function $-f$ is quasiconcave (strictly quasiconcave).
All the theorems in this section are formulated for quasiconcave and strictly quasiconcave functions. They have obvious analogies that hold for quasiconvex and strictly quasiconvex functions.

Remark

A function f is quasiconvex (strictly quasiconvex) if and only if the function $-f$ is quasiconcave (strictly quasiconcave).
All the theorems in this section are formulated for quasiconcave and strictly quasiconcave functions. They have obvious analogies that hold for quasiconvex and strictly quasiconvex functions.

Remark

- If a function f is strictly quasiconcave on M, then it is quasiconcave on M.
- Let f be a quasiconcave function on M. Then f is strictly quasiconcave on M if and only if the graph of f "does not contain a horizontal segment", i.e.

$$\neg\big(\exists \boldsymbol{a}, \boldsymbol{b} \in M, \boldsymbol{a} \neq \boldsymbol{b},\ \forall t \in [0,1] : f(t\boldsymbol{a} + (1-t)\boldsymbol{b}) = f(\boldsymbol{a})\big).$$

Remark

Let $M \subset \mathbb{R}^n$ be a convex set and f a function defined on M.

- If f is concave on M, then f is quasiconcave on M.
- If f is strictly concave on M, then f is strictly quasiconcave on M.

Remark

Let $M \subset \mathbb{R}^n$ be a convex set and f a function defined on M.

- If f is concave on M, then f is quasiconcave on M.
- If f is strictly concave on M, then f is strictly quasiconcave on M.

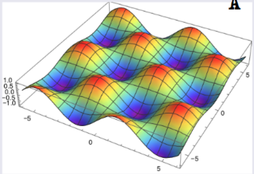
Theorem 27 (characterization of quasiconcave functions using level sets)

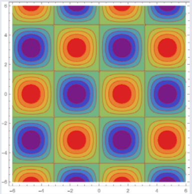
Let $M \subset \mathbb{R}^n$ be a convex set and f a function defined on M. Then f is quasiconcave on M if and only if for each $\alpha \in \mathbb{R}$ the set $Q_\alpha = \{\boldsymbol{x} \in M;\ f(\boldsymbol{x}) \geq \alpha\}$ is convex.

Exercise

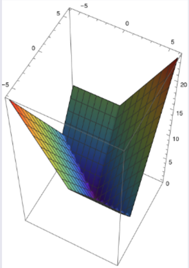
Find quasiconcave functions:

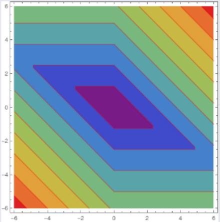
A



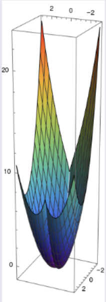


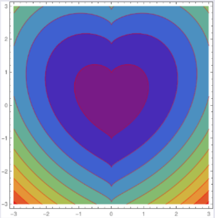
B





C





Theorem 28 (a uniqueness of an extremum)

Let f be a strictly quasiconcave function on a convex set $M \subset \mathbb{R}^n$. Then there exists at most one point of maximum of f.

Theorem 28 (a uniqueness of an extremum)

Let f be a strictly quasiconcave function on a convex set $M \subset \mathbb{R}^n$. Then there exists at most one point of maximum of f.

Corollary

Let $M \subset \mathbb{R}^n$ be a convex, closed, bounded and nonempty set and f a continuous and strictly quasiconcave function on M. Then f attains its maximum at exactly one point.

Theorem 29 (sufficient condition for concave and convex functions in $\mathbb{R}^2$)

Let $G \subset \mathbb{R}^2$ be convex and $f \in C^2(G)$.

If $\frac{\partial^2 f}{\partial x^2} \leq 0$, $\frac{\partial^2 f}{\partial y^2} \leq 0$, and $\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \geq 0$ hold on G, then f is concave on G.

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If $\frac{\partial^2 f}{\partial x^2} \geq 0$, $\frac{\partial^2 f}{\partial y^2} \geq 0$, and $\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \geq 0$ hold on G, then f is convex on G.

Exercise

Decide if the following functions are convex or concave on $\mathbb{R}^2$.

- A $f(x,y) = x^2 + y^2$
- B $f(x,y) = -x^4 - y^4$
- C $f(x,y) = -x^2 + y^2$