Graph sketching

https://www2.karlin.mff.cuni.cz/~kuncova/en/teachMat1.php

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Find the graphs of the following functions:

Exercise 1. $f(x) = |x| + \arctan(|x-1|)$

Instructions: 1. $D(f) = \mathbb{R}$.
2. The function is continuous on $\mathbb{R}$.
3. Function is not even, not odd, not periodic.
4. $\lim_{x\to\pm\infty} f(x) = +\infty$ (because of $|x|$).
5. The 1st derivative

$$f'(x) = \begin{cases} -1 + \frac{-1}{1+(1-x)^2} = -\frac{x^2-2x+3}{x^2-2x+2} = -\frac{(x-1)^2+2}{(x-1)^2+1} & x < 0 \\ 1 + \frac{-1}{1+(1-x)^2} = \frac{x^2-2x+1}{x^2-2x+2} = \frac{(x-1)^2}{(x-1)^2+1} & 0 < x < 1 \\ 1 + \frac{1}{1+(x-1)^2} = \frac{x^2-2x+3}{x^2-2x+2} = \frac{(x-1)^2+2}{(x-1)^2+1} & x > 1 \end{cases}$$

There is no derivative at the points 0, 1. Limits of derivatives (see Theorem) give:

$$f'_-(0) = -\frac{3}{2},\ f'_+(0) = \frac{1}{2},\ f'_-(1) = 0,\ f'_+(1) = 2.$$

Hence function is decreasing on $(-\infty, 0)$, increasing on $(0,1)$ a $(1,+\infty)$. Thus function is increasing on $(0,+\infty)$.

Extrema can be only at the points with no derivatives - 0 and 1. Since the function is decreasing on the left neighbourhood of 0 and increasing on the right neighbourhood of 0, we have local minimum at 0 and $f(0) = \arctan(1) = \frac{\pi}{4}$.

At the neighbourhood of the point 1 the function is increasing, therefore there is no extremum.

6. The 2nd derivative

$$f''(x) = \begin{cases} \frac{2(x-1)}{(x^2-2x+2)^2} & x < 1,\ x \neq 0 \\ -\frac{2(x-1)}{(x^2-2x+2)^2} & x > 1 \end{cases}$$

(Since the first derivative is not continuous at the points $0, 1$, then there can not be the second derivative defined.)

The function f is concave on $(-\infty, 0)$, $(0,1)$ and $(1,+\infty)$.

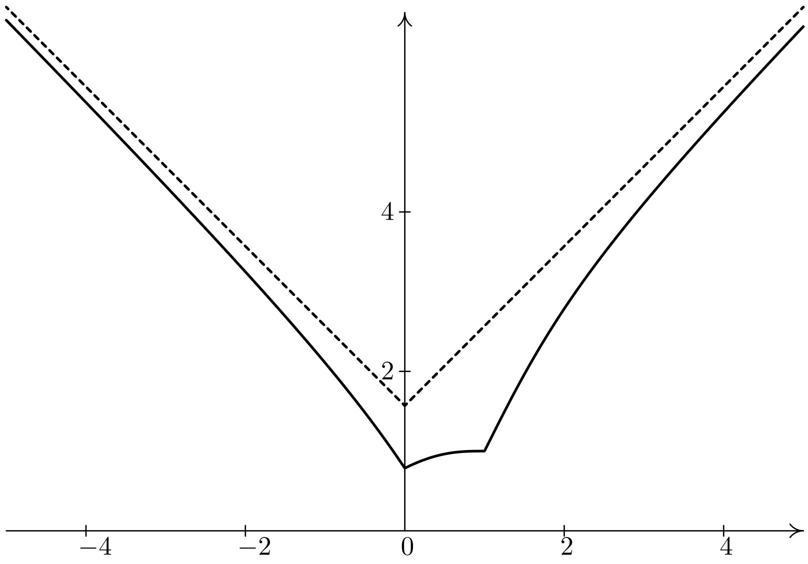
7. Asymptote at $-\infty$:

$$a_1 = \lim_{x\to-\infty} \frac{f(x)}{x} = -1, \qquad b_1 = \lim_{x\to-\infty} (f(x) - ax) = \frac{\pi}{2} \implies y = -x + \frac{\pi}{2}.$$

Asymptote at $+\infty$:

$$a_2 = \lim_{x\to+\infty} \frac{f(x)}{x} = 1, \qquad b_2 = \lim_{x\to+\infty} (f(x) - ax) = \frac{\pi}{2} \implies y = x + \frac{\pi}{2}.$$

8. Range: $H(f) = [f(0), +\infty) = [\frac{\pi}{4}, +\infty)$. Further, interesting point for graphing is $f(1) = 1$.



4
2
−4
−2
0
2
4

Exercise 2. $f(x) = \dfrac{x^3}{\sqrt{|x^4 - 1|}}$

Instructions: 1. $D(f) = \mathbb{R} \setminus \pm 1$.
2. The function is continuous on $D(f)$.
3. Function is odd, is not even, not periodic.
4. $\lim_{x\to\pm\infty} f(x) = \pm\infty$, $\lim_{x\to\pm 1} = \pm\infty$.
5. The 1st derivative

$$f'(x) = \begin{cases} \frac{x^2(x^4-3)}{(x^4-1)^{3/2}} & x \in (-\infty, -1) \cup (1, +\infty) \\ -\frac{x^2(x^4-3)}{(1-x^4)^{3/2}} & x \in (-1, +1) \end{cases}$$

There is no derivative at the points ± 1.

Hence the function is increasing on $(-\infty, -\sqrt[4]{3})$, $(-1, 1)$ and on $(\sqrt[4]{3}, +\infty)$. Function is decreasing on $(-\sqrt[4]{3}, -1)$ and $(1, +\sqrt[4]{3})$.

There is no global maximum or minimum. There is local maximum at the point $-\sqrt[4]{3}$ (with the value $-\sqrt{2}\sqrt[4]{3^3}$). There is local minimum at $\sqrt[4]{3}$ (the value is $\sqrt{2}\sqrt[4]{3^3}$).

6. The 2nd derivative

$$f''(x) = \begin{cases} \frac{6x(x^4+1)}{(x^4-1)^{5/2}} & x \in (-\infty, -1) \cup (1, +\infty) \\ \frac{6x(x^4+1)}{(1-x^4)^{5/2}} & x \in (-1, 1) \end{cases}$$

Hence f is concave on $(-\infty, -1)$, $(-1, 0)$ and convex on $(0, 1)$ and $(1, +\infty)$. There is the point of inflection at the point 0. ($f(0) = 0$.)

7. Asymptote at $-\infty$:

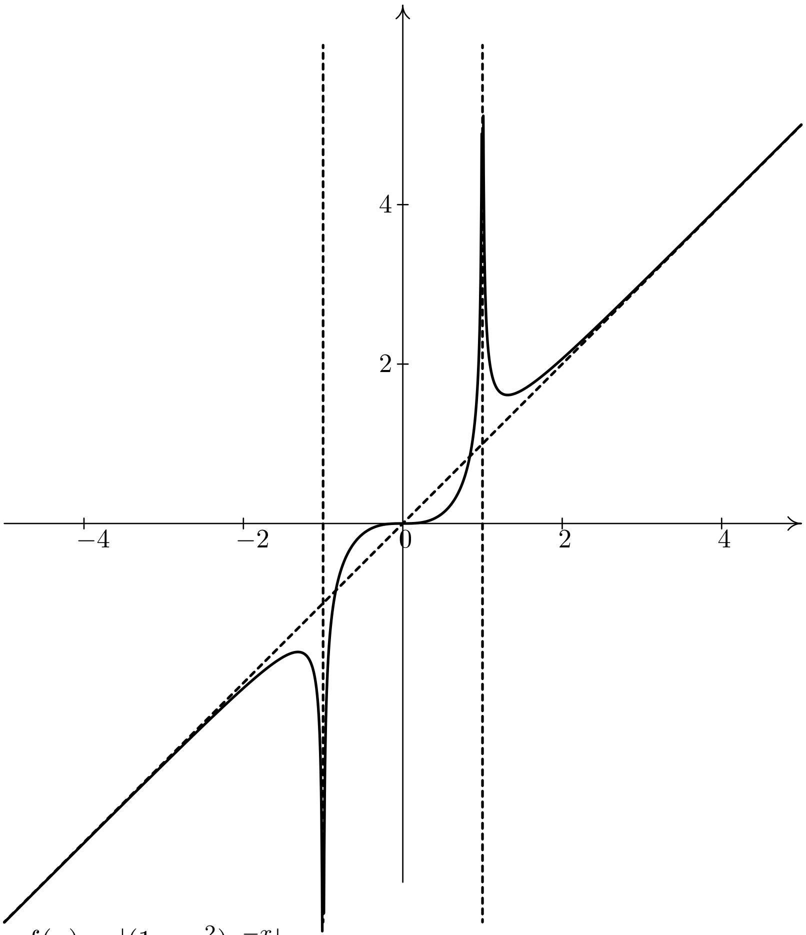
$$a_1 = \lim_{x\to-\infty} \frac{f(x)}{x} = 1, \qquad b_1 = \lim_{x\to-\infty} (f(x) - ax) = 0 \implies y = x.$$

Asymptote at $+\infty$:

$$a_2 = \lim_{x\to+\infty} \frac{f(x)}{x} = 1, \qquad b_2 = \lim_{x\to+\infty} (f(x) - ax) = 0 \implies y = x.$$

There are vertical asymptotes at ± 1.

8. Range: $H(f) = \mathbb{R}$



Exercise 3. $f(x) = |(1-x^2)e^{-x}|$

Instructions: 1. $D(f) = \mathbb{R}$.
2. The function is continuous on $\mathbb{R}$.
3. Function is not even, not odd, not periodic.
4. $\lim_{x\to+\infty} f(x) = 0$, $\lim_{x\to-\infty} f(x) = +\infty$.
5. We need to consider sign of the term $(1-x^2) = (1-x)(1+x)$. Then the 1st derivative is

$$f'(x) = \begin{cases} e^{-x}(x^2-2x-1) & x \in (-1,1) \\ -e^{-x}(x^2-2x-1) & x \in (-\infty,-1) \cup (1,+\infty) \end{cases}$$

There is no derivative at the points -1, 1 since limits of derivatives give:

$$f'_-(-1) = -2e,\ f'_+(-1) = -2e,\ f'_-(1) = -2e^{-1},\ f'_+(1) = 2e^{-1}.$$

because $e^{-x} > 0$ on $\mathbb{R}$ and

$$x^2 - 2x - 1 = 0 \Leftrightarrow x_{1,2} = 1 \pm \sqrt{2},$$

we have that the function is decreasing on $(-\infty,-1)$ and $(1-\sqrt{2},1)$ and increasing on $(-1,1-\sqrt{2})$ and $(1,1+\sqrt{2})$.

Extrema can be found at the points of zero derivatives or at the points without derivative. Hence: $-1, 1-\sqrt{2}, 1, 1+\sqrt{2}$.

Since $f(\pm 1) = 0$ and the function f is nonnegative, there is global maximum at the points ± 1. Because of the sign of derivative at its neighbourhood, we have local maxim at the points $1 \pm \sqrt{2}$.

6. The 2nd derivative

$$f''(x) = \begin{cases} e^{-x}(x^2-4x+1) & x \in (-\infty,-1) \cup (1,+\infty) \\ -e^{-x}(x^2-4x+1) & x \in (-1,1) \end{cases}$$

We have:

$$\begin{array}{ll} f''(x) = 0 & x \in \{2-\sqrt{3}, 2+\sqrt{3}\} \\ f''(x) > 0 & x \in (-\infty,-1) \cup (2-\sqrt{3},1) \cup (2+\sqrt{3},+\infty) \\ f''(x) < 0 & x \in (-1, 2-\sqrt{3}) \cup (1, 2+\sqrt{3}) \end{array}$$

Function is convex on the intervals of $f''(x) > 0$. Function is concave on the intervals with $f''(x) < 0$. There are points of inflection at $2 \pm \sqrt{3}$.

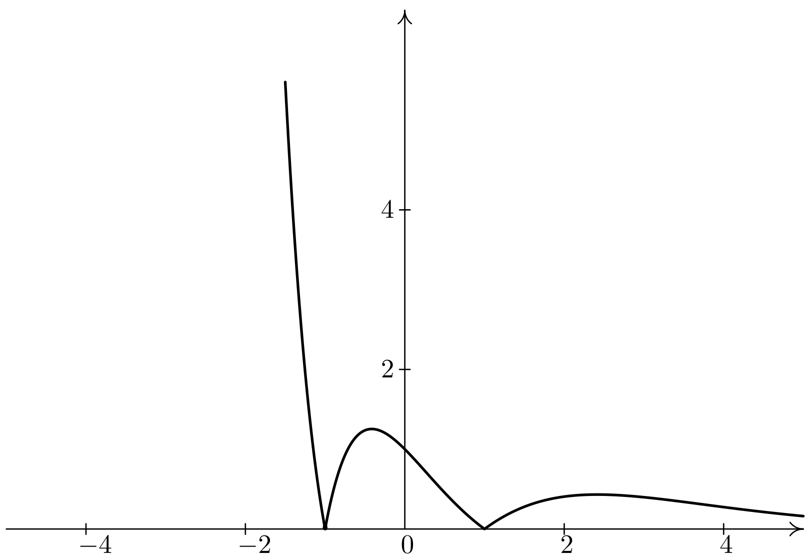
7. Asymptote at $-\infty$:

$$a_1 = \lim_{x \to -\infty} \frac{f(x)}{x} = -\infty, \qquad \text{asymptote does not exists.}$$

Asymptote at $+\infty$:

$$a_2 = \lim_{x \to +\infty} \frac{f(x)}{x} = 0, \qquad b_2 = \lim_{x \to +\infty} (f(x) - ax) = 0 \implies y = 0.$$

8. Range: $H(f) = [0, +\infty)$.



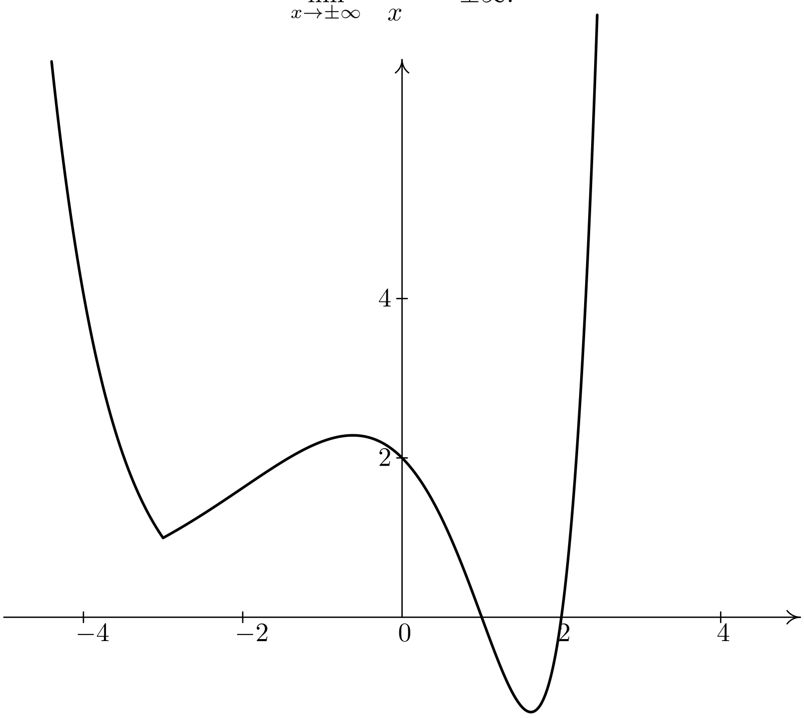
Exercise 4. $f(x) = (x^2 - 3x + 2)\exp(|x+3| - 3)$

Instructions:

$$f'(x) = \begin{cases} e^x(x^2 - x - 1) & x \in (-3, +\infty) \\ -e^{-6-x}(x^2 - 5x + 5) & x \in (-\infty, -3) \end{cases}$$

$$f''(x) = \begin{cases} e^x(x-1)(x+2) & x \in (-3, +\infty) \\ -e^{-6-x}(x-5)(x-2) & x \in (-\infty, -3) \end{cases}$$

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = \pm\infty.$$



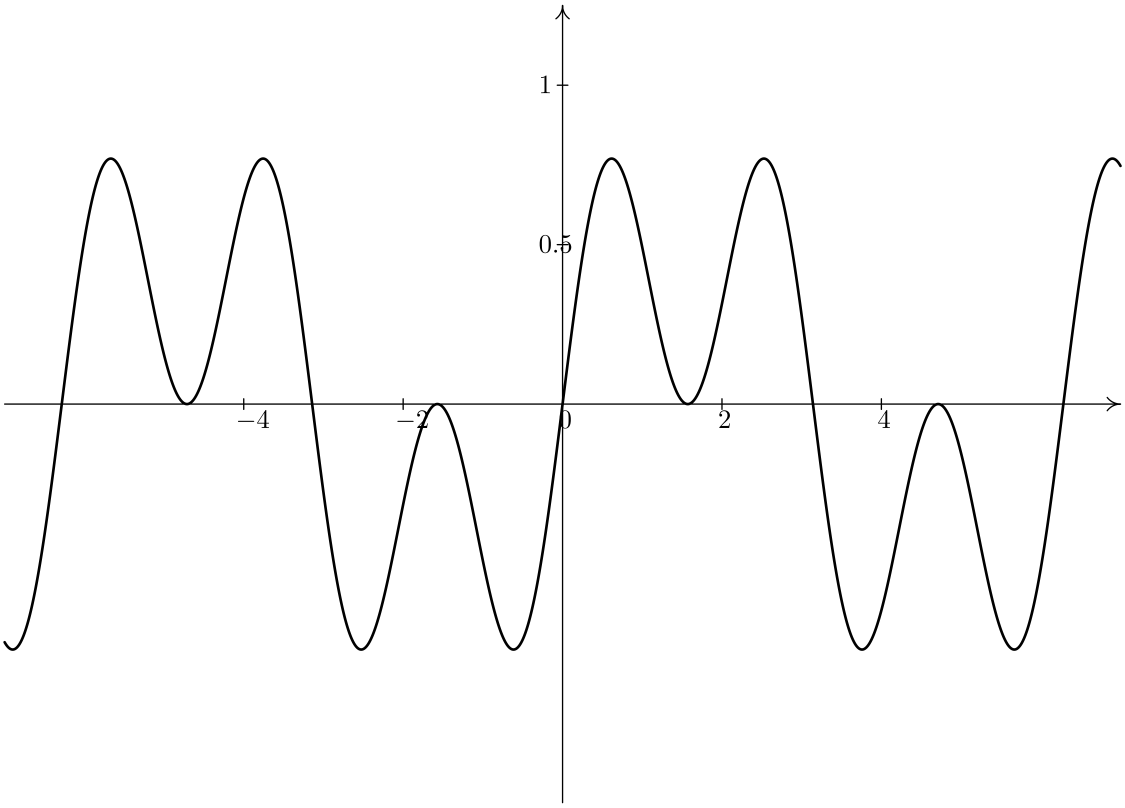
Exercise 5. $f(x) = \cos(x) \cdot \sin(2x)$

Instructions:

$$f'(x) = 2\cos x \cos 2x - \sin x \sin 2x$$

$$f''(x) = -4\cos 2x \sin x - 5\cos x \sin 2x$$

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \text{does not exist.}$$



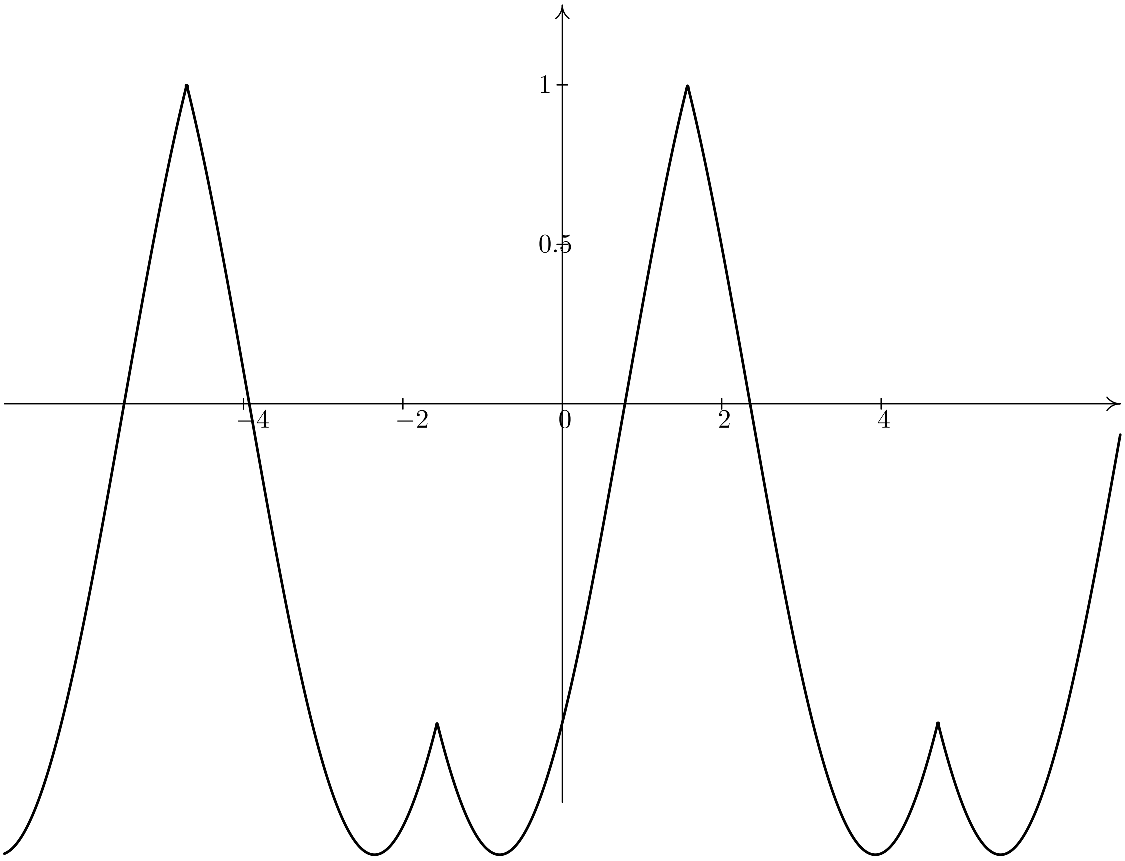
Exercise 6. $f(x) = \sin x - |\cos x|$

Instructions:

$$f'(x) = \begin{cases} \cos x + \sin x & \cos x > 0 \\ \cos x - \sin x & \cos x < 0 \end{cases}$$

$$f''(x) = \begin{cases} -\sin x + \cos x & \cos x > 0 \\ -\sin x - \cos x & \cos x < 0 \end{cases}$$

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = 0, \qquad \lim_{x\to\pm\infty} f(x) - 0\cdot x = \text{does not exist.}$$



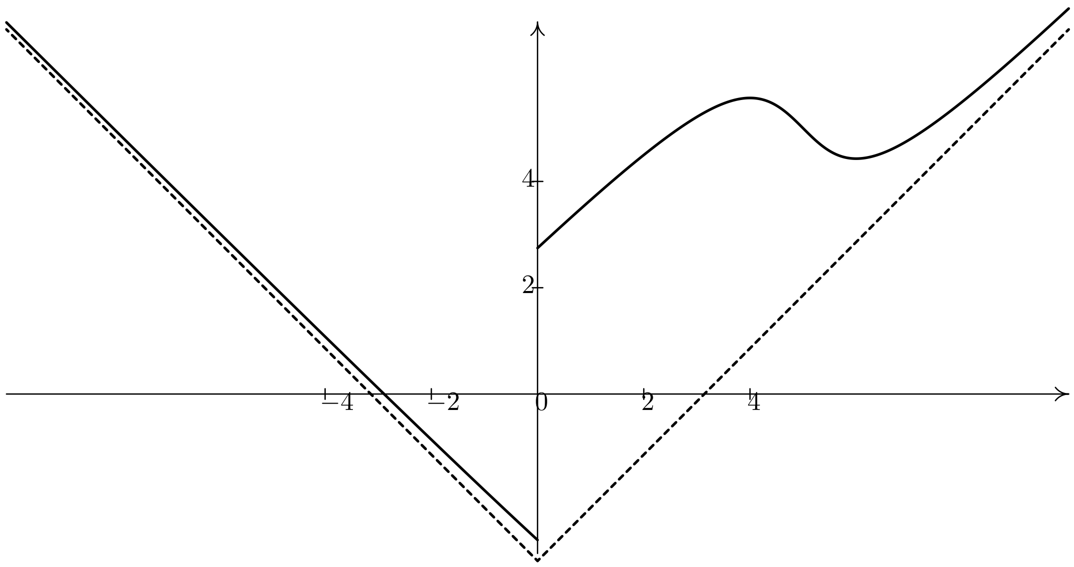
Exercise 7. $f(x) = (x - 2\arctan(x-5)) \cdot \operatorname{sgn}(x)$

Instructions:

$$f'(x) = \begin{cases} -\frac{(x-6)(x-4)}{x^2-10x+26} & x \in (-\infty, 0) \\ \frac{(x-6)(x-4)}{x^2-10x+26} & x \in (0, +\infty) \end{cases}$$

$$f''(x) = \begin{cases} -\frac{4(x-5)}{(x^2-10x+26)^2} & x \in (-\infty, 0) \\ \frac{4(x-5)}{(x^2-10x+26)^2} & x \in (0, +\infty) \end{cases}$$

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \pm 1, \qquad \lim_{x\to\pm\infty} f(x) \pm x = -\pi$$



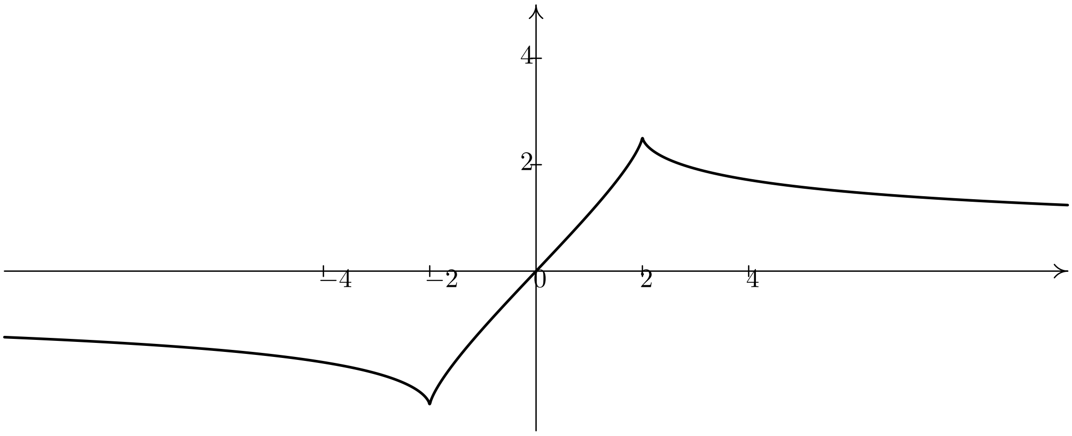
Exercise 8. $f(x) = \sqrt[3]{(x+2)^2} - \sqrt[3]{(x-2)^2}$

Instructions:

$$f'(x) = \frac{2}{3}\left(\frac{1}{\sqrt[3]{x+2}} - \frac{1}{\sqrt[3]{x-2}}\right) \qquad x \neq \pm 2$$

$$f''(x) = \frac{2}{9}\left(\frac{1}{\sqrt[3]{(x-2)^4}} - \frac{1}{\sqrt[3]{(x+2)^4}}\right) \qquad x \neq \pm 2$$

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = 0, \qquad \lim_{x\to\pm\infty} f(x) - 0\cdot x = 0.$$



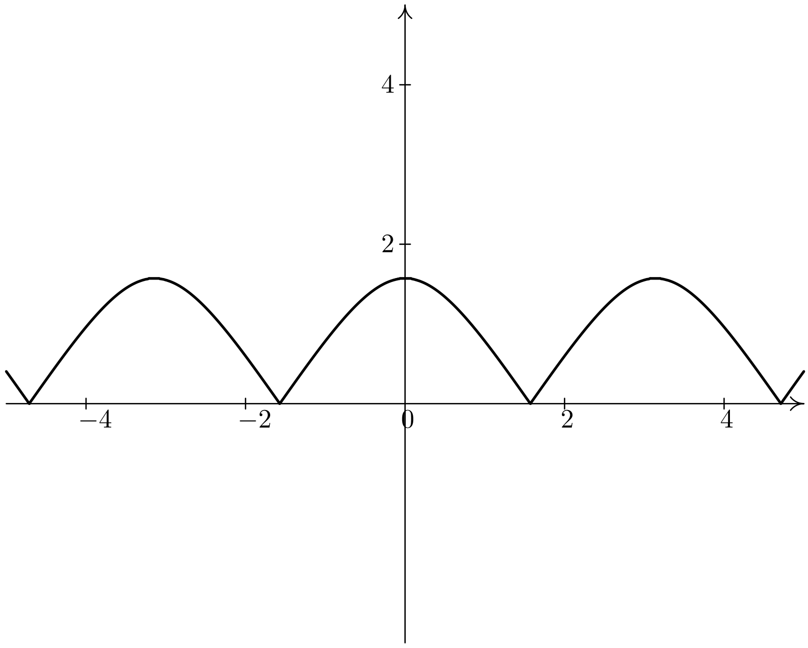
Exercise 9. $f(x) = \arcsin\left(\sqrt{1 - \sin^4 x}\right)$

Instructions:

$$f'(x) = -\frac{2\cos x \sin x}{\sqrt{1 - \sin^4 x}} \qquad x \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$$

$$f''(x) = -\frac{2\cos^4 x}{\sqrt{(1 - \sin^4 x)^3}} \qquad x \neq \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$$

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = 0, \qquad \lim_{x \to \pm\infty} f(x) - 0 \cdot x = \text{does not exist.}$$



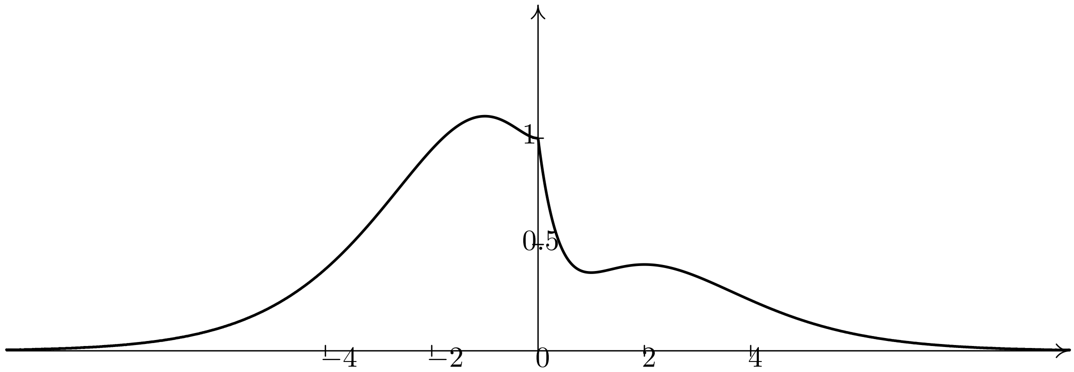
Exercise 10. $f(x) = (x^2 - x + 1)e^{-|x|}$

Instructions:

$$f'(x) = \begin{cases} e^x x(x+1) & x < 0 \\ -e^{-x}(x-2)(x-1) & x > 0 \end{cases}$$

$$f''(x) = \begin{cases} e^x(x^2+3x+1) & x < 0 \\ e^{-x}(x^2-5x+5) & x > 0 \end{cases}$$

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = 0, \qquad \lim_{x\to\pm\infty} f(x) - 0\cdot x = 0$$



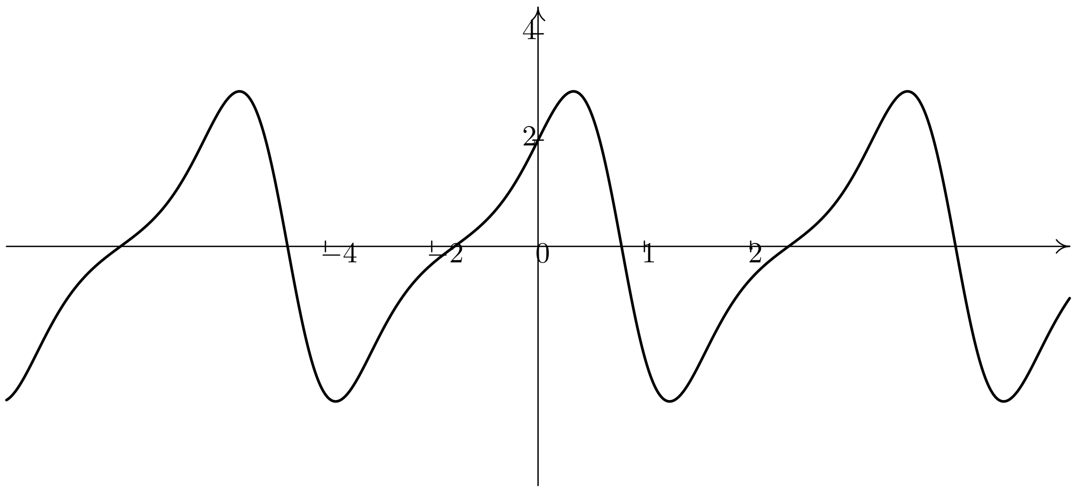
Exercise 11. $f(x) = e^{\sin x}\cos x$

Instructions:

$$f'(x) = -e^{\sin x}(\sin x - \cos^2 x)$$

$$f''(x) = e^{\sin x}\cos x(\cos^2 x - 3\sin x - 1)$$

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = 0, \qquad \lim_{x\to\pm\infty} f(x) - 0\cdot x = \text{does not exist.}$$



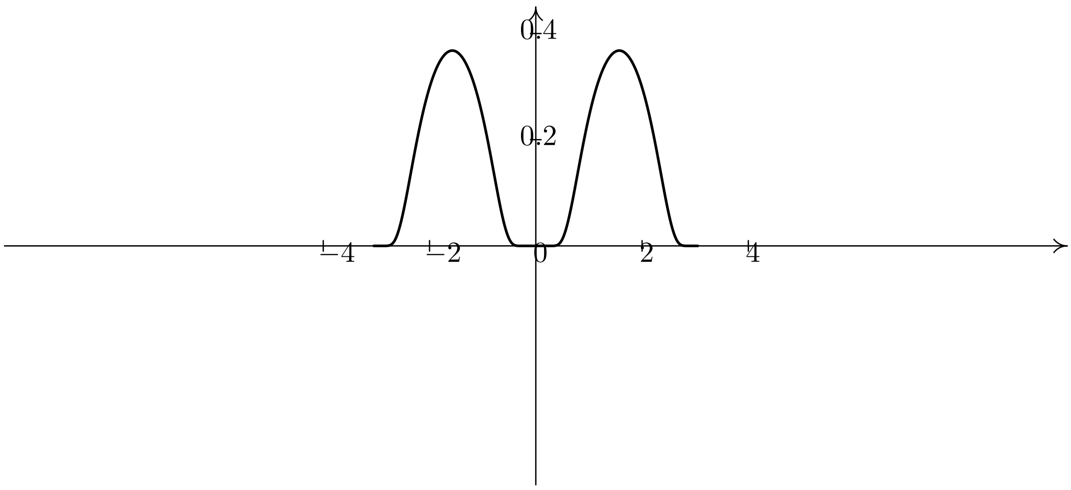
Exercise 12. $f(x) = \exp\left(-\frac{1}{\sin^2 x}\right)$ pro $x \in \mathbb{R} \setminus \{k\pi,\ k \in \mathbb{Z}\}$, $f(k\pi) = 0$ pro $k \in \mathbb{Z}$.

Instructions:

$$f'(x) = 2 \cdot e^{-\frac{1}{\sin^2 x}} \frac{\operatorname{cotg} x}{\sin^2 x} \quad x \neq k\pi, \qquad f'(k\pi) = 0.$$

$$f''(x) = \frac{1}{2} e^{-\frac{1}{\sin^2 x}} \frac{1}{\sin^6 x} (1 + 6\cos 2x + \cos 4x) \quad x \neq k\pi$$

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = 0, \qquad \lim_{x \to \pm\infty} f(x) - 0 \cdot x = \text{does not exist.}$$



Exercise 13. $f(x) = \sqrt[3]{x^2} e^{-x}$

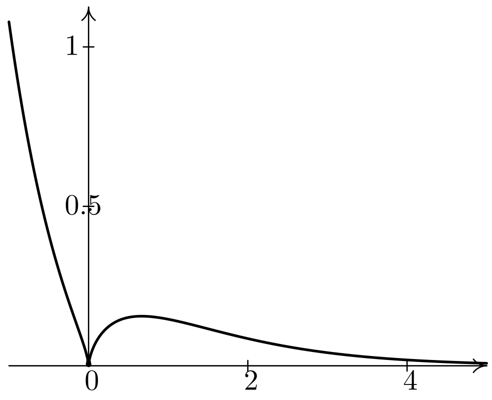
Instructions:

$$f'(x) = -\frac{e^{-x} x (3x - 2)}{3x^{4/3}}$$

$$f''(x) = \frac{e^{-x}(9x^2 - 12x - 2)}{9x^{4/3}}$$

$$\lim_{x \to -\infty} \frac{f(x)}{x} = +\infty, \qquad \text{asymptota v } -\infty \text{ není}$$

$$\lim_{x \to +\infty} \frac{f(x)}{x} = 0, \qquad \lim_{x \to +\infty} f(x) - 0 \cdot x = 0.$$



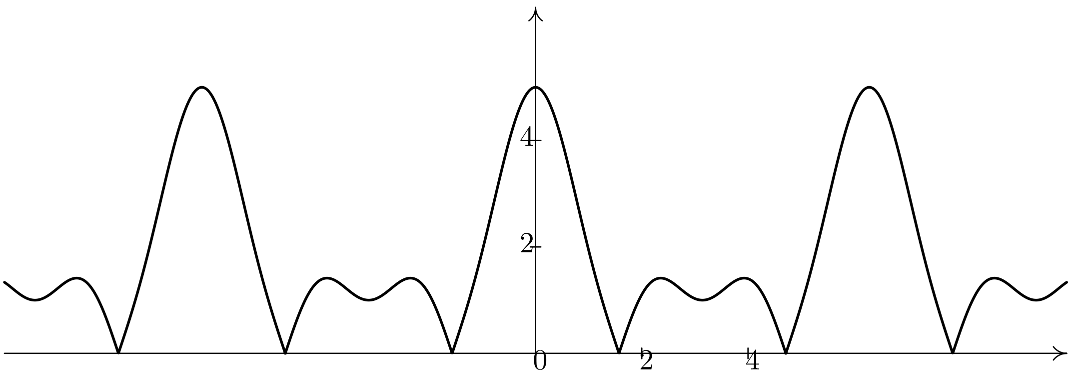
Exercise 14. $f(x) = |3\cos x| + 2\cos^3 x$

Instructions:

$$f'(x) = \begin{cases} -3\sin x - 6\cos^2 x \sin x & \cos x > 0 \\ 3\sin x - 6\cos^2 x \sin x & \cos x < 0 \end{cases}$$

$$f''(x) = \begin{cases} -3\cos x + 12\cos x \sin^2 x - 6\cos^3 x & \cos x > 0 \\ 3\cos x + 12\cos x \sin^2 x - 6\cos^3 x & \cos x < 0 \end{cases}$$

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = 0, \qquad \lim_{x\to\pm\infty} f(x) - 0\cdot x = \text{does not exist.}$$



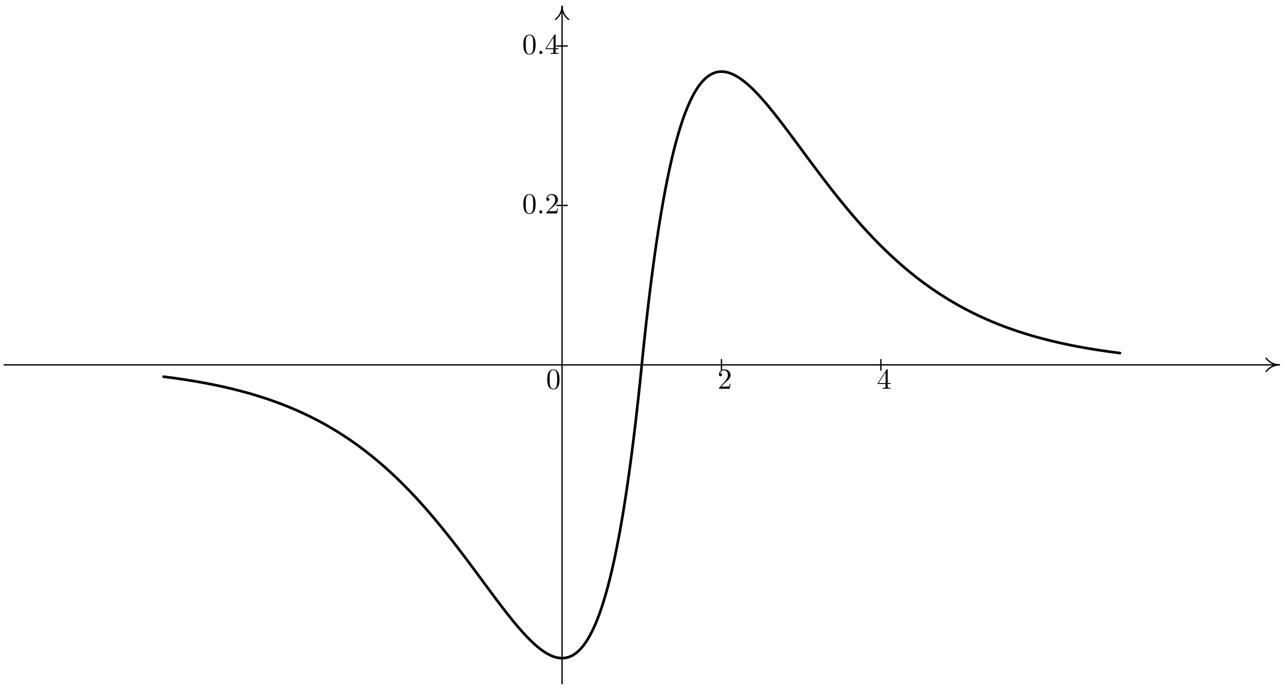
Exercise 15. $f(x) = (x-1)e^{-|x-1|}$

Instructions:

$$f'(x) = \begin{cases} -e^{1-x}(x-2) & x > 1 \\ xe^{x-1} & x < 1 \end{cases}$$

$$f''(x) = \begin{cases} e^{1-x}(x-3) & x > 1 \\ e^{x-1}(x+1) & x < 1 \end{cases}$$

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = 0, \qquad \lim_{x\to\pm\infty} f(x) - 0\cdot x = 0.$$


0.4
0.2
0
2
4

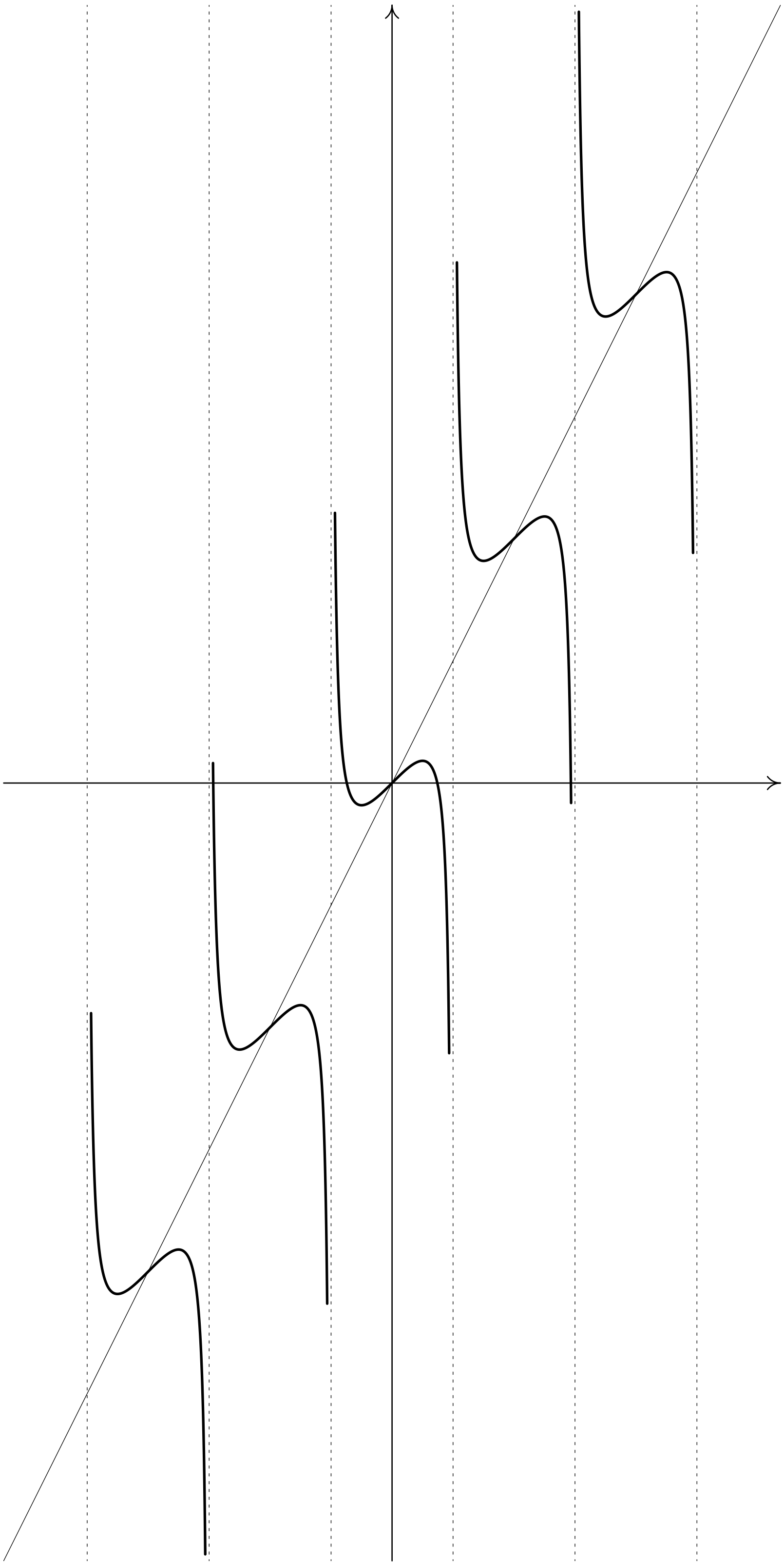
Exercise 16. $f(x) = 2x - \operatorname{tg} x$

Instructions:

$$f'(x) = 2 - \frac{1}{\cos^2 x} \qquad x \neq \frac{\pi}{2} + \pi k,\ k \in \mathbb{Z}$$

$$f''(x) = -2\frac{\sin x}{\cos^3 x} \qquad x \neq \frac{\pi}{2} + \pi k,\ k \in \mathbb{Z}$$

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \text{does not exist.}$$



Exercise 17. $f(x) = \arcsin \left|\frac{1-x}{1-2x}\right|$

Instructions: Domain:

$$\left|\frac{1-x}{1-2x}\right| \leq 1$$

$$-1 \leq \frac{1-x}{1-2x} \leq 1$$

$$\frac{2-3x}{1-2x} \geq 0 \ \wedge \ \frac{x}{1-2x} \leq 0$$

hence

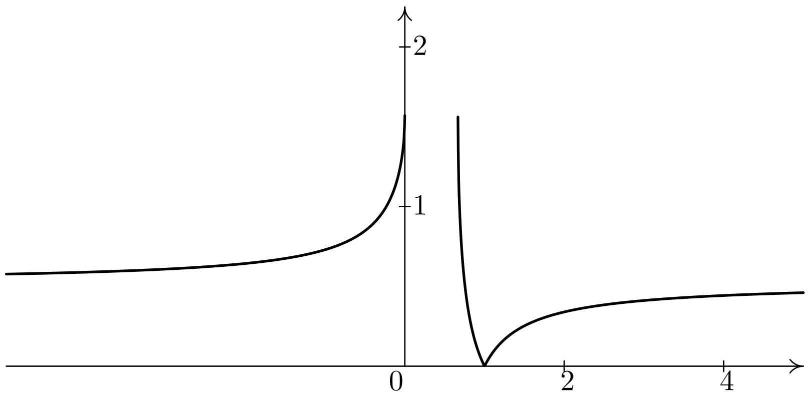
$$D(f) = (-\infty, 0] \cup [\frac{2}{3}, +\infty)$$

Derivative:

$$f'(x) = \begin{cases} \frac{1}{(2x-1)^2\sqrt{\frac{x(3x-2}{(2x-1)^2}}} & 0 < \frac{1-x}{1-2x} < 1 \\ -\frac{1}{(2x-1)^2\sqrt{\frac{x(3x-2}{(2x-1)^2}}} & -1 < \frac{1-x}{1-2x} < 0 \end{cases}$$

$$f''(x) = \begin{cases} -\frac{12x^2-9x+1}{(2x-1)^5\left(\frac{x(3x-2)}{(2x-1)^2}\right)^{3/2}} & 0 < \frac{1-x}{1-2x} < 1 \\ \frac{12x^2-9x+1}{(2x-1)^5\left(\frac{x(3x-2)}{(2x-1)^2}\right)^{3/2}} & -1 < \frac{1-x}{1-2x} < 0 \end{cases}$$

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = 0, \qquad \lim_{x\to\pm\infty} f(x) - 0\cdot x = \frac{\pi}{6}.$$



Exercise 18. $f(x) = 2\text{arctan } x + \arcsin \left(\frac{2x}{1+x^2}\right)$

Instructions: Use the substitution $x = \text{tg } y$ (it is possible on the whole $\mathbb{R}$), the fact that

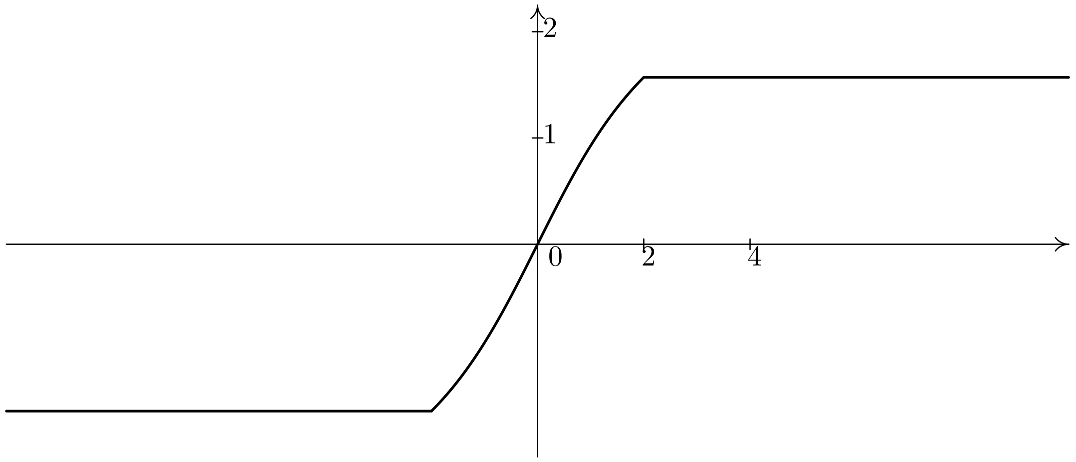
$$\frac{2\text{tg } y}{1+\text{tg }^2 y} = \sin(2y)$$

and the fact that

$$\arcsin(\sin z) = \begin{cases} z & z \in (-\frac{\pi}{2}, \frac{\pi}{2}) \\ -\pi - z & z \in (-\pi, -\frac{\pi}{2}) \\ \pi - z & z \in (\frac{\pi}{2}, \pi) \end{cases}$$

Then we obtain

$$f(x) = \begin{cases} -\pi & x \in (-\infty, -1) \\ 4\arctan x & x \in (-\infty, -1) \\ \pi & x \in (1, +\infty) \end{cases}$$



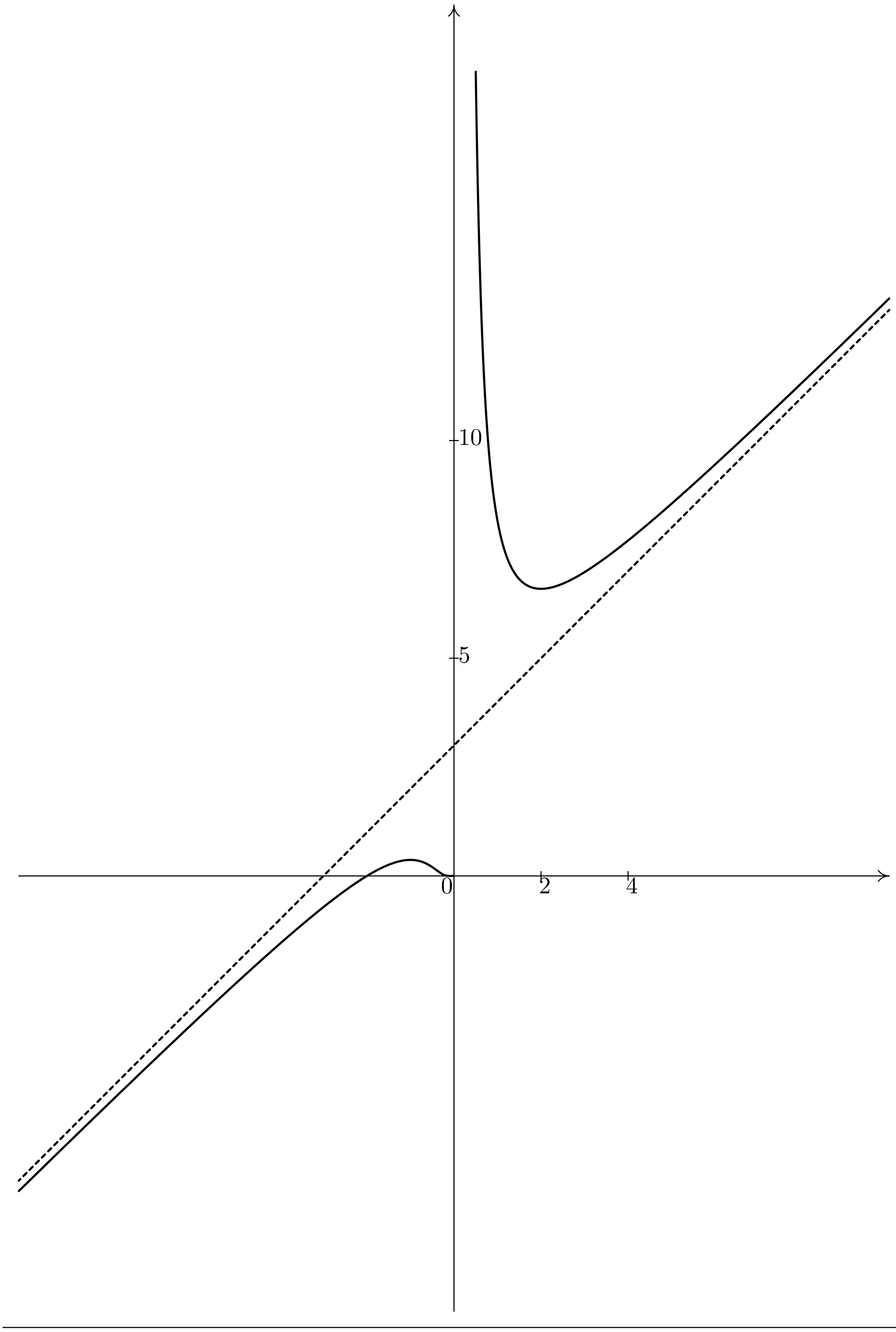
Exercise 19. $f(x) = (x+2)e^{1/x}$

Instructions:

$$f'(x) = \frac{e^{1/x}(x-2)(x+1)}{x^2},\ x \neq 0$$

$$f''(x) = \frac{e^{1/x}(5x+2)}{x^4},\ x \neq 0$$

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = 1, \qquad \lim_{x\to\pm\infty} f(x) - 1 \cdot x = 3.$$


10
5
0
2
4