1a)

Example 1: For the function $f(x) = -x^3 + 3x^2 - 4$:

a) Find the intervals where the function is increasing, decreasing.
b) Find the local maximum and minimum points and values.
c) Find the inflection points.
d) Find the intervals where the function is concave up, concave down.
e) Sketch the graph

I) Using the First Derivative:

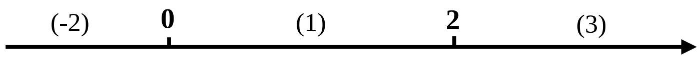
- **Step 1**: Locate the **critical points** where the derivative is = 0:

$$f'(x) = -3x^2 + 6x$$

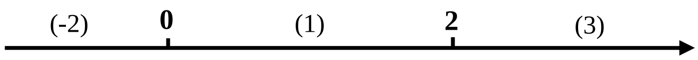
$$f'(x) = 0 \text{ then } \quad 3x(x - 2) = 0.$$

Solve for x and you will find $\mathbf{x = 0}$ and $\mathbf{x = 2}$ as the critical points

- **Step 2:** Divide $f'(x)$ into intervals using the critical points found in the previous step, then choose a **test points** in each interval such as (-2), (1), (3).

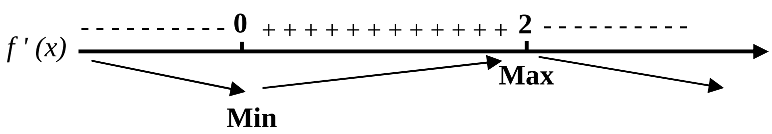


- **Step 3:** Find the derivative for the function in each test point: *(It is recommended to create a table underneath)*



$f'(x) = -3x^2 + 6x$	$f'(-2) = -24$	$f'(1) = +3$	$f'(3) = -9$
Sign	- - - - - - - -	+ + + + + + + + + +	- - - - - - - - - - - - -
Shape	**Decreasing**	**Increasing**	**Decreasing**
Intervals	$x < 0$	$0 < x < 2$	$x > 2$

- **Step 4:** Look at both sides of each critical point:



Local Minimum at $x = 0$, Minimum = $f(0) = -(0)^3 + 3(0)^2 - 4 = -4$; or **Min (0 , -4)**

Local Maximum at $x = 2$, Maximum = $f(2) = -(2)^3 + 3(2)^2 - 4 = 0$; or **Max (2 , 0)**

Increasing or $f'(x) > 0$ in: $0 < x < 2$

Decreasing or $f'(x) < 0$ in: $x < 0$ and $x > 2$

Example 1, continue

<u>II) Using the Second Derivative:</u>

- **Step 5:** Locate the **inflection points** where the second derivative is = 0; find $f''(x)$ and make it = 0

  $f'(x) = -3x^2 + 6x$

  $f''(x) = -6x + 6$

  $f''(x) = 0$ then $-6x + 6 = 0$

  Solve for x and you will find $\boldsymbol{x = 1}$ as the inflection point

- **Step 6:** Divide $f''(x)$ into intervals using the inflection points found in the previous step, then choose a **test point** in each interval such as (0) and (2).

- **Step 7:** Find the second derivative for the function in each test point: *(It is recommended to create a table underneath)*

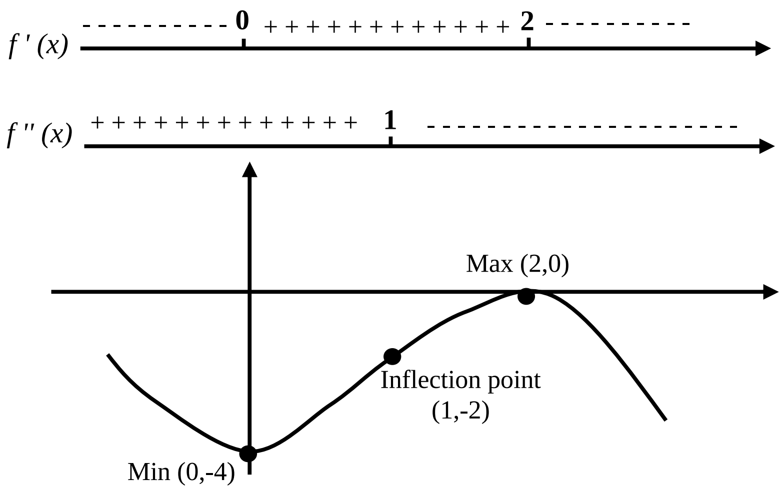
$f''(x) = -6x + 6$	$f''(0) = 6$	$f''(2) = -6$
Sign	+ + + + + + + + + + +	- - - - - - - - - - - - - - - - - -
Shape	**Concave up**	**Concave Down**
Intervals	$\boldsymbol{x < 1}$	$\boldsymbol{x > 1}$

- **Step 8:** Summarize all results in the following table:

Increasing in the intervals:	$f'(x) > 0$ in $\boldsymbol{0 < x < 2}$
Decreasing in the intervals:	$f'(x) < 0$ in $\boldsymbol{x < 0}$ and $\boldsymbol{x > 2}$
Local Max. points and Max values:	Max. at $x = 2$, **Max (2 , 0)**
Local Min. points and Min values:	Min. at $x = 0$, **Min (0 , -4)**
Inflection points at:	$x = 1$, $f(1) = -2$ or at **(1,-2)**
Concave Up in the intervals:	$f''(x) > 0$ in $\boldsymbol{x < 1}$
Concave Down in the intervals:	$f''(x) < 0$ in $\boldsymbol{x > 1}$

- **Step 9:** Sketch the graph:

1a)



15)

Example 2: Analyze the function $f(x) = 3x^5 - 20x^3$

a) Find the intervals where the function is increasing, decreasing.
b) Find the local maximum and minimum points and values.
c) Find the inflection points.
d) Find the intervals where the function is concave up, concave down.
e) Sketch the graph

I) Using the First Derivative:

- **Step 1**: The **critical points** where the derivative is = 0:

  $f'(x) = 15x^4 - 60x^2$

  $f'(x) = 0$ then $15x^2(x^2 - 4) = 0.$

  Solve for x and you will find $\mathbf{x = -2}$, $\mathbf{x = 0}$ and $\mathbf{x = 2}$ as the critical points

- **Step 2:** Intervals & test points in $f'(x)$:

  (-3) **-2** (-1) **0** (1) **2** (3)

- **Step 3:** Derivative for the function in each test point:

  (-3) **-2** (-1) **0** (1) **2** (3)

$f'(x) = 15x^4 - 60x^2$	$f'(-3) = 675$	$f'(-1) = -45$	$f'(1) = -45$	$f'(3) = 675$
Sign	+ + + + + +	- - - - - - - - -	- - - - - - - - -	+ + + + + +
Shape	**Increasing**	**Decreasing**	**Decreasing**	**Increasing**
Intervals	$\mathbf{x < -2}$	$\mathbf{-2 < x < 0}$	$\mathbf{0 < x < 2}$	$\mathbf{x > 2}$

- **Step 4:** $f'(x)$

  + + + + + + + **-2** - - - - - - - - - - **0** - - - - - - - - - - **2** + + + + + + +

  Max **Min**

Local Maximum at $x = -2$, Maximum = $f(-2) = 3(-2)^5 - 20(-2)^3 = 64$; or **Max (-2 , 64)**

Local Minimum at $x = 2$, Minimum = $f(2) = 3(2)^5 - 20(2)^3 = -64$; or **Min (2 , -64)**

Increasing or $f'(x) > 0$ in: $x < -2$ and $x > 2$

Decreasing or $f'(x) < 0$ in: $-2 < x < 0$ and $0 < x < 2$, or $-2 < x < 2$

Example 2, continue

<u>II) Using the Second Derivative:</u>

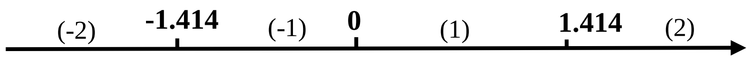
- **Step 5:** Locate the inflection points by making $f''(x) = 0$:

  $f''(x) = 60x^3 - 120x$

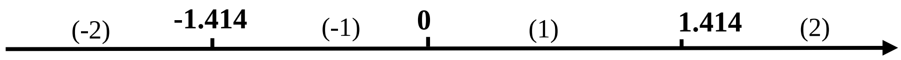
  $f''(x) = 0$ then $60x(x^2 - 2) = 0$.

  Solve for x and you will find $\mathbf{x = 0}$, $\mathbf{x} = \pm\sqrt{2} = \pm 1.414$

- **Step 6:** Intervals & test points



- **Step 7:**

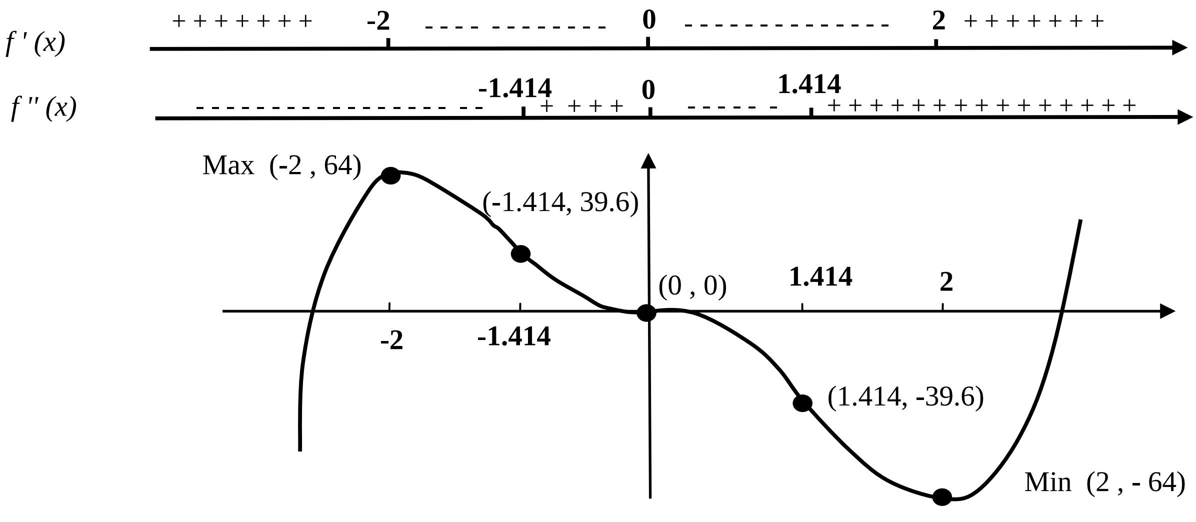


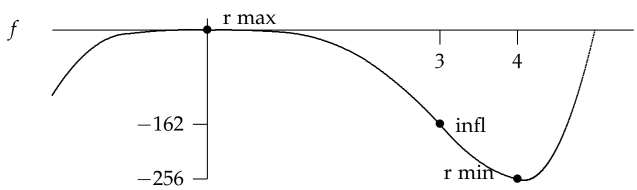
$f''(x) = 60x^3 - 120x$	$f''(-2) = -$	$f''(-1) = +$	$f''(1) = -$	$f''(2) = +$
Sign	- - - - - - - -	+ + + + + +	- - - - - - - -	+ + + + + +
Shape	**Concave Down**	**Concave Up**	**Concave Down**	**Concave Up**
Intervals	$\mathbf{x < -1.414}$	$\mathbf{-1.414 < x < 0}$	$\mathbf{0 < x < 1.414}$	$\mathbf{x > 1.414}$

- **Step 8:** Summarize all results in the following table:

Increasing in the intervals:	$\mathbf{x < -2}$ and $\mathbf{x > 2}$
Decreasing in the intervals:	$\mathbf{-2 < x < 2}$
Local Max. points and Max values:	Max. at $x = -2$, **Max (-2 , 64)**
Local Min. points and Min values:	Min. at $x = 2$, **Min (2 , -64)**
Inflection points at:	**(-1.414 , 39.6) , (0 , 0) , (-1.414 , -39.6)**
Concave Up in the intervals:	$\mathbf{-1.414 < x < 0}$ and $\mathbf{x > 1.414}$
Concave Down in the intervals:	$\mathbf{x < -1.414}$ and $\mathbf{0 < x < 1.414}$

- **Step 9:** Sketch the graph: *(Make sure the scale is consistent between f'(x) and f''(x) intervals)*





1c)

EXAMPLE 32.3. Do a complete graph of $y = f(x) = \frac{x^2}{x^2+1}$. Indicate all extrema, inflections, etc.

SOLUTION. First locate critical numbers:

$$f'(x) = \frac{2x(x^2+1) - x^2(2x)}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2} = 0 \quad \text{at } x = 0.$$

	dec	l min	inc
	$----$	0	$++++$
f'		0	

Check concavity with the second derivative.

$$f''(x) = \frac{2(x^2+1)^2 - 2x(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{2(x^2+1) - 8x^2}{(x^2+1)^3} = \frac{2-6x^2}{(x^2+1)^3} = 0$$

at $x = \pm 1/\sqrt{3} \approx \pm 0.577$.

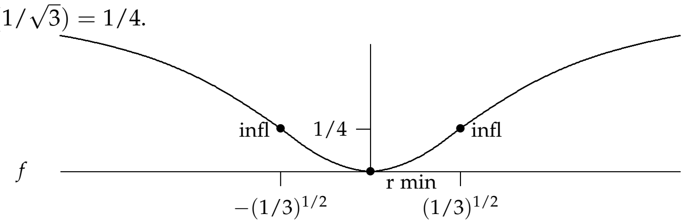
	concave down	inflect	conc up	inflect	concave down
	$----$	0	$+++$	0	$----$
f''		$-(1/3)^{1/2}$		$(1/3)^{1/2}$	

Plot the critical numbers and the inflections and then connect them appropriately based on the information from the two derivatives.

(*a*) $f(0) = 0$.

(*b*) $f(-1/\sqrt{3}) = \frac{1/3}{4/3} = 1/4$.

(*c*) $f(1/\sqrt{3}) = 1/4$.



4d

EXAMPLE 32.4. Do a complete graph of $y = f(x) = e^{-x^2/2}$. Indicate all extrema, inflections, etc.

SOLUTION. Locate critical numbers: $f'(x) = -xe^{-x^2/2} = 0$ at $x = 0$.

	inc	l max	dec
	$++++$	0	$----$
f'		0	

Check concavity with the second derivative.

$$f''(x) = -e^{-x^2/2} - xe^{-x^2/2}(-x) = (x^2-1)e^{-x^2/2} = 0 \quad \text{at} \quad x = \pm 1.$$

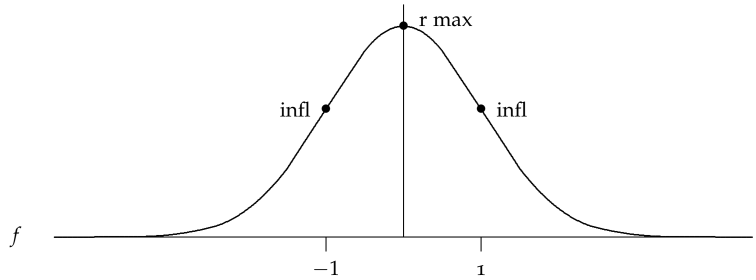
	concave up	inflect	conc dn	inflect	concave up
	$++++$	0	$----$	0	$++++$
f''		-1		1	

Plot the critical numbers and the inflections and then connect them appropriately based on the information from the two derivatives.

(*a*) $f(0) = e^0 = 1$.

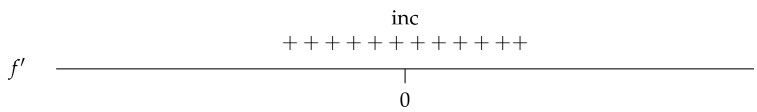
(*b*) $f(-1) = e^{-1/2} \approx 0.607$.

(*c*) $f(1) = e^{-1/2} \approx 0.607$.

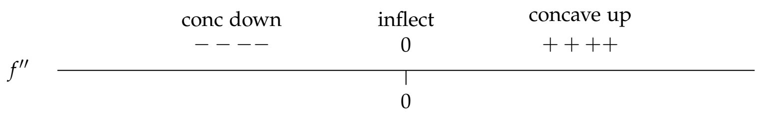


EXAMPLE 32.5. Here's a quick one with an interesting point: Do a complete graph of $y = f(x) = x^3 + x$.

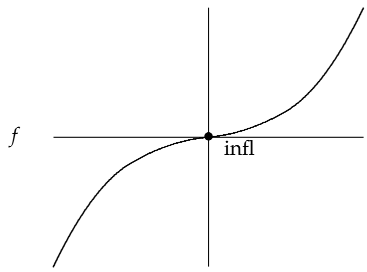
SOLUTION. Locate critical numbers: $f'(x) = 3x^2 + 1 \neq 0$; there are no critical numbers.



Check concavity with the second derivative: $f''(x) = 6x = 0 \quad \text{at} \quad x = 0$.



The only point to plot is $f(0) = 0$. The shape of the curve is determined by the first and second derivatives.



YOU TRY IT 32.2. Graph each of the following:

(*a*) $f(x) = \frac{1}{x^2+1}$. Hint: The graph will look very similar to the exponential graph we did above.

(*b*) $f(x) = x^4 - 4x^3$. Hint: It has critical numbers at $x = 0$ and 3 and inflections at $x = 0$ and 2.

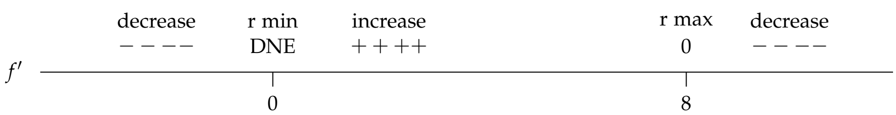
(*c*) $f(x) = x - \cos x$. It has critical numbers but no extrema. It has lots of inflections.

(*d*) Challenge: Graph $f(x) = (x^3 - 8)^{1/3}$.

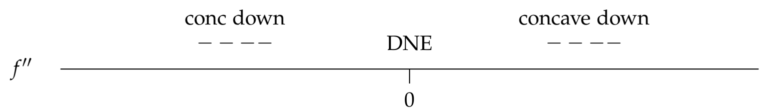
(e)

EXAMPLE 32.6. Here's another quick one with a twist: Do a complete graph of $y = f(x) = 3x^{2/3} - x$.

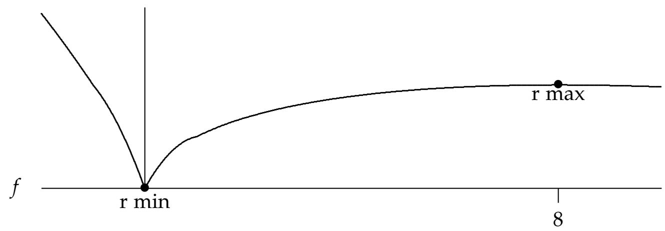
SOLUTION. Locate critical numbers: $f'(x) = 2x^{-1/3} - 1 = \frac{2}{x^{1/3}} - 1 = 0$, so $\frac{2}{x^{1/3}} = 1$ so $\frac{1}{x^{1/3}} = \frac{1}{2}$ so $x^{1/3} = 2$ or $x = 8$. Also $f'(x)$ DNE at $x = 0$.



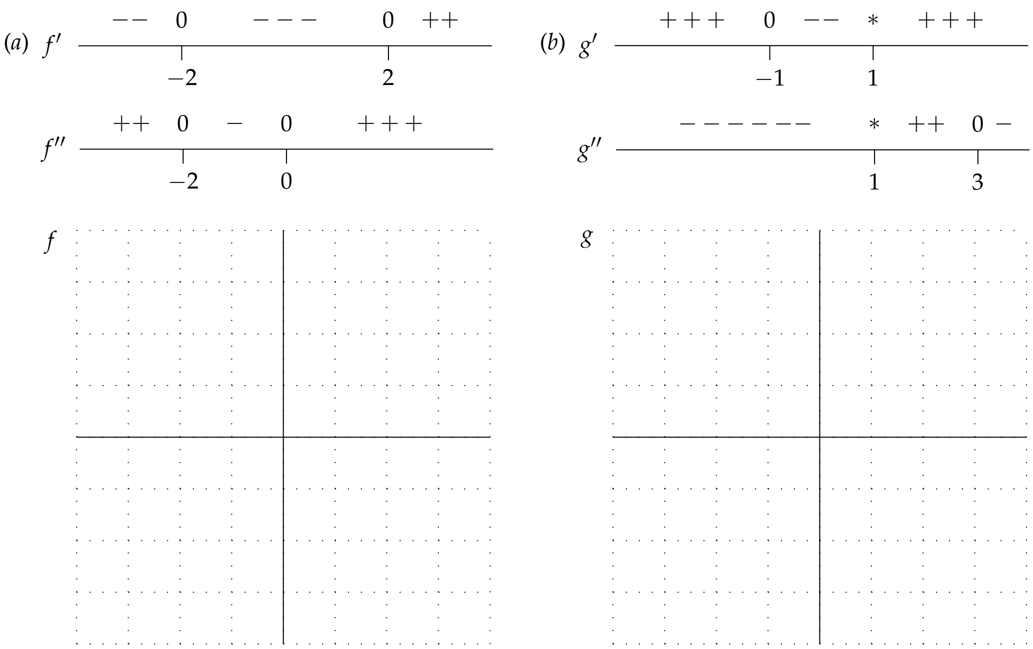
Check concavity with the second derivative: $f''(x) = \frac{2/3}{x^{4/3}} \neq 0$, but $f''(x)$ DNE at $x = 0$.



The only points to plot is $f(0) = 0$ and $f(8) = 4$. The shape of the curve is determined by the first and second derivatives.

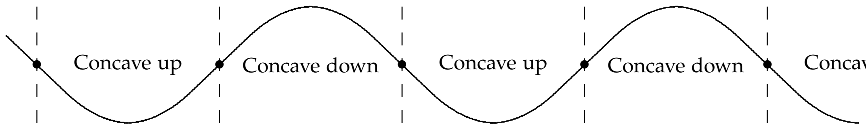


EXAMPLE 32.7. Below I give you information about the first and second derivatives of two *continuous* functions. For each, sketch a function that would have dervatives like those given. Indicate on your graph which points are local extrema and which are inflections. $*$ indicates that the derivative does not exist at the point (though the original function does). You will need to make up values for the critical and inflection points consistent with the information supplied.



THEOREM 32.1 (The Concavity Test). Let f be a function whose second derivative exists on an interval I.

1. If $f''(x) > 0$ for all points in I, then f is concave up on I.
2. If $f''(x) < 0$ for all points in I, then f is concave down on I.



Notice in the curve above that there are several places where the concavity switches. The second derivative must be 0 (or else it does not exist) at the point since the curve is not bending either way. We give these points where the concavity changes a special name.

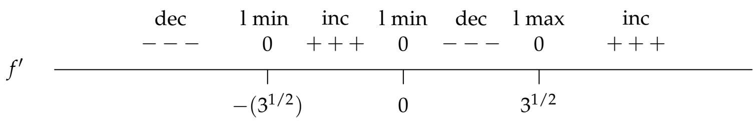
DEFINITION 32.2. A point P is called a **point of inflection** for f if f changes concavity there.

We can find such points by looking for places where the second derivative, $f''(x)$, changes sign.

11 **EXAMPLE 32.1.** Find the intervals of concavity and the inflection points for $f(x) = x^4 - 6x^2 + 1$. Sketch a graph that includes relative extrema, critical numbers, and inflections.

SOLUTION. Begin with the first derivative. Previously we saw

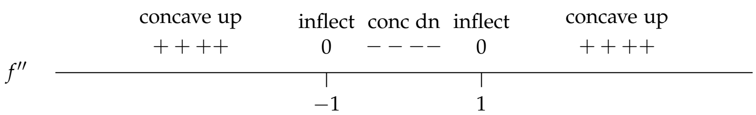
$$f'(x) = 4x^3 - 12x = 4x(x^2 - 3) = 4x(x - \sqrt{3})(x + \sqrt{3}) = 0 \quad \text{at} \quad x = \pm\sqrt{3}, 0.$$



Now we can take the second derivative.

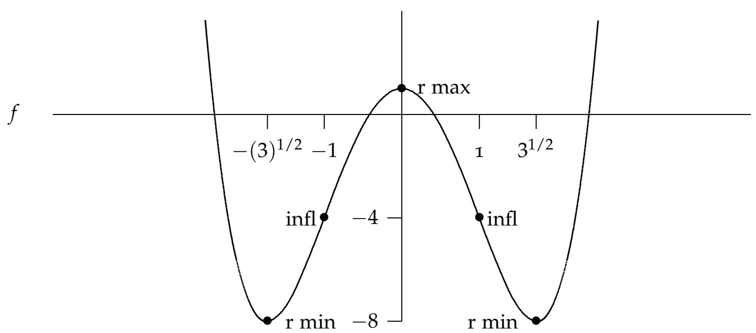
$$f''(x) = 12x^2 - 12 = 12(x^2 - 1) = 0 \quad \text{at} \quad x = \pm 1.$$

These are the potential inflection points; we still have to determine whether the sign of f'' actually changes at these points (so that the concavity changes).

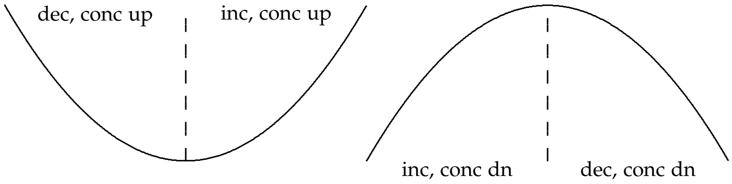


To make a graph we will plot just the critical numbers and the inflections and then connect them appropriately based on the information from the two derivatives.

- $f(-\sqrt{3}) = 9 - 18 + 1 = -8 = f(\sqrt{3}) = 9 - 18 + 1 = -8$.
- $f(0) = 1$.
- $f(1) = 1 - 6 + 1 = -4 = f(-1)$.



Note The various combinations of increasing/decreasing concave up/down are illustrated below.



The graph indicates that for smooth (differentiable twice) functions, that at a local min the function must be concave up and at a local max the function is concave down. Interpreting concavity by using the second derivative leads to

THEOREM 32.2 (Second Derivative Test). Assume that f is a function so that $f'(c) = 0$ and f'' exists on an open interval containing c.

(*a*) If $f''(c) > 0$, then f has a local min at c.

(*b*) If $f''(c) < 0$, then f has a local max at c.

(*c*) If $f''(c) = 0$, then the test fails there may be a local max, or min, or neither at c.

Possibility (3) is why the First Derivative Test is more useful when classifying critical numbers. In this regard, the Second Derivative Test acts mostly as a check to ensure that you have not made an error in classification.

YOU TRY IT 32.1. From Test 3A: Do a complete graph of $f(x) = x^4 + 4x^3 + 10$. Indicate all extrema, inflections, etc.

EXAMPLE 32.2. Do a complete graph of $y = f(x) = x^5 - 5x^4$. Indicate all extrema, inflections, etc.

SOLUTION. Critical numbers: $f'(x) = 5x^4 - 20x^3 = 5x^3(x-4) = 0$ at $x = 0, 4$.

	inc	r max	dec	r min	inc
f'	$+++$	0	$----$	0	$+++$
		0		4	

Now take the second derivative.

$$f''(x) = 20x^3 - 60x^2 = 20x^2(x-3) = 0 \quad \text{at} \quad x = 0, 3.$$

These are the potential inflection points; we still have to determine whether the sign of f'' actually changes at these points.

	conc dn	No Inf	conc dn	Inflect	concave up
f''	$----$	0	$----$	0	$++++$
		0		3	

Aside: Apply the second derivative test to the critical numbers. Notice that $f''(4) > 0$, therefore by the second derivative test there is a local min at $x = 4$.

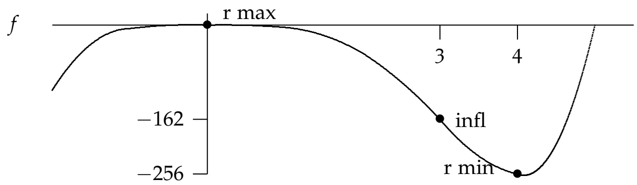
However, $f''(0) = 0$, so the second derivative test fails. Nonetheless, the first derivative test tell us that there is relative max at 0.

To make a graph we will plot just the critical numbers and the inflections and then connect them appropriately based on the information from the two derivatives.

(*a*) $f(0) = 0$.

(*b*) $f(4) = 1024 - 1280 = -256$.

(*c*) $f(3) = 243 - 405 = -162$.



EXAMPLE 32.3. Do a complete graph of $y = f(x) = \frac{x^2}{x^2+1}$. Indicate all extrema, inflections, etc.

SOLUTION. First locate critical numbers:

$$f'(x) = \frac{2x(x^2+1) - x^2(2x)}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2} = 0 \quad \text{at } x = 0.$$

	dec	l min	inc
	$----$	0	$++++$
f'		0	

Check concavity with the second derivative.

$$f''(x) = \frac{2(x^2+1)^2 - 2x(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{2(x^2+1) - 8x^2}{(x^2+1)^3} = \frac{2-6x^2}{(x^2+1)^3} = 0$$

at $x = \pm 1/\sqrt{3} \approx \pm 0.577$.

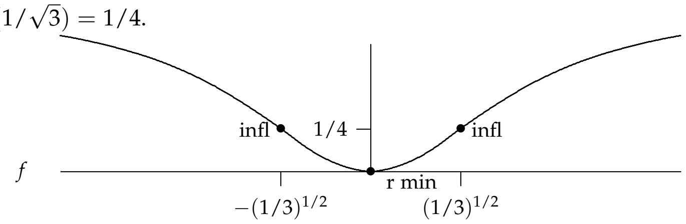
	concave down	inflect	conc up	inflect	concave down
	$----$	0	$+++$	0	$----$
f''		$-(1/3)^{1/2}$		$(1/3)^{1/2}$	

Plot the critical numbers and the inflections and then connect them appropriately based on the information from the two derivatives.

(*a*) $f(0) = 0$.

(*b*) $f(-1/\sqrt{3}) = \frac{1/3}{4/3} = 1/4$.

(*c*) $f(1/\sqrt{3}) = 1/4$.



EXAMPLE 32.4. Do a complete graph of $y = f(x) = e^{-x^2/2}$. Indicate all extrema, inflections, etc.

SOLUTION. Locate critical numbers: $f'(x) = -xe^{-x^2/2} = 0$ at $x = 0$.

	inc	l max	dec
	$++++$	0	$----$
f'		0	

Check concavity with the second derivative.

$$f''(x) = -e^{-x^2/2} - xe^{-x^2/2}(-x) = (x^2-1)e^{-x^2/2} = 0 \quad \text{at} \quad x = \pm 1.$$

	concave up	inflect	conc dn	inflect	concave up
	$++++$	0	$----$	0	$++++$
f''		-1		1	

Plot the critical numbers and the inflections and then connect them appropriately based on the information from the two derivatives.

(*a*) $f(0) = e^0 = 1$.

Example 1 Find the largest open intervals on which the graph of $f(x) = x^3 - 3x^2 + 4$ is concave up and on which it is concave down.

SOLUTION The first and second derivatives of $f(x) = x^3 - 3x^2 + 4$ are

$$f'(x) = \frac{d}{dx}(x^3 - 3x^2 + 4) = 3x^2 - 6x$$

$$f''(x) = \frac{d}{dx}(3x^2 - 6x) = 6x - 6 = 6(x-1). \tag{1}$$

The second derivative **(1)** is zero at $x = 1$, negative for $x < 1$, and positive for $x > 1$ (Figure 3). By Theorem 1 above, the graph of $y = f(x)$ is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$. This can be seen from its graph in Figure 4. □

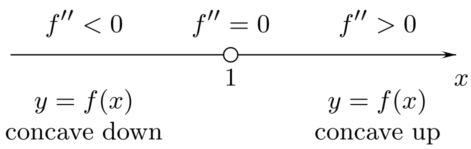


FIGURE 3

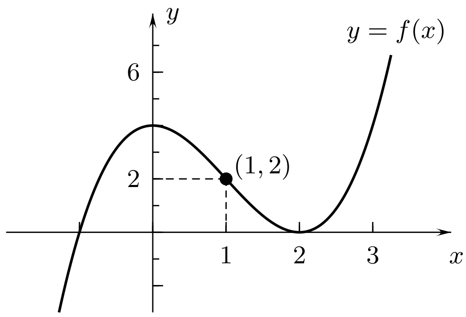


FIGURE 4

Inflection points

The point $(1, 2)$ at $t = 1$ on the graph of $f(x) = x^3 - 3x^2 + 4$ in Figure 4 where the curve switches from concave up to concave down is called an INFLECTION POINT of the graph, according to the following definition:

Definition 2 (Inflection points) *An inflection point on the graph of $y = f(x)$ is a point $(x_0, f(x_0))$ where the graph has a tangent line and is such that either the graph is concave up on an open interval (a, x_0) to the left of x_0 and concave down on an open interval (x_0, b) to the right of x_0 or the graph is concave down on an open interval to the left of x_0 and concave up on an open interval to the right of x_0.*

Example 2 Find the inflection point of $y = 3 + x^2 - 8/x$.

SOLUTION The first derivative of $y = 3 + x^2 - 8x^{-1}$ is $y' = \dfrac{d}{dx}(x^2 - 8x^{-1}) = 2x + 8x^{-2}$, and its second derivative is

$$y'' = \frac{d}{dx}(2x + 8x^{-2}) = 2 - 16x^{-3} = 2 - \frac{16}{x^3} = \frac{2(x^3 - 8)}{x^3}. \tag{2}$$

The second derivative **(1)** can change sign only at $x = 0$ where its denominator is zero or at $x = 2$ where its numerator is zero. These values set off three open intervals on which the sign of $f''(x)$ is constant. We could determine the signs by studying the signs of the numerator and denominator of **(2)** or by calculating sample values. We will use the latter approach and calculate $f''(x)$ at $x = -1$ in $(-\infty, 0)$, at $x = 1$ in $(0, 2)$, and at $x = 3$ in $(2, \infty)$:

$$\begin{aligned}
y''(-1) &= \frac{2[(-1)^3 - 8]}{(-1)^3} &&= 18 > 0 \implies y''(x) > 0 \text{ for } x < 0 \\
y''(1) &= \frac{2[1^3 - 8]}{1^3} &&= -14 < 0 \implies y''(x) < 0 \text{ for } 0 < x < 2 \\
y''(3) &= \frac{2[3^3 - 8]}{3^3} &&= \tfrac{38}{27} > 0 \implies y''(x) > 0 \text{ for } x > 2.
\end{aligned}$$

This information is shown above the x-axis in Figure 5. By Theorem 1, the graph is concave up on $(-\infty, 0)$, concave down on $(0, 2)$, and concave up on $(2, \infty)$. The one inflection point is $(2, 3)$ where $x = 2$ and the value of the function is $y(2) = 3 + 2^2 - 8/2 = 3$ (Figure 6). There is no inflection point at $x = 0$ because the function is not defined there. □

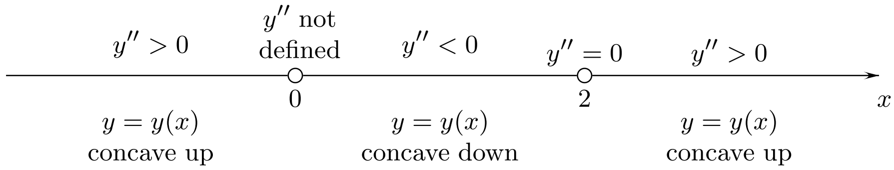
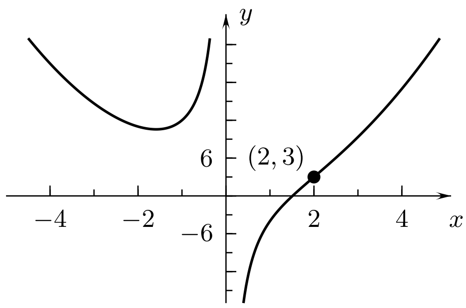
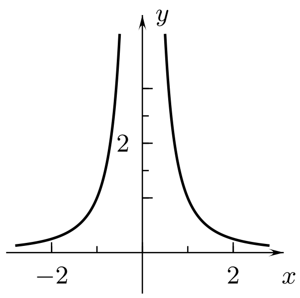


FIGURE 5



$y = 3 + x^2 - 8/x$

FIGURE 6



$y = 1/x^2$

FIGURE 7

Question 1 Figure 7 shows the graph of $y = x^{-2}$, whose first derivative is $y' = -2x^{-3}$ and whose second derivative $y'' = 6x^{-4}$ is positive for $x < 0$ and for $x > 0$. The graph is concave up in $(-\infty, 0)$ and in $(0, \infty)$ but not in $(-\infty, \infty)$. Explain.

The next example combines Theorem 1 with techniques for analyzing graphs from Section 3.2.

Example 3 Draw the graph of $g(x) = 4x^3 - x^4$ by studying the formula for the function, by determinmg the most extensive open intervals on which the function is increasing and decreasing, and by finding the most extensive open intervals on which its graph is concave up and concave down. Show any local maxima or minima and inflection points.

SOLUTION USING THE FORMULA FOR THE FUNCTION: The polynomial $g(x) = 4x^3 - x^4$ is continuous for all x. It has the same limits as $x \to \pm\infty$ as its highest order term $y = -x^4$:

$$\lim_{x\to\pm\infty} g(x) = \lim_{x\to\pm\infty} (4x^3 - x^4) = \lim_{x\to\pm\infty} (-x^4) = -\infty.$$

The graph of g looks approximately like the graph $y = 4x^3$ of its lowest-order term near $x = 0$ where x^4 is much smaller than $4x^3$.

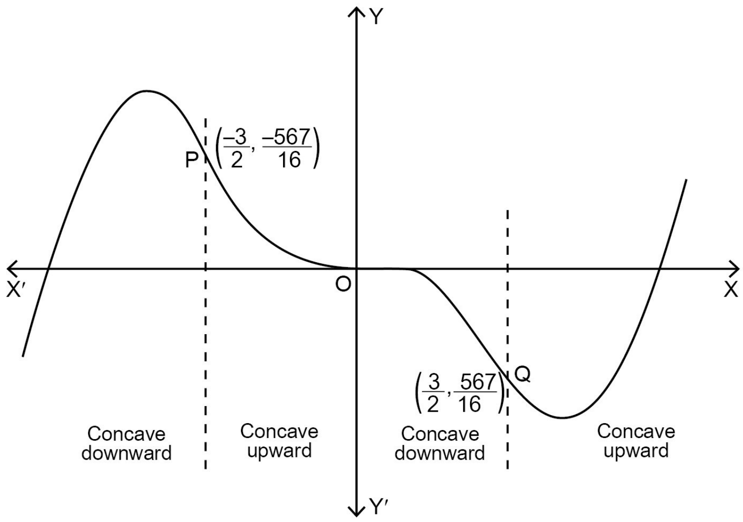


Fig. 11: Graph of $y = 2x^5 - 15x^3$.

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Example 9: Determine the concavity and point(s) of inflection for the function f given by $f(x) = (2x-5)^{1/3} + 1$.

Solution: Derivaties of f are

$$f'(x) = \frac{1}{3}(2x-5)^{-2/3} \cdot 2 = \frac{2}{3}(2x-5)^{-2/3}$$

$$f''(x) = -\frac{4}{9}(2x-5)^{-5/3} \cdot 2 = -\frac{8}{9}(2x-5)^{-5/3}$$

When $x < \frac{5}{2}, f''(x) > 0$, so, f is concave upward on $\left]-\infty, \frac{5}{2}\right[$. When $x > \frac{5}{2}, f''(x) < 0,$ so, f is concave downward on $\left]\frac{5}{2}, \infty\right[$. To find the point of inflection, we find where $f''(x) = 0$ and where $f''(x)$ does not exist. Since, $f''(x)$ is never 0, we only need to find where $f''(x)$ does not exist. Thus, the possible inflection point is $\left(\frac{5}{2}, f\left(\frac{5}{2}\right)\right)$, that is $\left(\frac{5}{2}, 1\right)$. The graph is shown in Fig. 12.

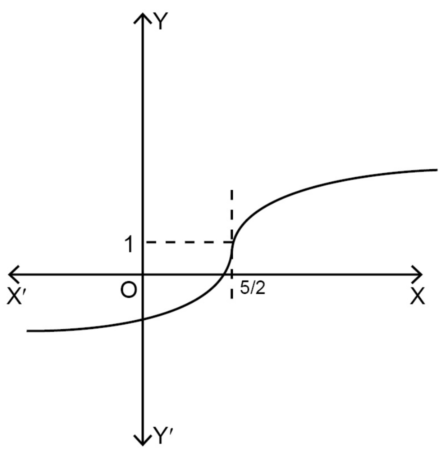

Fig. 12: Graph of $y = (2x-5)^{\frac{1}{3}} + 1$.

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1j

Example For $f(x) = \ln(1-\ln x)$,
(a) find any vertical and horizontal asymptotes
(b) find the intervals of increase or decrease
(c) find any local maximum or minimum values
(d) find the intervals of concavity and any inflection points
(e) sketch the graph of $f(x)$

The goal here is to sketch the function using calculus, without the aid of a computer. We will need the derivatives, so let's get them first:

$$\begin{aligned}
f(x) &= \ln(1-\ln x) \\
f'(x) &= \frac{d}{dx}[\ln(1-\ln x)] \\
&= \frac{1}{1-\ln x}\frac{d}{dx}[1-\ln x] \ \ \text{(chain rule)} \\
&= \frac{1}{1-\ln x}\left(-\frac{1}{x}\right) \\
&= -\frac{1}{x(1-\ln x)} \\
f''(x) &= -\frac{d}{dx}\left[\frac{1}{x(1-\ln x)}\right] \\
&= -\frac{x(1-\ln x)\frac{d}{dx}[1] - 1\frac{d}{dx}[x(1-\ln x)]}{x^2(1-\ln x)^2} \\
&= -\frac{x(1-\ln x)(0) - (1-\ln x)\frac{d}{dx}[x] - x\frac{d}{dx}[(1-\ln x)]}{x^2(1-\ln x)^2} \\
&= -\frac{-(1-\ln x) - x\left(-\frac{1}{x}\right)}{x^2(1-\ln x)^2} \\
&= -\frac{-1+\ln x+1}{x^2(1-\ln x)^2} \\
&= -\frac{\ln x}{x^2(1-\ln x)^2}
\end{aligned}$$

• Horizontal Asymptotes:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}(\ln(1-\ln x)) \longrightarrow \ln(-\infty)$$

To get the horizontal asymptotes, we need to know what happens to our function as $x \to \infty$ and $x \to -\infty$. We tried to do that above, and ran into a problem, since $\ln(-\infty)$ is not defined. This clues us in that maybe we should look at the domain of our function before proceeding.

Since $\ln x$ is only defined for $x > 0$, we know our function must have $x > 0$ due to the red part in $f(x) = \ln(1-\ln x)$. Also, because of the blue part of $f(x) = \ln(1-\ln x)$, we must have that $1-\ln x > 0$. This means

$$\begin{aligned}
1-\ln x &> 0 \\
\ln x &< 1 \\
x &< e^1 = e
\end{aligned}$$

So the domain of our function is $0 < x < e$, and there are no horizontal asymptotes since the function is not defined outside this region.

• Vertical Asymptotes:

$$\lim_{x\to a} f(x) = \pm\infty \rightsquigarrow x = a \text{ is a vertical asymptote}$$

Our function $f(x)$ is continuous, so the only place we might have a vertical asymptote is is at the endpoints. Let's check them:

$$\begin{aligned}\lim_{x\to 0} f(x) &= \lim_{x\to 0} \ln(1-\ln x) \to \ln(1-(-\infty)) \to +\infty \\ \lim_{x\to e} f(x) &= \lim_{x\to e} \ln(1-\ln x) \to \ln(1-1) \to \ln 0 \to -\infty\end{aligned}$$

We have vertical asymptotes at both endpoints, $x = 0$ and $x = e$.

• Intervals of Increasing/Decreasing:
Solve $f'(c) = -\frac{1}{c(1-\ln c)} = 0$. This condition does not occur inside our interval. Also, $f'(x)$ exists for all x. There are no critical numbers for $f'(x)$.

Write down a table showing where $f(x)$ is increasing and decreasing:

Interval	$f'(a)$ (a is in interval)	Sign of f'	f
$(0, e)$	$f'(1) = -\frac{1}{(1)(1-\ln 1)} = -1$	$-$	decreasing

• Max/Min:
Since the function is always decreasing on $(0, e)$, there are no max or mins.

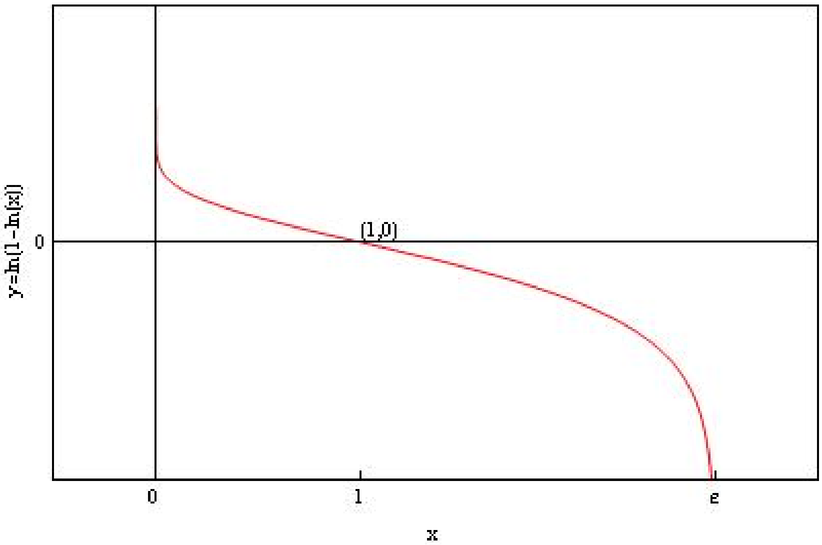
• Intervals of Concave Up/Concave Down:
Solve $f''(c) = -\frac{\ln c}{c^2(1-\ln c)^2} = 0$. The only solution is $c = +1$, since the numerator is zero there and the denominator is finite. This is the only critical number for $f''(x)$ since $f''(x)$ exists for all x.

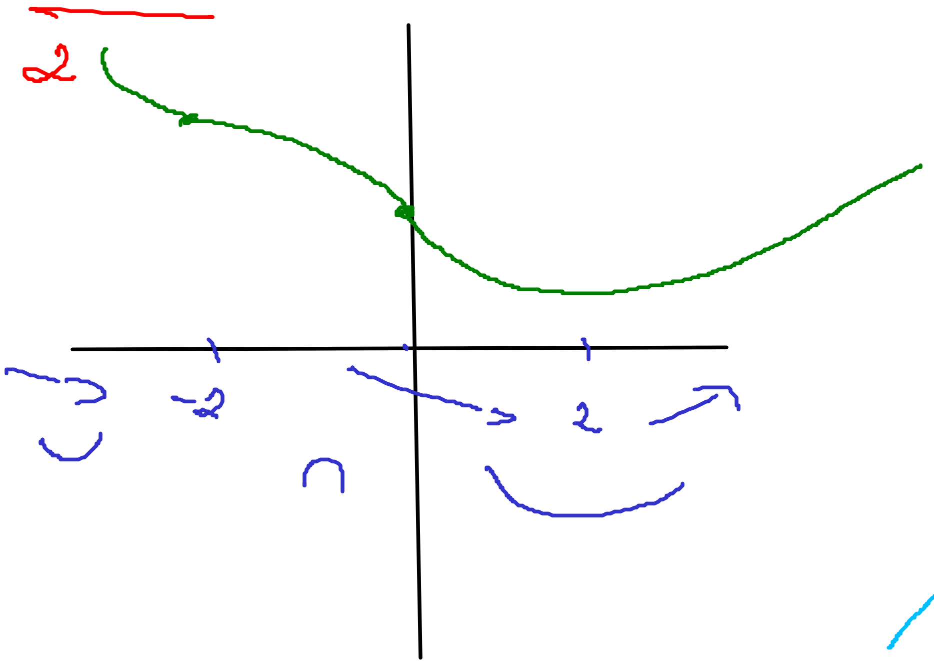
Write down a table showing where $f(x)$ is concave up and down. We will need to use the fact that $\ln x < 0$ if $x < 1$, and $\ln x > 0$ is $x > 1$ to help us get the sign of f'' is the intervals.

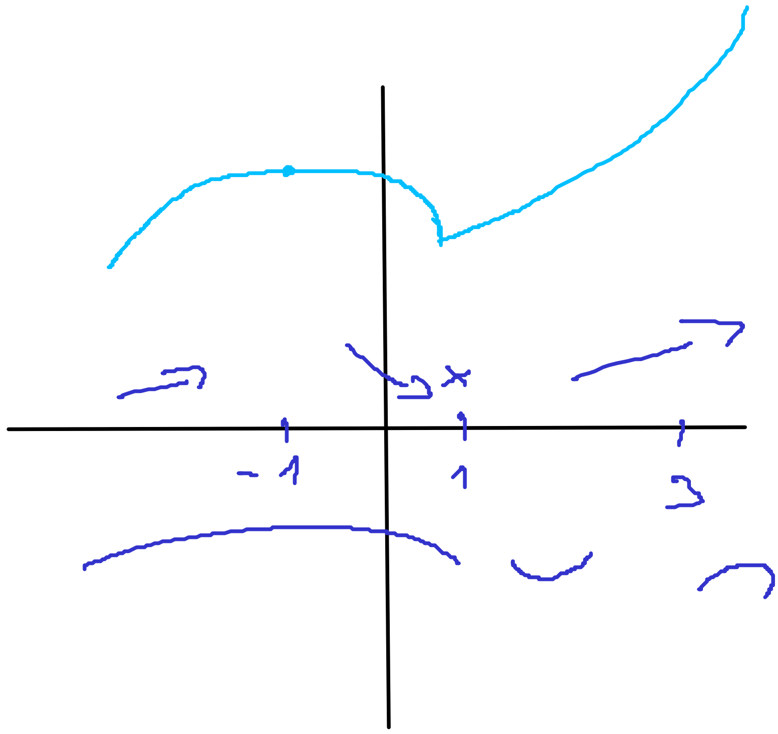
Interval	$f''(a)$ (a is in interval)	Sign of f''	f
$(0, 1)$	$f''(1/2) = -\frac{\ln 1/2}{(1/2)^2(1-\ln 1/2)^2} = -\frac{-}{+} > 0$	$+$	Concave Up
$(1, e)$	$f''(3/2) = -\frac{\ln 3/2}{(3/2)^2(1-\ln 3/2)^2} = -\frac{+}{+} < 0$	$-$	Concave Down

• Points of Inflection:
The function f goes from concave up to concave down at $x = 1 \longrightarrow$ point of inflection. $f(1) = \ln(1-\ln 1) = \ln(1-0) = 0$. Point: $(1, f(1)) = (1, 0)$ (Hey! This means $x = 1$ is a root of f!)

• Sketch: Putting everything together from our detailed analysis, we get


y=ln(1-ln(x))
0
(1,0)
0
1
e
x


2
-2
2


-1
1
3

3a

2) Oblique and horizontal asymptotes.
We will start with x approaching $+\infty$. By Theorem 6.46

$$a = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{\frac{4+x^3}{4-x^2}}{x} = \lim_{x\to+\infty} \frac{4+x^3}{x\cdot(4-x^2)} = \lim_{x\to+\infty} \frac{4+x^3}{4x-x^3} =$$

$$= \lim_{x\to+\infty} \frac{x^3\cdot\left(\frac{4}{x^3}+1\right)}{x^3\cdot\left(\frac{4}{x^2}-1\right)} = \lim_{x\to+\infty} \frac{\frac{4}{x^3}+1}{\frac{4}{x^2}-1} = \frac{0+1}{0-1} = -1.$$

Further

$$b = \lim_{x\to+\infty} [f(x)-ax] = \lim_{x\to+\infty} \left[\frac{4+x^3}{4-x^2} - (-1)\cdot x\right] =$$

$$= \lim_{x\to+\infty} \left[\frac{4+x^3}{4-x^2} + x\right] = \lim_{x\to+\infty} \frac{4+x^3+4x-x^3}{4-x^2} = \lim_{x\to+\infty} \frac{4+4x}{4-x^2} =$$

$$= \lim_{x\to+\infty} \frac{x^2\cdot\left(\frac{4}{x^2}+\frac{4}{x}\right)}{x^2\cdot\left(\frac{4}{x^2}-1\right)} = \lim_{x\to+\infty} \frac{\frac{4}{x^2}+\frac{4}{x}}{\frac{4}{x^2}-1} = \frac{0+0}{0-1} = 0.$$

Therefore, $y=-x$ is the oblique asymptote of the graph of f as x approaches $+\infty$.
It is easy to persuade oneself that the same result is obtained for x approaching $-\infty$. The graph of f and all asymptotes are displayed in Fig. 6.15 a). ▲

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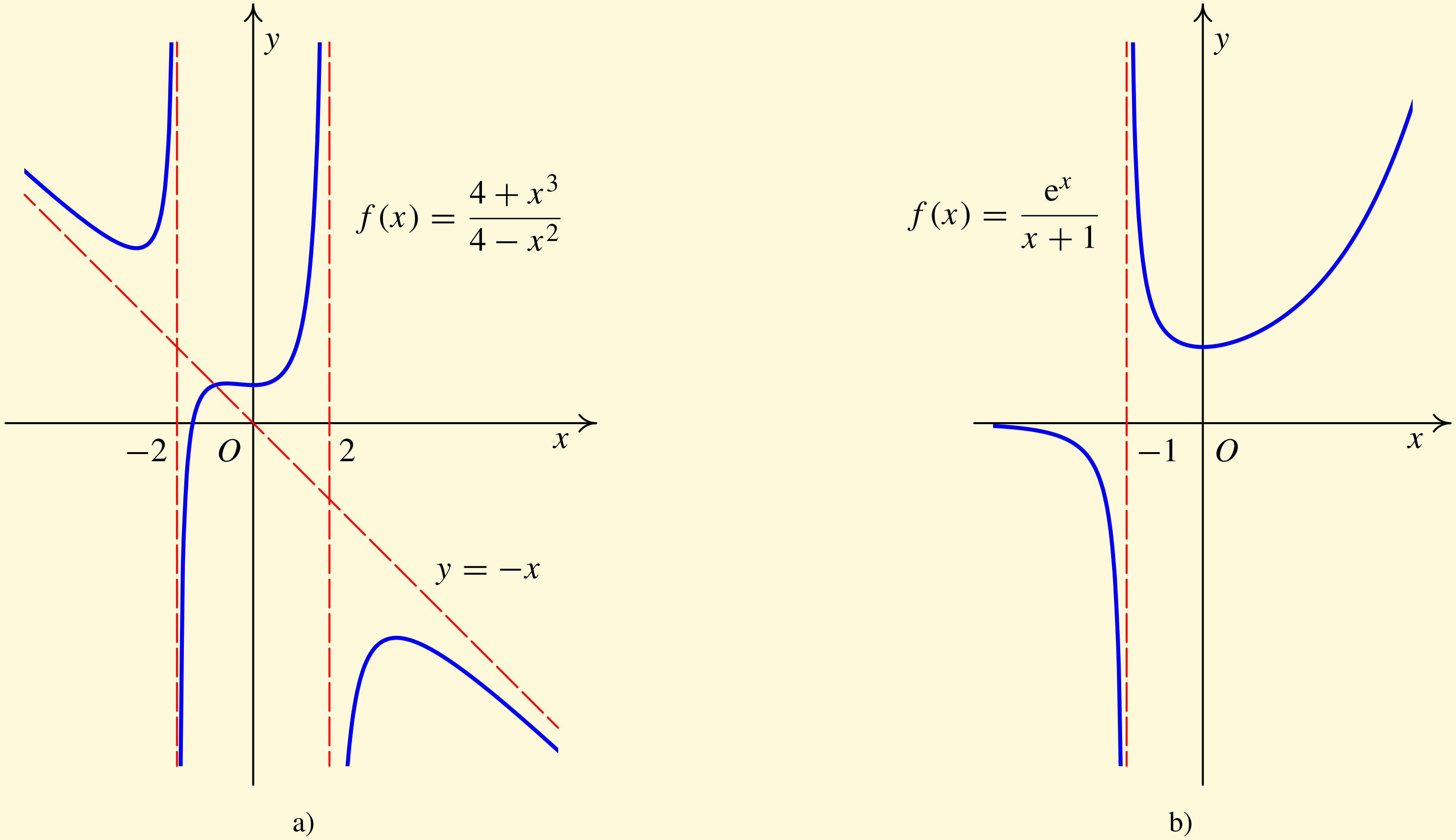
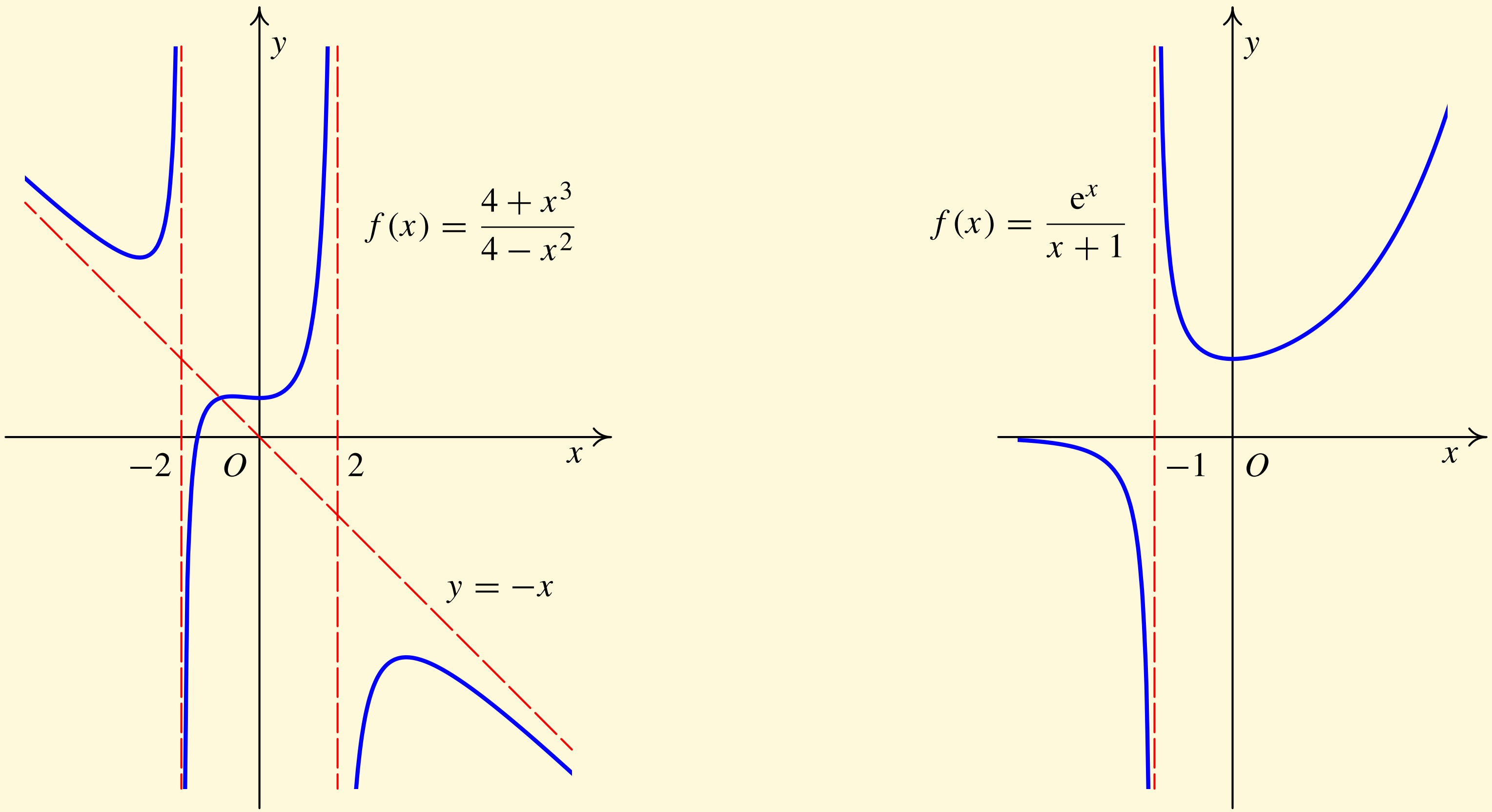


Fig. 6.15

3b



a) b)

Fig. 6.15

Comment 6.48 It is easy to verify that if a *rational function* has an asymptote as x approaches $+\infty$, it has the same asymptote as x approaches $-\infty$, and vice versa. For other types of functions this may not be valid, see the following example.

Example 6.49 Find all asymptotes of the graph of function $f(x) = \dfrac{e^x}{x+1}$. 3b

Solution.

1) Vertical asymptotes.
Since $\operatorname{Dom}(f) = \mathbb{R} \setminus \{-1\}$ and f is continuous on $\operatorname{Dom}(f)$ a vertical asymptote can only occur at the point -1. We use Theorem 4.41 to find necessary limits. Since $x+1 > 0$ for $x > -1$ and $x+1 < 0$ for $x < -1$ we obtain

$$\lim_{x\to-1^+} \frac{e^x}{x+1} = \lim_{x\to-1^+} e^x \lim_{x\to-1^+} \frac{1}{x+1} = e^{-1} \cdot \left(\frac{1}{0^+}\right) = e^{-1} \cdot (+\infty) = +\infty,$$

$$\lim_{x\to-1^-} \frac{e^x}{x+1} = \lim_{x\to-1^-} e^x \lim_{x\to-1^-} \frac{1}{x+1} = e^{-1} \cdot \left(\frac{1}{0^-}\right) = e^{-1} \cdot (-\infty) = -\infty.$$

Therefore, the graph of f has the vertical asymptote $x = -1$.

2) Oblique and horizontal asymptotes.
We will start with x approaching $+\infty$. By Theorem 6.46

$$a = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{\frac{e^x}{x+1}}{x} = \lim_{x\to+\infty} \frac{e^x}{x^2+x} = \left(\frac{+\infty}{+\infty}\right) \overset{LH}{=}$$

$$\overset{LH}{=} \lim_{x\to+\infty} \frac{e^x}{2x+1} = \left(\frac{+\infty}{+\infty}\right) \overset{LH}{=} \lim_{x\to+\infty} \frac{e^x}{2} = \frac{+\infty}{2} = +\infty.$$

This limit is improper hence the graph of f has no asymptote as x approaches $+\infty$.

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Now we will consider x approaching $-\infty$. By Theorem 6.46

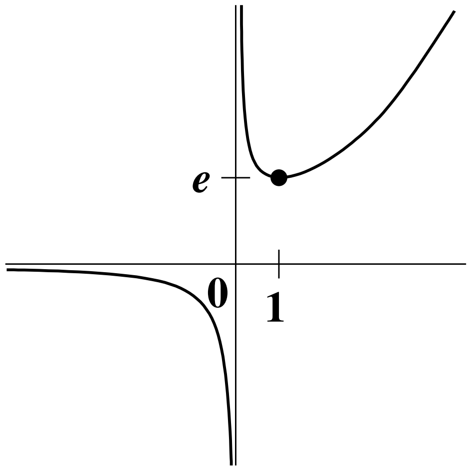
$$a = \lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{\frac{e^x}{x+1}}{x} = \lim_{x\to-\infty} \frac{e^x}{x^2+x} = \frac{0}{+\infty} = 0.$$

Further

$$b = \lim_{x\to-\infty} [f(x) - ax] = \lim_{x\to-\infty} \left[\frac{e^x}{x+1} - 0x\right] = \lim_{x\to-\infty} \frac{e^x}{x+1} = \frac{0}{-\infty} = 0.$$

Therefore, $y = 0$ is the horizontal asymptote of the graph of f as x approaches $-\infty$. The graph of f and all asymptotes are displayed in Fig. 6.15 b). ▲

Comment 6.50 The graph of function f has a horizontal asymptote $y = b$ (i.e. the slope $a = 0$) as x approaches $+\infty$ if and only if $\lim\limits_{x\to+\infty} f(x) = b$. The condition is evidently necessary. It is also sufficient because then $a = \lim\limits_{x\to+\infty} \frac{f(x)}{x} = \frac{b}{+\infty} = 0$. The same holds as x approaches $-\infty$.



3c

4. Solution: The domain is easy, the only problem is the fraction in the exponential. Thus $D_f = (-\infty,0)\cup(0,\infty)$. We see that there are three possibilities for asymptotes. There might be asymptotes at $\pm\infty$ and there might also be a vertical asymptote at 0. We start with this one, and to decide on it, we need to check one-sided limits at 0:

$$\lim_{x\to 0^+}\big(x\,e^{\frac{2}{x}}\big) = \Big\langle\!\Big\langle 0\,e^{2/0^+} = 0\,e^{\infty} = 0\cdot\infty \implies \text{ zmna ve zlomek}\Big\rangle\!\Big\rangle = \lim_{x\to 0^+}\Big(\frac{e^{\frac{2}{x}}}{x^{-1}}\Big)$$
$$= \Big\langle\!\Big\langle \frac{\infty}{\infty} \implies \text{ LH}\Big\rangle\!\Big\rangle = \lim_{x\to 0^+}\Big(\frac{e^{\frac{2}{x}}\frac{-2}{x^2}}{-x^{-2}}\Big) = \lim_{x\to 0^+}\big(2e^{\frac{2}{x}}\big) = \Big\langle\!\Big\langle 2e^{1/0^+} = 2e^{\infty} = 2\infty\Big\rangle\!\Big\rangle = \infty.$$

Since we have an infinite one-sided limit, there is a vertical asymptote at $x=0$. It is not necessary to check the limit from the left because the vertical asymptote is already decided, but we will show it anyway as it is a nice and easy exercise:

$$\lim_{x\to 0^-}\big(x\,e^{\frac{2}{x}}\big) = 0\,e^{2/0^-} = 0\,e^{-\infty} = 0\cdot 0 = 0.$$

Note that in the first calculation (the limit from the right) there was an opportunity to simplify it; we will show it now:

$$\lim_{x\to 0^+}\Big(\frac{e^{\frac{2}{x}}}{x^{-1}}\Big) = \Big\langle\!\Big\langle \text{substitute } y=\frac{1}{x} \implies y\to\infty\Big\rangle\!\Big\rangle = \lim_{y\to\infty}\Big(\frac{e^{2y}}{y}\Big)$$
$$= \Big\langle\!\Big\langle \frac{\infty}{\infty} \implies \text{ LH}\Big\rangle\!\Big\rangle = \lim_{y\to\infty}\Big(\frac{2e^{2y}}{1}\Big) = 2e^{\infty} = \infty.$$

Now we check whether there is an asymptote at ∞. First the limit:

$$\lim_{x\to\infty}\Big(x\,e^{\frac{2}{x}}\Big) = \infty\,e^{2/\infty} = \infty\,e^0 = \infty\cdot 1 = \infty.$$

This means that there is no horizontal asymptote at ∞, but the infitity leaves open the chance that there might be an oblique asymptote. To find its slope (if it exists at all) we calculate

$$k = \lim_{x\to\infty}\Big(\frac{x\,e^{\frac{2}{x}}}{x}\Big) = \lim_{x\to\infty}\big(e^{\frac{2}{x}}\big) = e^0 = 1.$$

Since this limit converges, there is an oblique asymptote with slope $k=1$ at infinity. To find the shift q we use the appropriate formula:

$$q = \lim_{x\to\infty}\big(f(x)-k\cdot x\big) = \lim_{x\to\infty}\big(x\,e^{\frac{2}{x}} - x\big) = \Big\langle\!\Big\langle \infty-\infty \implies \text{ put together}\Big\rangle\!\Big\rangle$$
$$= \lim_{x\to\infty}\big(x(e^{\frac{2}{x}}-1)\big) = \Big\langle\!\Big\langle \infty\cdot 0 \implies \text{ change into fraction}\Big\rangle\!\Big\rangle = \lim_{x\to\infty}\Big(\frac{e^{\frac{2}{x}}-1}{x^{-1}}\Big)$$
$$= \Big\langle\!\Big\langle \text{substitution } y=\frac{1}{x} \implies y\to 0^+\Big\rangle\!\Big\rangle = \lim_{y\to 0^+}\Big(\frac{e^{2y}-1}{y}\Big) = \Big\langle\!\Big\langle \frac{0}{0} \implies \text{ LH}\Big\rangle\!\Big\rangle$$
$$= \lim_{y\to 0^+}\Big(\frac{2e^{2y}}{1}\Big) = 2.$$

Thus there is an oblique asymptote $y = x + 2$ at infinity.
The calculations at negative infinity are similar, so we will show them briefly:

$$\lim_{x\to-\infty}\big(x\,e^{\frac{2}{x}}\big) = -\infty\,e^{-2/\infty} = -\infty\,e^0 = -\infty\cdot 1 = -\infty.$$

So no horizontal, but a chance for an oblique asymptote.

3c

$$k = \lim_{x\to-\infty}\Big(\frac{x\,e^{\frac{2}{x}}}{x}\Big) = \lim_{x\to-\infty}\big(e^{\frac{2}{x}}\big) = e^0 = 1.$$

$$q = \lim_{x\to-\infty}\big(x\,e^{\frac{2}{x}} - x\big) = \lim_{x\to-\infty}\big(x(e^{\frac{2}{x}} - 1)\big) = \lim_{y\to 0^-}\Big(\frac{e^{2y}-1}{y}\Big) = 2.$$

The line $y = x + 2$ is also an asymptote at $-\infty$.

5a. Solution: Domain: $D_f = I\!R\setminus\{0\} = (-\infty,0)\cup(0,\infty)$. There is no symmetry, the function is continuous on D_f. Intercepts: $f(x) = 0 \implies x = 2$. Limits at endpoints:

$$\lim_{x\to\infty}\Big(\frac{|x-2|}{x}\Big) = \Big\langle\!\Big\langle x\to\infty \implies x>2\Big\rangle\!\Big\rangle = \lim_{x\to\infty}\Big(\frac{x-2}{x}\Big) = 1;$$
$$\lim_{x\to-\infty}\Big(\frac{|x-2|}{x}\Big) = \Big\langle\!\Big\langle x\to-\infty \implies x<2\Big\rangle\!\Big\rangle = \lim_{x\to-\infty}\Big(\frac{-(x-2)}{x}\Big) = -1;$$
$$\lim_{x\to 0^+}\Big(\frac{|x-2|}{x}\Big) = \Big\langle\!\Big\langle \tfrac{2}{0^+}\Big\rangle\!\Big\rangle = \infty;$$
$$\lim_{x\to 0^-}\Big(\frac{|x-2|}{x}\Big) = \Big\langle\!\Big\langle \tfrac{2}{0^-}\Big\rangle\!\Big\rangle = -\infty.$$

From the limits we see three things: There is a vertical asymptote at $x = 0$, there is a horizontal asymptote $y = 1$ at ∞ and a horizontal asymptote $y = -1$ at $-\infty$, and therefore there is no oblique asymptote at ∞ and $-\infty$.
Derivative: First we need to get rid of the absolute value:

$$f(x) = \begin{cases} \frac{x-2}{x}; & x\ge 2\\ \frac{2-x}{x}; & x\le 2\end{cases}.$$

Thus

$$f'(x) = \begin{cases} \frac{2}{x^2}; & x>2\\ -\frac{2}{x^2}; & x<2\end{cases}.$$

There are two critical points, $x = 0$ and $x = 2$, both coming from f' DNE. So

	$(-\infty,0)$	$(0,2)$	$(2,\infty)$
$f'(x)$	$-$	$-$	$+$
$f(x)$	$\searrow$	$\searrow$	$\nearrow$

There is a local minimum $f(2) = 0$.
Now the second derivative:

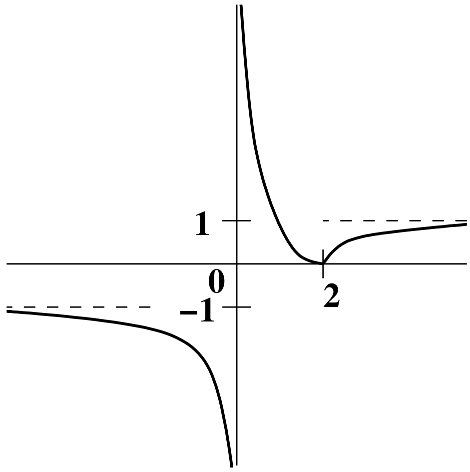
$$f''(x) = \begin{cases} -\frac{4}{x^3}; & x>2\\ \frac{4}{x^3}; & x<2\end{cases}.$$

There are two dividing points again, $x = 0$ and $x = 2$, so

	$(-\infty,0)$	$(0,2)$	$(2,\infty)$
$f''(x)$	$-$	$+$	$-$
$f(x)$	$\frown$	$\smile$	$\frown$

Thus $f(2) = 0$ is also an inflection point.
We put the info together and then draw the graph:

$(-\infty,0)$	$(0,2)$	$(2,\infty)$
↘	↘	↗
⌢	⌣	⌢



3d

5b. Solution: Domain: $D_f = I\!R$, since $e^{3x}+1 \geq 1 > 0$ always; this means that there are no vertical asymptotes. There is no symmetry, the function is continuous on D_f. Intercepts: $f(0) = \ln(2)$. Limits at endpoints:

$$\lim_{x\to\infty}\Big(\ln\big(e^{3x}+1\big)\Big) = \Big\langle\!\Big\langle \ln\big(e^{\infty}+1\big) = \ln(\infty)\Big\rangle\!\Big\rangle = \infty;$$

$$\lim_{x\to-\infty}\Big(\ln\big(e^{3x}+1\big)\Big) = \Big\langle\!\Big\langle \ln\big(e^{-\infty}+1\big) = \ln(0+1)\Big\rangle\!\Big\rangle = 0.$$

Thus there is a horizontal asymptote $y = 0$ at $-\infty$. There is no horizontal symptote at ∞ (there might be an oblique asymptote there).
Derivative:

$$f'(x) = \frac{3e^{3x}}{e^{3x}+1}.$$

This derivative exists everywhere and is always positive (as $e^{3x} > 0$), so there are no critical points (hence no local extrema) and the function f is increasing on $I\!R$.
Now the second derivative:

$$f''(x) = \frac{9e^{3x}}{(e^{3x}+1)^2}.$$

Again, the second derivative exists everywhere and is always positive, so there are no inflection points and the function f is concave up on $I\!R$.
So the function is always increasing and concave up. It remains to check on an oblique asymptote at ∞. First:

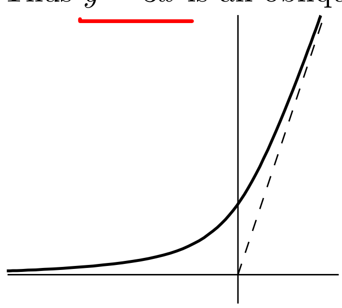
$$k = \lim_{x\to\infty}\big(f'(x)\big) = \lim_{x\to\infty}\Big(\frac{3e^{3x}}{e^{3x}+1}\Big) = 3.$$

Since this limit converges, there is an oblique asymptote at ∞. We find q:

$$q = \lim_{x\to\infty}\big(f(x)-kx\big) = \lim_{x\to\infty}\Big(\ln\big(e^{3x}+1\big) - 3x\Big) = \Big\langle\!\Big\langle \infty-\infty \implies \text{ put together}\Big\rangle\!\Big\rangle$$

$$= \lim_{x\to\infty}\Big(\ln\big(e^{3x}+1\big) - \ln\big(e^{3x}\big)\Big) = \lim_{x\to\infty}\Big(\ln\Big(\frac{e^{3x}+1}{e^{3x}}\Big)\Big) = \ln(1) = 0.$$

Thus $y = 3x$ is an oblique asymptote at ∞. Graph:



Therefore, the slant asymptote is $y = x$.

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Example 14: Find the slant asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Solution: $y^2 = b^2\left(\frac{x^2}{a^2} - 1\right) = \frac{b^2}{a^2}(x^2 - a^2)$

$$m = \lim_{x\to\pm\infty} \frac{y}{x} = \lim_{x\to\pm\infty} \frac{1}{x}\left[\pm\frac{b}{a}\sqrt{x^2 - a^2}\right]$$

$$= \pm\frac{b}{a} \lim_{x\to\pm\infty} \sqrt{1 - \frac{a^2}{x^2}}$$

$$= \pm\frac{b}{a}(1) = \pm\frac{b}{a}.$$

$$c = \lim_{x\to\pm\infty}\left(y - \left(\pm\frac{b}{a}x\right)\right) = \pm\frac{b}{a} \lim_{x\to\pm\infty}\left[\sqrt{x^2 - a^2} - x\right]$$

$$= \pm\frac{b}{a} \lim_{x\to\pm\infty} \frac{-a^2}{\sqrt{x^2 - a^2} + x} = 0.$$

Thus, the slant asymptotes are $y = \pm\frac{b}{a}x$. Fig. 17 shows these asymptotes.

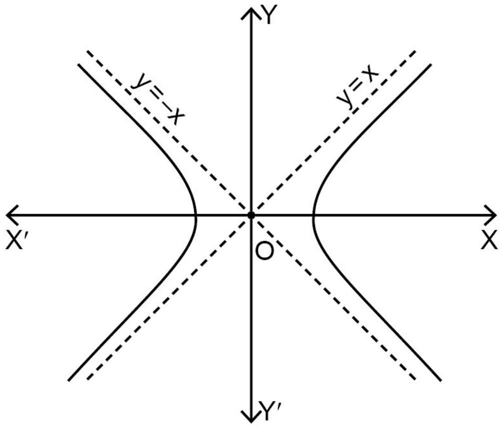

Fig. 17: Slant asymptotes in a Hyperbola.

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Example 15: Find the slant asymptote to the curve $y = \sqrt{x^2 + 9x}$.

Solution: Slope $m = \lim_{x\to\pm\infty} \frac{\sqrt{x^2 + 9x}}{x}$

$$= \lim_{x\to\pm\infty} \frac{\sqrt{(x^2)\left(1 + \frac{9}{x}\right)}}{x}$$

$$= \lim_{x\to\pm\infty} \frac{|x|\sqrt{1 + \frac{9}{x}}}{x}$$

$$= \lim_{x\to+\infty} \frac{x\sqrt{1+\frac{9}{x}}}{x} \quad \text{and} \quad \lim_{x\to-\infty} \frac{-x\sqrt{1+\frac{9}{x}}}{x}$$

$$= \lim_{x\to+\infty} \sqrt{1+\frac{9}{x}} \quad \text{and} \quad \lim_{x\to-\infty} -\sqrt{1+\frac{9}{x}}$$

$$= 1 \text{ and } -1$$

Now, let us find c for both the values of m. Where $m = 1$, we get:

$$c = \lim_{x\to\pm\infty} \left[\sqrt{x^2+9x} - x\right]$$

$$= \lim_{x\to\pm\infty} \frac{x^2+9x-x^2}{\sqrt{x^2+9x}+x}$$

$$= \lim_{x\to\pm\infty} \frac{9x}{\sqrt{x^2+9x}+x}$$

$$= \lim_{x\to\pm\infty} \frac{9x}{x\left(\sqrt{1+\frac{9}{x}}+1\right)}$$

$$= \lim_{x\to\pm\infty} \frac{9}{\sqrt{1+\frac{9}{x}}+1}$$

$$= \frac{9}{2}$$

Similarly, when $m = -1$, we get $c = \lim\limits_{x\to\pm\infty} \left[\sqrt{x^2+9x}+x\right] = -\frac{9}{2}$

Hence, the slant asymptote to f are $y = x + \frac{9}{2}$ and $y = -x - \frac{9}{2}$.

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Example 16: Show that $f(x) = x + \sqrt{x}$ does not have a slant asymptote at ∞.

Solution: We shall do a proof by contradiction. Suppose f has a slant asymptote $y = mx + c$. Then we must have

$$m = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \left(\frac{x+\sqrt{x}}{x}\right) = \lim_{x\to\pm\infty} \left(1 + \frac{1}{\sqrt{x}}\right) = 1 \text{ or does not exist.}$$

so, $y = x + c$.

And then, we get

$$c = \lim_{x\to\pm\infty} \left(f(x) - x\right) = \lim_{x\to\pm\infty} \left(x + \sqrt{x} - x\right) = \lim_{x\to\pm\infty} \sqrt{x} = \infty \text{ or does not exist.}$$

Which is a contradiction (since c must be finite).

Hence, f cannot have a slant asymptote at ∞.

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Let us find the slant asymptote to a curve, where the equation of the curve is of the form $f(x, y) = 0$.

Example 17: Find the oblique asymptotes for curve $x^3 - y^3 = 3xy$.

Solution: Suppose that the given curve has an oblique asymptote $y = mx + c$. The equation of the curve can be written as that, as $x \to \infty$, the graph of f approaches the line $y = x - 1$, so $y = x - 1$ is an oblique asymptote to the graph of f at ∞. Similarly, as $x \to -\infty$, the graph of f approaches the line $y = -\frac{x}{2} + 1$, so $y = -\frac{x}{2} + 1$ is an oblique asymptote to the graph of f at $-\infty$, as shown in Fig. 16.

Going back to the definition of the oblique asymptotes, we can say that in the first case, the line $y = mx + c$ is an oblique asymptote of $f(x)$ when x tends to ∞, and in the second case the line $y = mx + c$ is an oblique asymptote of $f(x)$ when x tends to $-\infty$. The oblique asymptote, for the function $f(x)$ will be given by the equation $y = mx + c$. The value of m is computed first and is given by the following limit:

Suppose $y = mx + c$ is a slant asymptote to f at $\pm\infty$, then
$\lim\limits_{x \to \pm\infty} [f(x) - (mx + c)] = 0$.

On dividing this equation both the sides by x, we get

$$\lim_{x \to \pm\infty} \left[\frac{f(x)}{x} - \frac{mx + c}{x} \right] = 0$$

$$\lim_{x \to \pm\infty} \left[\frac{f(x)}{x} - m - \frac{c}{x} \right] = 0$$

$$\lim_{x \to \pm\infty} \left[\frac{f(x)}{x} - m \right] = 0 \qquad \left[\because \lim_{x \to \pm\infty} \frac{c}{x} = 0 \right]$$

Thus, $m = \lim\limits_{x \to \pm\infty} \frac{f(x)}{x}$.

We can solve m separately for two cases as $x \to \infty$ and as $x \to -\infty$. If this limit does not exist or is equal to zero, then, there is no oblique asymptote in that direction.

Having m, then the value of c can be computed by $c = \lim\limits_{x \to \pm\infty} [f(x) - mx]$. If this limit does not exist, then there is no oblique asymptote in that direction, even if a limit defining m exists.

Let us find slant asymptotes in the following examples:

Example 13: Find the slant asymptotes of $y = \frac{x^3}{x^2 - 1}$.

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Solution: We shall find m and c.

$$m = \lim_{x \to \pm\infty} \frac{y}{x} = \lim_{x \to \pm\infty} \frac{1}{x} \left(\frac{x^3}{x^2 - 1} \right)$$

$$= \lim_{x \to \pm\infty} \frac{1}{1 - \frac{1}{x^2}} = 1$$

$$c = \lim_{x \to \pm\infty} (y - mx) = \lim_{x \to \pm\infty} \left(\frac{x^3}{x^2 - 1} - (1)x \right)$$

$$= \lim_{x \to \pm\infty} \frac{x}{x^2 - 1} = 0$$

Therefore, the slant asymptote is $y = x$. 3g

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Example 14: Find the slant asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Solution: $y^2 = b^2\left(\frac{x^2}{a^2} - 1\right) = \frac{b^2}{a^2}(x^2 - a^2)$

$$m = \lim_{x\to\pm\infty} \frac{y}{x} = \lim_{x\to\pm\infty} \frac{1}{x}\left[\pm\frac{b}{a}\sqrt{x^2 - a^2}\right]$$

$$= \pm\frac{b}{a}\lim_{x\to\pm\infty}\sqrt{1 - \frac{a^2}{x^2}}$$

$$= \pm\frac{b}{a}(1) = \pm\frac{b}{a}.$$

$$c = \lim_{x\to\pm\infty}\left(y - \left(\pm\frac{b}{a}x\right)\right) = \pm\frac{b}{a}\lim_{x\to\pm\infty}\left[\sqrt{x^2 - a^2} - x\right]$$

$$= \pm\frac{b}{a}\lim_{x\to\pm\infty}\frac{-a^2}{\sqrt{x^2 - a^2} + x} = 0.$$

Thus, the slant asymptotes are $y = \pm\frac{b}{a}x$. Fig. 17 shows these asymptotes.

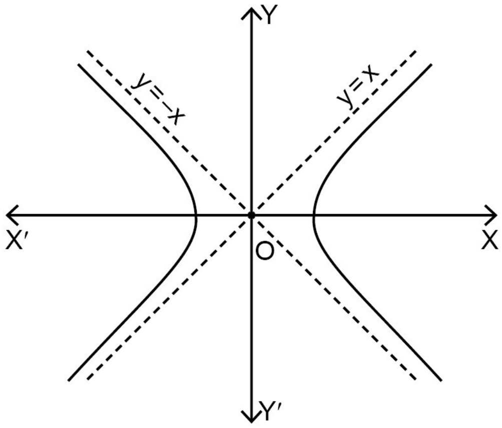

Fig. 17: Slant asymptotes in a Hyperbola.

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Example 15: Find the slant asymptote to the curve $y = \sqrt{x^2 + 9x}$.

Solution: Slope $m = \lim_{x\to\pm\infty}\frac{\sqrt{x^2 + 9x}}{x}$

$$= \lim_{x\to\pm\infty}\frac{\sqrt{(x^2)\left(1 + \frac{9}{x}\right)}}{x}$$

$$= \lim_{x\to\pm\infty}\frac{|x|\sqrt{1 + \frac{9}{x}}}{x}$$