20th lesson

https://www2.karlin.mff.cuni.cz/~kuncova/en/teachMat1.php
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Theory

Definition 1. We say that a function f is

- *convex* on an interval I if
$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2),$$
for each $x_1, x_2 \in I$ and each $\lambda \in [0,1]$.
- *concave* on an interval I if
$$f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda) f(x_2),$$
for each $x_1, x_2 \in I$ and each $\lambda \in [0,1]$.

Theorem 2 (Second derivative and convexity)**.** Let I be an open interval and f has a finite second derivative on I.

(i) if $f''(x) \geq 0$ for every $x \in I$, then f is **convex** on I;

(ii) if $f''(x) \leq 0$ for every $x \in I$, then f is **concave** on I.

Theorem 3. Let $a \in \mathbb{R}$ be an inflection point of a function f. Then $f''(a)$ either does not exists or equals zero.

Theorem 4. Suppose that a function f has a continuous first derivative on a an interval (a,b) and $z \in (a,b)$. Suppose further that

- $\forall x \in (a,z) : f''(x) > 0,$
- $\forall x \in (z,b) : f''(x) < 0.$

Then z is an inflection point of f.

Theorem 5. A function f has an asymptote at ∞ given by the affine function $x \mapsto kx+q$ if and only if
$$\lim_{x\to\infty} \frac{f(x)}{x} = k \in \mathbb{R} \quad \text{and} \quad \lim_{x\to\infty} (f(x) - kx) = q \in \mathbb{R}.$$

Exercises

1. Find the (local) maxima/minima. Find the intervals of monotonicity. Find the intervals of convexity/concavity. Sketch the graph.

   (a) $f(x) = -x^3 + 3x^2 - 4$

   (b) $f(x) = 3x^5 - 20x^3$

   (c) $f(x) = \dfrac{x^2}{x^2+1}$

   (d) $f(x) = e^{-x^2/2}$

   (e) $f(x) = 3\sqrt[3]{x^2} - x$

   (f) $f(x) = x^4 - 6x^2 + 1$

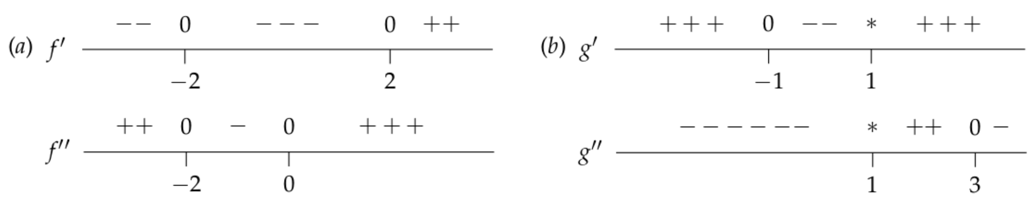
   (g) $f(x) = x^5 - 5x^4$

   (h) $f(x) = 3 + x^2 - \dfrac{8}{x}$

   (i) $f(x) = \sqrt[3]{2x-5} + 1$

   (j) $f(x) = \ln(1 - \ln x)$

2. Sketch the graphs of following continuous functions:



Source 1: `http://math.hws.edu/~mitchell/Math130F12/tufte-latex/Lecture32.pdf`

3. Find the asymptotes. (Find the domain, then find (horizontal and slant) asymptotes at $\pm\infty$, if possible.)

   (a) $f(x) = \dfrac{4+x^3}{4-x^2}$

   (b) $f(x) = \dfrac{e^x}{x+1}$

   (c) $f(x) = xe^{2/x}$

   (d) $f(x) = \ln(1 + e^{3x})$

   (e) $f(x) = \sqrt{x^2 + 9x}$

   (f) $f(x) = x + \sqrt{x}$

   (g) $f(x) = \dfrac{x^3}{x^2-1}$