1a)

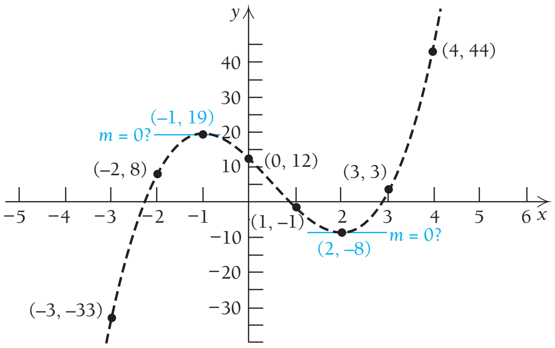
Exampl E 1 Graph the function f given by

$$f(x) = 2x^3 - 3x^2 - 12x + 12,$$

and find the relative extrema.

Solution Suppose that we are trying to graph this function but don't know any calculus. What can we do? We could plot several points to determine in which direction the graph seems to be turning. Let's pick some x-values and see what happens.

x	$f(x)$
-3	-33
-2	8
-1	19
0	12
1	-1
2	-8
3	3
4	44



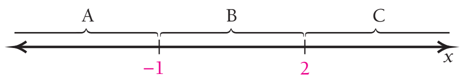
We plot the points and use them to sketch a "best guess" of the graph, shown as the dashed line in the figure above. According to this rough sketch, it appears that the graph has a tangent line with slope 0 somewhere around $x = -1$ and $x = 2$. But how do we know for sure? We use calculus to support our observations. We begin by finding a general expression for the derivative:

$$f'(x) = 6x^2 - 6x - 12.$$

We next determine where $f'(x)$ does not exist or where $f'(x) = 0$. Since we can evaluate $f'(x) = 6x^2 - 6x - 12$ for any real number, there is no value for which $f'(x)$ does not exist. So the only possibilities for critical values are those where $f'(x) = 0$, locations at which there are horizontal tangents. To find such values, we solve $f'(x) = 0$:

$$\begin{aligned} 6x^2 - 6x - 12 &= 0 \\ x^2 - x - 2 &= 0 && \text{Dividing both sides by 6} \\ (x + 1)(x - 2) &= 0 && \text{Factoring} \\ x + 1 = 0 \quad \text{or} \quad x - 2 &= 0 && \text{Using the Principle of Zero Products} \\ x = -1 \quad \text{or} \quad x &= 2. \end{aligned}$$

The critical values are -1 and 2. Since it is at these values that a relative maximum or minimum might exist, we examine the intervals on each side of the critical values: A is $(-\infty, -1)$, B is $(-1, 2)$, and C is $(2, \infty)$, as shown below.



Next, we analyze the sign of the derivative on each interval. If $f'(x)$ is positive for one value in the interval, then it will be positive for all values in the interval. Similarly, if it is negative for one value, it will be negative for all values in the interval. Thus, we choose a test value in each interval and make a substitution. The test values we choose are -2, 0, and 4.

1e)

A: Test -2, $f'(-2) = 6(-2)^2 - 6(-2) - 12$
$= 24 + 12 - 12 = 24 > 0$;

B: Test 0, $f'(0) = 6(0)^2 - 6(0) - 12 = -12 < 0$;

C: Test 4, $f'(4) = 6(4)^2 - 6(4) - 12$
$= 96 - 24 - 12 = 60 > 0$.

Interval: A, B, C; x: -1, 2

	A	B	C
Test Value	$x = -2$	$x = 0$	$x = 4$
Sign of $f'(x)$	$f'(-2) > 0$	$f'(0) < 0$	$f'(4) > 0$
Result	f is increasing on $(-\infty, -1)$	f is decreasing on $(-1, 2)$	f is increasing on $(2, \infty)$

Change indicates a relative maximum.

Change indicates a relative minimum.

Therefore, by the First-Derivative Test,

f has a relative maximum at $x = -1$ given by

$$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 12$$ Substituting into the *original* function

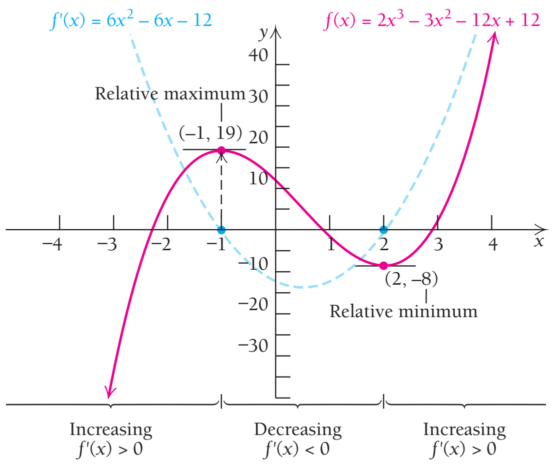
$$= 19$$ This is a relative maximum.

and f has a relative minimum at $x = 2$ given by

$$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 12 = -8.$$ This is a relative minimum.

Thus, there is a relative maximum at $(-1, 19)$ and a relative minimum at $(2, -8)$, as we suspected from the sketch of the graph.

The information we have obtained from the first derivative can be very useful in sketching a graph of the function. We know that this polynomial is continuous, and we know where the function is increasing, where it is decreasing, and where it has relative extrema. We complete the graph by using a calculator to generate some additional function values. The graph of the function, shown below in red, has been scaled to clearly show its curving nature.



TECHNOLOGY CONNECTION

Exploratory

Consider the function f given by

$$f(x) = x^3 - 3x + 2.$$

Graph both f and f' using the same set of axes. Examine the graphs using the TABLE and TRACE features. Where do you think the relative extrema of $f(x)$ occur? Where is the derivative equal to 0? Where does $f(x)$ have critical values?

Theorem 9.2 (The First Derivative Test): *Let f be continuous on the interval (a, b) and suppose that c is the only critical point of (a, b). Suppose f is differentiable on (a, b) except possibly at c. Then:*

1. *If $f'(x)$ is negative to the left of c and positive to the right of c, then f has a local min at c.*
2. *If $f'(x)$ is positive to the left of c and negative to the right of c, then f has a local max at c.*
3. *If $f'(x)$ is the same sign to the right and left of c then c is neither a max nor a min.*

In cases like 3 above we call the point c a **saddle point** if $f'(c) = 0$.

Examples: Find the critical points and identify their type for the following functions.

15)

1. $f(x) = 2x^3 + 9x^2 - 108x + 30$.

   solution: We first have to find the critical points

   $$f'(x) = 6x^2 + 18x - 108 = 6(x^2 + 3x - 18) = 6(x + 6)(x - 3)$$

   So the critical points are $x = -6, x = 3$. To fill in our real line we plug in some test points from $(-\infty, -6), (-6, 3)$ and $(3, \infty)$:

   $$f'(-7) = 6(-7 + 6)(-7 - 3) = 60 > 0$$

   $$f'(0) = 6(6)(-3) = -108 < 0$$

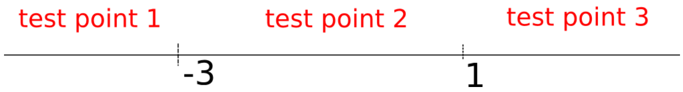
   $$f'(4) = 6(4 + 6)(4 - 3) = 60 > 0$$

   Thus we have the following

		-6		3	
$f'(x)$	$+$	0	$-$	0	$+$
$f(x)$	$\nearrow$		$\searrow$		$\nearrow$

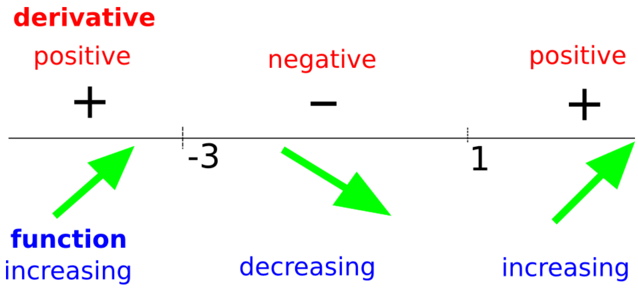
   So by the first derivative test we can conclude that $f(x)$ has a local max at $x = -6$ and a local min at $x = 3$.

Recall that in the precalculus course you may have used the test points from each of the subintervals of the number line. For example, the points -3 and 1 divide the number line to three parts. Take a test point from each part.



For example, with $-4, 0$ and 2 as the test points, you obtain that $f'(-4) = 15 > 0$, $f'(0) = -9 < 0$ and $f'(2) = 15 > 0$. *Careful:* you want to plug the test points in the derivative f' not in the function f since you are determining the sign of f', not f.

Thus we have the number line on the right.

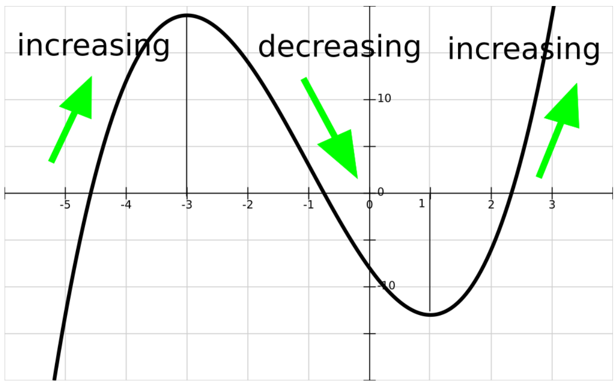


From the number line, we conclude that $f(x)$ is increasing for $x < -3$ and $x > 1$ and decreasing for $-3 < x < 1$. Alternatively, you can write your answer using interval notation as follows.

$f(x)$ is increasing on $(-\infty, -3)$ and $(1, \infty)$,

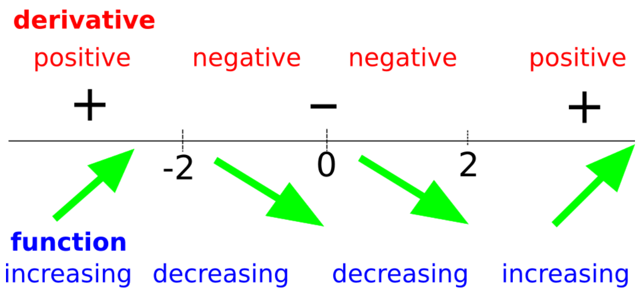
$f(x)$ is decreasing on $(-3, 1)$.

Finally, graph the function and make sure that the graph agrees with your findings.



1c) (b) Use the quotient rule to find the derivative $f'(x) = \frac{2x(2x)-2(x^2+4)}{(2x)^2} = \frac{4x^2-2x^2-8}{4x^2} = \frac{2x^2-8}{4x^2} = \frac{x^2-4}{2x^2} = \frac{(x-2)(x+2)}{2x^2}$.

Three terms impact the sign of $f'(x)$: $x-2$, $x+2$ and $2x^2$. These terms change the sign for $x = 2, -2$ and 0 so those are the numbers relevant for the number line. Since these three values divide the number line to four pieces, you need four test points. Test the points and obtain the number line on the right.

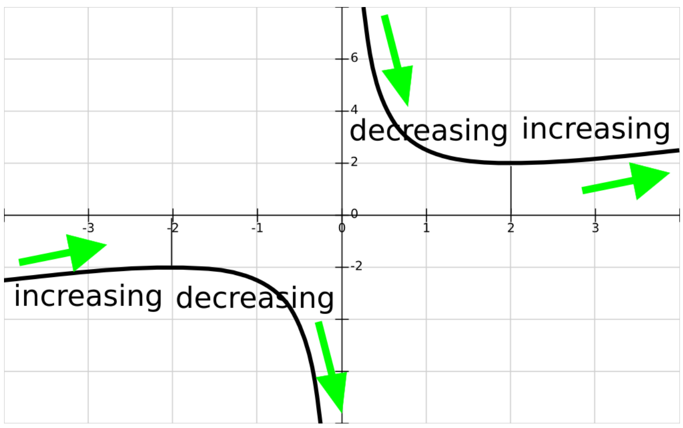


From the number line, we conclude that $f(x)$ is increasing for $x < -2$ and $x > 2$ and decreasing for $-2 < x < 0$ and $0 < x < 2$. Alternatively, using interval notation you have that.

$f(x)$ is increasing on $(-\infty, -2)$ and $(2, \infty)$,

$f(x)$ is decreasing on $(-2, 0)$ and $(0, 2)$.

Finally, graph the function and make sure that the graph agrees with your findings.



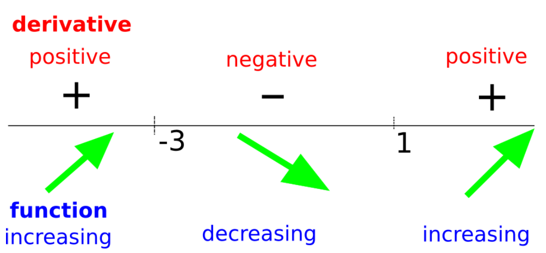
Using the number line test just as when determining increasing/decreasing intervals, one can readily classify the critical points into three categories matching the three cases above and determine the points at which a function has extreme values. This procedure is known as the **First Derivative Test.** Let us summarize.

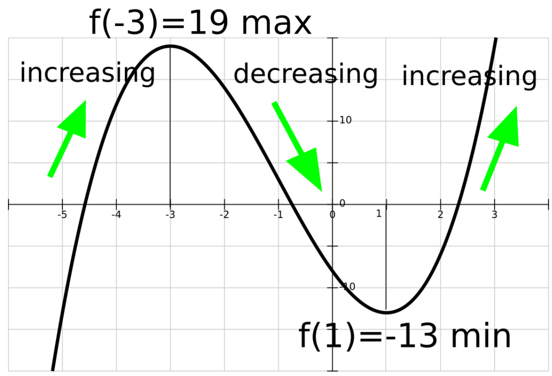
The First Derivative Test. To determine the extreme values of a continuous function $f(x)$:
1. Find $f'(x)$.
2. Set it to zero and find all the critical points.
3. Use the number line to classify the critical points into the three cases.
   - if $f'(x)$ changes from negative to positive at c, f has a minimum at c,
   - if $f'(x)$ changes from positive to negative at c, f has a maximum at c,
   - if $f'(x)$ does not change the sign at c, f has no extreme value c.

Example 2 revisited. Determine the extreme values of the following functions.

(a) $f(x) = x^3 + 3x^2 - 9x - 8$ (b) $f(x) = \dfrac{x^2+4}{2x}$

Solutions. (a) Recall that the derivative is $f'(x) = 3(x^2 + 2x - 3) = 3(x-1)(x+3)$ so that $x = 1$ and $x = -3$ are the critical points. In Example 2, we have performed the number line test for the derivative and obtained the number line below. Thus, both critical values are extreme values and there is a minimum at $x = 1$ and a maximum at $x = -3$. Compute the y-values to determine the minimal value $f(1) = -13$ and the maximal value $f(-3) = 19$. Consider the graph again to make sure that the graph agrees with your findings.





12

(b) Recall that the derivative is $f'(x) = \frac{(x-2)(x+2)}{2x^2}$. Thus the critical points are $x = 2, x = -2$ (at which f' is zero) and $x = 0$ at which f' is not defined. In Example 2, we have performed the number line test for the derivative and obtained the number line below. Thus, both 2 and -2 are extreme values and there is a minimum at $x = 2$ and a maximum at $x = -2$. The derivative is not changing sign at 0 so it is not an extreme value. f is also undefined at 0 so it cannot have an extreme value at 0 for that reason too. Compute the y-values to determine the minimal value $f(2) = 2$ and the maximal value $f(-2) = -2$. Consider the graph again to make sure that the graph agrees with your findings.

For reference, the graph of the derivative is shown in blue. Note that $f'(x) = 0$ where $f(x)$ has relative extrema. We summarize the behavior of this function as follows, by noting where it is increasing or decreasing, and by characterizing its critical points:

- The function f is increasing over the interval $(-\infty, -1)$.
- The function f has a relative maximum at the point $(-1, 19)$.
- The function f is decreasing over the interval $(-1, 2)$.
- The function f has a relative minimum at the point $(2, -8)$.
- The function f is increasing over the interval $(2, \infty)$.

❯ Quick Check 1

Graph the function g given by $g(x) = x^3 - 27x - 6$, and find the relative extrema.

❮ Quick Check 1

Interval notation and point notation look alike. Be clear when stating your answers whether you are identifying an interval or a point.

To use the first derivative for graphing a function f:

1. Find all critical values by determining where $f'(x)$ is 0 and where $f'(x)$ is undefined (but $f(x)$ is defined). Find $f(x)$ for each critical value.
2. Use the critical values to divide the x-axis into intervals and choose a test value in each interval.
3. Find the sign of $f'(x)$ for each test value chosen in step 2, and use this information to determine where $f(x)$ is increasing or decreasing and to classify any extrema as relative maxima or minima.
4. Plot some additional points and sketch the graph.

The *derivative* f' is used to find the critical values of f. The test values are substituted into the *derivative* f', and the function values are found using the *original* function f.

1d

example 2 Find the relative extrema of the function f given by

$$f(x) = 2x^3 - x^4.$$

Then sketch the graph.

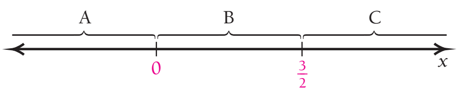
Solution First, we must determine the critical values. To do so, we find $f'(x)$:

$$f'(x) = 6x^2 - 4x^3.$$

Next, we find where $f'(x)$ does not exist or where $f'(x) = 0$. Since $f'(x) = 6x^2 - 4x^3$ is a polynomial, it exists for all real numbers x. Therefore, the only candidates for critical values are where $f'(x) = 0$, that is, where the tangent line is horizontal:

$$\begin{aligned} 6x^2 - 4x^3 &= 0 && \text{Setting } f'(x) \text{ equal to } 0 \\ 2x^2(3 - 2x) &= 0 && \text{Factoring} \\ 2x^2 = 0 \quad &\text{or} \quad 3 - 2x = 0 \\ x^2 = 0 \quad &\text{or} \quad 3 = 2x \\ x = 0 \quad &\text{or} \quad x = \tfrac{3}{2}. \end{aligned}$$

The critical values are 0 and $\frac{3}{2}$. We use these values to divide the x-axis into three intervals as shown below: A is $(-\infty, 0)$; B is $(0, \frac{3}{2})$; and C is $(\frac{3}{2}, \infty)$.



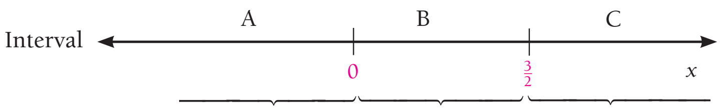
Note that $f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - \left(\frac{3}{2}\right)^4 = \frac{27}{16}$ and $f(0) = 2 \cdot 0^3 - 0^4 = 0$ are possible extrema.

We now determine the sign of the derivative on each interval by choosing a test value in each interval and substituting. We generally choose test values for which it is easy to compute $f'(x)$.

$$\text{A:}\quad \text{Test } -1,\quad f'(-1) = 6(-1)^2 - 4(-1)^3 = 6 + 4 = 10 > 0;$$

$$\text{B:}\quad \text{Test } 1,\quad f'(1) = 6(1)^2 - 4(1)^3 = 6 - 4 = 2 > 0;$$

$$\text{C:}\quad \text{Test } 2,\quad f'(2) = 6(2)^2 - 4(2)^3 = 24 - 32 = -8 < 0.$$



1d

Test Value	$x = -1$	$x = 1$	$x = 2$
Sign of $f'(x)$	$f'(-1) > 0$	$f'(1) > 0$	$f'(2) < 0$
Result	f is increasing on $(-\infty, 0)$	f is increasing on $\left(0, \frac{3}{2}\right)$	f is decreasing on $\left(\frac{3}{2}, \infty\right)$

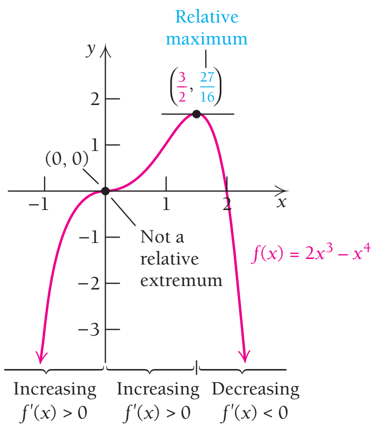
No change

Change indicates a relative maximum.

Therefore, by the First-Derivative Test, f has no extremum at $x = 0$ (since $f(x)$ is increasing on both sides of 0) and has a relative maximum at $x = \frac{3}{2}$. Thus, $f\left(\frac{3}{2}\right)$, or $\frac{27}{16}$, is a relative maximum.

We use the information obtained to sketch the graph below. Other function values are listed in the table.

x	$f(x)$, approximately
-1	-3
-0.5	-0.31
0	0
0.5	0.19
1	1
1.25	1.46
2	0



We summarize the behavior of f:

- The function f is increasing over the interval $(-\infty, 0)$.
- The function f has a critical point at $(0, 0)$, which is neither a minimum nor a maximum.

- The function f is increasing over the interval $\left(0, \frac{3}{2}\right)$.
- The function f has relative maximum at the point $\left(\frac{3}{2}, \frac{27}{16}\right)$.
- The function f is decreasing over the interval $\left(\frac{3}{2}, \infty\right)$.

1d

Since f is increasing over the intervals $(-\infty, 0)$ and $\left(0, \frac{3}{2}\right)$, we can say that f is increasing over $\left(-\infty, \frac{3}{2}\right)$ despite the fact that $f'(0) = 0$ within this interval. In this case, we can observe that any secant line connecting two points within this interval will have a positive slope.

❮ Quick Check 2

❯ Quick Check 2

Find the relative extrema of the function h given by $h(x) = x^4 - \frac{8}{3}x^3$. Then sketch the graph.

example 3 Find the relative extrema of the function f given by

$$f(x) = (x - 2)^{2/3} + 1.$$

1e

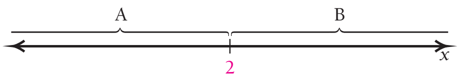
Then sketch the graph.

Solution First, we determine the critical values. To do so, we find $f'(x)$:

$$f'(x) = \frac{2}{3}(x - 2)^{-1/3}$$

$$= \frac{2}{3\sqrt[3]{x - 2}}.$$

Next, we find where $f'(x)$ does not exist or where $f'(x) = 0$. Note that $f'(x)$ does not exist at 2, although $f(x)$ does. Thus, 2 is a critical value. Since the only way for a fraction to be 0 is if its numerator is 0, we see that $f'(x) = 0$ has no solution. Thus, 2 is the only critical value. We use 2 to divide the x-axis into the intervals A, which is $(-\infty, 2)$, and B, which is $(2, \infty)$. Note that $f(2) = (2 - 2)^{2/3} + 1 = 1$.



To determine the sign of the derivative, we choose a test value in each interval and substitute each value into the derivative. We choose test values 0 and 3. It is not necessary to find an exact value of the derivative; we need only determine the sign. Sometimes we can do this by just examining the formula for the derivative:

$$\text{A:} \quad \text{Test } 0, \quad f'(0) = \frac{2}{3\sqrt[3]{0 - 2}} < 0;$$

$$\text{B:} \quad \text{Test } 3, \quad f'(3) = \frac{2}{3\sqrt[3]{3 - 2}} > 0.$$

Test Value	$x = 0$	$x = 3$
Sign of $f'(x)$	$f'(0) < 0$	$f'(3) > 0$
Result	f is decreasing on $(-\infty, 2)$	f is increasing on $(2, \infty)$

Change indicates a relative minimum.

TECHNOLOGY CONNECTION

EXERCISES

In Exercises 1 and 2, consider the function f given by

$$f(x) = 2 - (x - 1)^{2/3}.$$

1. Graph the function using the viewing window $[-4, 6, -2, 4]$.

2. Graph the first derivative. What happens to the graph of the derivative at the critical values?

13 (c) $f(x) = \frac{2x}{x^2+4} \Rightarrow f'(x) = \frac{2(x^2+4)-2x(2x)}{(x^2+4)^2} = \frac{8-2x^2}{(x^2+4)^2} = \frac{2(2-x)(2+x)}{(x^2+4)^2}$. Since the denominator is never zero, the only critical points are $pm2$.

The number line test gives you that the function f is increasing on $(-2,2)$ and decreasing on $(-\infty,2)$ and $(2,\infty)$. At $x=2$ there is a maximum of $\frac{1}{2}$ and at $x=-2$ there is a minimum of $\frac{-1}{2}$.

(d) $f(x) = e^x(x^2-x-5) \Rightarrow f'(x) = e^x(x^2-x-5) + e^x(2x-1) = e^x\left(x^2-x-5+2x-1\right) = e^x(x^2+x-6) = e^x(x+3)(x-2)$. Since e^x is never zero, 2 and -3 are the only critical points. From the number line test we have that f is increasing on $(-\infty,-3)$ and $(2,\infty)$ and decreasing on $(-3,2)$. At $x=-3$ there is a maximum of $f(-3) = 7e^{-3} \approx .348$ and at $x=2$ there is a minimum of $f(2) = -3e^2 \approx -22.17$.

4. The object changes the direction of the motion when the distance function is changing from increasing to decreasing or vice versa. So, those times corresponds to critical points at which there are extreme values.

   $s(t) = t^3 - 7t^2 + 13t \Rightarrow v(t) = s'(t) = 3t^2 - 14t + 13$. Use the calculator program or the quadratic equation formula to determine that $3t^2 - 14t + 13 = 0 \Rightarrow t = \frac{14\pm\sqrt{40}}{6} = \frac{7\pm\sqrt{10}}{3} \Rightarrow t = 3.39$ and $t \approx 1.30$. Perform the number line test and obtain that the distance changes from forwards to backwards at 1.30 and from backwards to forwards at 3.39. 1.3 second after the movement the object is about 7.27 meters from the starting point and 3.39 seconds after the start of the movement, the object is 2.58 meters from the starting point.

   Thus the distance increases from 0 to 7.27 meters in the first 1.3 seconds, then decreases from 7.27 to 2.58 meters from 1.3 to 3.39 seconds and then increases again after 3.39 seconds.

5. $p(t) = \frac{230t}{t^2+6t+9} \Rightarrow p'(t) = \frac{230(t^2+6t+9)-(2t+6)230t}{(t^2+6t+9)^2} = \frac{230(t^2+6t+9-2t^2-6t)}{(t^2+6t+9)^2} = \frac{230(9-t^2)}{(t^2+6t+9)^2} = \frac{230(3-t)(3+t)}{(t+3)^4} = \frac{230(3-t)}{(t+3)^3}$. Thus the critical points are ±3. Perform the number line test and determine that there is a maximum at 3. There is no extreme value at -3 but this time is not relevant in the context of the problem anyway. Thus the concentration is maximal 3 hours after the medication is administered and the maximal percent concentration is $p(3) = \frac{115}{6} \approx 19.17$ %.

6. $R(x) = 15.22xe^{-.015x} \Rightarrow R'(x) = 15.22e^{-.015x} + 15.22xe^{-.015x}(-.015) = 15.22e^{-.015x}\left(1-.015x\right)$. Since the term in front of parenthesis is never zero, the only critical point is $1-.015x = 0 \Rightarrow x = \frac{200}{3} \approx 66.67$. Perform the number line test to show that the function is changing from increasing to decreasing at this point. Thus, there is a maximal value at this point. The maximum value is $R(66.67) \approx 373.27$. When interpreting this answer you can round the integer x-value assuming that the entire items are produced. So, the maximal revenue of \$373.27 is obtained when 67 items are sold.

- The function f is increasing over the interval $\left(0, \frac{3}{2}\right)$.
- The function f has relative maximum at the point $\left(\frac{3}{2}, \frac{27}{16}\right)$.
- The function f is decreasing over the interval $\left(\frac{3}{2}, \infty\right)$.

1d

Since f is increasing over the intervals $(-\infty, 0)$ and $\left(0, \frac{3}{2}\right)$, we can say that f is increasing over $\left(-\infty, \frac{3}{2}\right)$ despite the fact that $f'(0) = 0$ within this interval. In this case, we can observe that any secant line connecting two points within this interval will have a positive slope.

❮ Quick Check 2

❯ Quick Check 2

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example 3 Find the relative extrema of the function f given by

$$f(x) = (x - 2)^{2/3} + 1.$$

1e

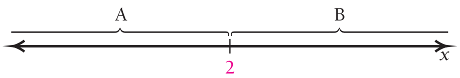
Then sketch the graph.

Solution First, we determine the critical values. To do so, we find $f'(x)$:

$$f'(x) = \frac{2}{3}(x - 2)^{-1/3}$$

$$= \frac{2}{3\sqrt[3]{x - 2}}.$$

Next, we find where $f'(x)$ does not exist or where $f'(x) = 0$. Note that $f'(x)$ does not exist at 2, although $f(x)$ does. Thus, 2 is a critical value. Since the only way for a fraction to be 0 is if its numerator is 0, we see that $f'(x) = 0$ has no solution. Thus, 2 is the only critical value. We use 2 to divide the x-axis into the intervals A, which is $(-\infty, 2)$, and B, which is $(2, \infty)$. Note that $f(2) = (2 - 2)^{2/3} + 1 = 1$.

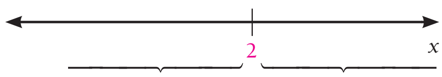


To determine the sign of the derivative, we choose a test value in each interval and substitute each value into the derivative. We choose test values 0 and 3. It is not necessary to find an exact value of the derivative; we need only determine the sign. Sometimes we can do this by just examining the formula for the derivative:

$$\text{A:} \quad \text{Test } 0, \quad f'(0) = \frac{2}{3\sqrt[3]{0 - 2}} < 0;$$

$$\text{B:} \quad \text{Test } 3, \quad f'(3) = \frac{2}{3\sqrt[3]{3 - 2}} > 0.$$

Interval



Test Value	$x = 0$	$x = 3$
Sign of $f'(x)$	$f'(0) < 0$	$f'(3) > 0$
Result	f is decreasing on $(-\infty, 2)$	f is increasing on $(2, \infty)$

Change indicates a relative minimum.

TECHNOLOGY CONNECTION

EXERCISES

In Exercises 1 and 2, consider the function f given by

$$f(x) = 2 - (x - 1)^{2/3}.$$

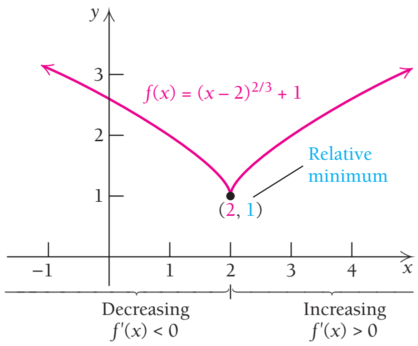
1. Graph the function using the viewing window $[-4, 6, -2, 4]$.

2. Graph the first derivative. What happens to the graph of the derivative at the critical values?

Since we have a change from decreasing to increasing, we conclude from the First-Derivative Test that a relative minimum occurs at $(2, f(2))$, or $(2, 1)$. The graph has *no* tangent line at $(2, 1)$ since $f'(2)$ does not exist.

We use the information obtained to sketch the graph. Other function values are listed in the table.

x	$f(x)$, approximately
-1	3.08
-0.5	2.84
0	2.59
0.5	2.31
1	2
1.5	1.63
2	1
2.5	1.63
3	2
3.5	2.31
4	2.59



We summarize the behavior of f:

- The function f is decreasing over the interval $(-\infty, 2)$.
- The function f has a relative minimum at the point $(2, 1)$.
- The function f is increasing over the interval $(2, \infty)$.

❯ Quick Check 3

Find the relative extrema of the function g given by $g(x) = 3 - x^{1/3}$. Then sketch the graph.

❮ Quick Check 3

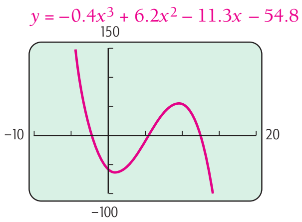
TECHNOLOGY CONNECTION

Finding Relative Extrema

To explore some methods for approximating relative extrema, let's find the relative extrema of

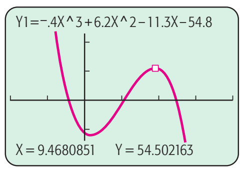
$$f(x) = -0.4x^3 + 6.2x^2 - 11.3x - 54.8.$$

We first graph the function, using a window that reveals the curvature.



Method 1: TRACE

Beginning with the window shown at left, we press TRACE and move the cursor along the curve, noting where relative extrema might occur.



A relative maximum seems to be about $y = 54.5$ at $x = 9.47$. We can refine the approximation by zooming in to obtain the following window. We press TRACE and move

(continued)

12

2. $h(x) = x^2 e^{3x}$.

<u>*solution:*</u> We first have to find the critical points:

$$h'(x) = 3x^2 e^{3x} + 2xe^{3x} = x(3x+2)e^{3x}$$

So the critical points are $x = 0, x = -\frac{2}{3}$. Notice that the e^{3x} doesn't give us any critical points because it is defined everywhere and is never 0. To fill in our real line we plug in some test points from $(-\infty, -\frac{2}{3}), (-\frac{2}{3}, 0)$ and $(0, \infty)$:

$$h'(-1) = (-1)(-3+2)e^{-3} = e^{-3} > 0$$

$$h'\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}+2\right)e^{-3/2} = -\frac{1}{4}e^{-3/2} < 0$$

$$h'(1) = (1)(3+2)e^3 = 5e^3 > 0$$

Thus we have the following

		$-\frac{2}{3}$		0	
$h'(x)$	$+$	0	$-$	0	$+$
$h(x)$	$\nearrow$		$\searrow$		$\nearrow$

So by the first derivative test we can conclude that $h(x)$ has a local max at $x = -\frac{2}{3}$ and a local min at $x = 0$.

3. $f(x) = x^{1/3} e^{x/6}$.

12

<u>*solution:*</u> Notice that this function is defined everywhere, i.e. its domain is $\mathbb{R}$. We once again find the critical points:

$$\begin{aligned} f'(x) &= x^{1/3}\left(\frac{1}{6}e^{x/6}\right) + \frac{1}{3}x^{-2/3}e^{x/6} \\ &= e^{x/6}\left(\frac{x^{1/3}}{6} + \frac{1}{3x^{2/3}}\right) \\ &= e^{x/6}\left(\frac{x}{6x^{2/3}} + \frac{2}{6x^{2/3}}\right) \\ &= e^{x/6}\left(\frac{x+2}{6x^{2/3}}\right) \end{aligned}$$

So $f'(x) = 0$ at $x = 2$ and is undefined at $x = 0$, and thus we have two critical points. To fill in our real line we plug in some test points from $(-\infty, 0), (0, 2)$ and $(2, \infty)$:

$$f'(-3) = e^{-3/6}\left(\frac{-1}{6(-3)^{2/3}}\right) < 0$$

12)

Example For $f(x) = \ln(1 - \ln x)$,
(a) find any vertical and horizontal asymptotes
(b) find the intervals of increase or decrease
(c) find any local maximum or minimum values
(d) find the intervals of concavity and any inflection points
(e) sketch the graph of $f(x)$

The goal here is to sketch the function using calculus, without the aid of a computer. We will need the derivatives, so let's get them first:

$$\begin{aligned}
f(x) &= \ln(1 - \ln x) \\
f'(x) &= \frac{d}{dx}[\ln(1 - \ln x)] \\
&= \frac{1}{1 - \ln x}\frac{d}{dx}[1 - \ln x] \;\; \text{(chain rule)} \\
&= \frac{1}{1 - \ln x}\left(-\frac{1}{x}\right) \\
&= -\frac{1}{x(1 - \ln x)} \\
f''(x) &= -\frac{d}{dx}\left[\frac{1}{x(1 - \ln x)}\right] \\
&= -\frac{x(1 - \ln x)\frac{d}{dx}[1] - 1\frac{d}{dx}[x(1 - \ln x)]}{x^2(1 - \ln x)^2} \\
&= -\frac{x(1 - \ln x)(0) - (1 - \ln x)\frac{d}{dx}[x] - x\frac{d}{dx}[(1 - \ln x)]}{x^2(1 - \ln x)^2} \\
&= -\frac{-(1 - \ln x) - x\left(-\frac{1}{x}\right)}{x^2(1 - \ln x)^2} \\
&= -\frac{-1 + \ln x + 1}{x^2(1 - \ln x)^2} \\
&= -\frac{\ln x}{x^2(1 - \ln x)^2}
\end{aligned}$$

• Horizontal Asymptotes:

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} (\ln(1 - \ln x)) \longrightarrow \ln(-\infty)$$

To get the horizontal asymptotes, we need to know what happens to our function as $x \to \infty$ and $x \to -\infty$. We tried to do that above, and ran into a problem, since $\ln(-\infty)$ is not defined. This clues us in that maybe we should look at the domain of our function before proceeding.

Since $\ln x$ is only defined for $x > 0$, we know our function must have $x > 0$ due to the red part in $f(x) = \ln(1 - \ln x)$. Also, because of the blue part of $f(x) = \ln(1 - \ln x)$, we must have that $1 - \ln x > 0$. This means

$$\begin{aligned}
1 - \ln x &> 0 \\
\ln x &< 1 \\
x &< e^1 = e
\end{aligned}$$

So the domain of our function is $0 < x < e$, and there are no horizontal asymptotes since the function is not defined outside this region.

- Vertical Asymptotes:

$$\lim_{x\to a} f(x) = \pm\infty \rightsquigarrow x = a \text{ is a vertical asymptote}$$

Our function $f(x)$ is continuous, so the only place we might have a vertical asymptote is is at the endpoints. Let's check them:

$$\begin{aligned}\lim_{x\to 0} f(x) &= \lim_{x\to 0} \ln(1-\ln x) \to \ln(1-(-\infty)) \to +\infty \\ \lim_{x\to e} f(x) &= \lim_{x\to e} \ln(1-\ln x) \to \ln(1-1) \to \ln 0 \to -\infty\end{aligned}$$

We have vertical asymptotes at both endpoints, $x = 0$ and $x = e$.

- Intervals of Increasing/Decreasing:

Solve $f'(c) = -\frac{1}{c(1-\ln c)} = 0$. This condition does not occur inside our interval. Also, $f'(x)$ exists for all x. There are no critical numbers for $f'(x)$.

Write down a table showing where $f(x)$ is increasing and decreasing:

Interval	$f'(a)$ (a is in interval)	Sign of f'	f
$(0, e)$	$f'(1) = -\frac{1}{(1)(1-\ln 1)} = -1$	$-$	decreasing

- Max/Min:

Since the function is always decreasing on $(0, e)$, there are no max or mins.

- Intervals of Concave Up/Concave Down:

Solve $f''(c) = -\frac{\ln c}{c^2(1-\ln c)^2} = 0$. The only solution is $c = +1$, since the numerator is zero there and the denominator is finite. This is the only critical number for $f''(x)$ since $f''(x)$ exists for all x.

Write down a table showing where $f(x)$ is concave up and down. We will need to use the fact that $\ln x < 0$ if $x < 1$, and $\ln x > 0$ is $x > 1$ to help us get the sign of f'' is the intervals.

Interval	$f''(a)$ (a is in interval)	Sign of f''	f
$(0, 1)$	$f''(1/2) = -\frac{\ln 1/2}{(1/2)^2(1-\ln 1/2)^2} = -\frac{-}{+} > 0$	$+$	Concave Up
$(1, e)$	$f''(3/2) = -\frac{\ln 3/2}{(3/2)^2(1-\ln 3/2)^2} = -\frac{+}{+} < 0$	$-$	Concave Down

- Points of Inflection:

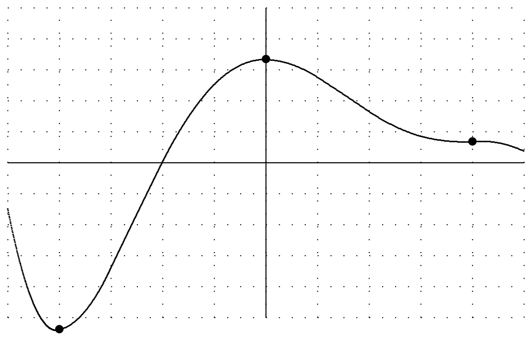
The function f goes from concave up to concave down at $x = 1 \longrightarrow$ point of inflection. $f(1) = \ln(1-\ln 1) = \ln(1-0) = 0$. Point: $(1, f(1)) = (1, 0)$ (Hey! This means $x = 1$ is a root of f!)

- Sketch: Putting everything together from our detailed analysis, we get

SOLUTION. All we need to do is pay attention to the **sign** of the derivative.

dec	r min	inc	r max	dec	none	dec
$--$	0	$+++$	0	$----$	0	$--$
	-4		0		4	

Here's one function that satisfies these conditions. Notice that $x = 4$ is not a relative extreme point.



More Examples

EXAMPLE 31.24. Let $f(x) = xe^{2x}$. Where is f increasing? Decreasing? Where does it have relative extrema?

SOLUTION. Use the Increasing/Decreasing Test. Find the derivative and the critical numbers.

$$f'(x) = e^{2x} + 2xe^{2x} = e^{2x}[1 + 2x] = 0 \quad \text{at} \quad x = -1/2.$$

Set up the number line and determine the sign of $f'(x)$ on either side of the critical point. $f'(-1) = -e^{-2} < 0$ and $f'(0) = 1$.

	decreasing	r min	increasing
f'	$---$	0	$+++$
		$-1/2$	

Using interval notation: f is increasing on $(-1/2, \infty)$ and it is decreasing on $(-\infty, -1/2)$. From the First Derivative Test, there is a relative min at $x = -1/2$.

1d

EXAMPLE 31.25. Let $f(x) = x - \sin x$. Where is f increasing? Decreasing? Where does it have relative extrema?

SOLUTION. Use the Increasing/Decreasing Test. Find the derivative and the critical numbers.

$$f'(x) = 1 - \cos x = 0 \text{ at } x = 0, \pm 2\pi, \pm 4\pi \dots.$$

Since $\cos x \leq 1$ the sign of $f'(x)$ between the critical points is always positive. So the $f(x)$ is always increasing and by the First Derivative Test and there are no relative extrema.

	increase	Not Extr	increase	Not Extr	increase	Not Extr	increase
f'	$+++$	0	$+++$	0	$+++$	0	$+++$
		-2π		0		2π	

EXAMPLE 31.26. Let $f(x) = (x^2 - 4)^{2/3}$. Where is f increasing? Decreasing? Where does it have relative extrema?

SOLUTION. Use the Increasing/Decreasing Test. Find the derivative and the critical points.

$$f'(x) = \frac{2}{3}(x^2 - 4)^{-1/3} 2x = \frac{4x}{3(x^2 - 4)^{1/3}} = 0 \text{ at } x = 0 \text{ DNE at } x = \pm 2.$$

1j

(c) $f(x) = \frac{2x}{x^2+4} \Rightarrow f'(x) = \frac{2(x^2+4)-2x(2x)}{(x^2+4)^2} = \frac{8-2x^2}{(x^2+4)^2} = \frac{2(2-x)(2+x)}{(x^2+4)^2}$. Since the denominator is never zero, the only critical points are $pm2$.

The number line test gives you that the function f is increasing on $(-2, 2)$ and decreasing on $(-\infty, 2)$ and $(2, \infty)$. At $x = 2$ there is a maximum of $\frac{1}{2}$ and at $x = -2$ there is a minimum of $\frac{-1}{2}$.

1e

(d) $f(x) = e^x(x^2 - x - 5) \Rightarrow f'(x) = e^x(x^2 - x - 5) + e^x(2x - 1) = e^x\left(x^2 - x - 5 + 2x - 1\right) = e^x(x^2 + x - 6) = e^x(x + 3)(x - 2)$. Since e^x is never zero, 2 and -3 are the only critical points. From the number line test we have that f is increasing on $(-\infty, -3)$ and $(2, \infty)$ and decreasing on $(-3, 2)$. At $x = -3$ there is a maximum of $f(-3) = 7e^{-3} \approx .348$ and at $x = 2$ there is a minimum of $f(2) = -3e^2 \approx -22.17$.

4. The object changes the direction of the motion when the distance function is changing from increasing to decreasing or vice versa. So, those times corresponds to critical points at which there are extreme values.

   $s(t) = t^3 - 7t^2 + 13t \Rightarrow v(t) = s'(t) = 3t^2 - 14t + 13$. Use the calculator program or the quadratic equation formula to determine that $3t^2 - 14t + 13 = 0 \Rightarrow t = \frac{14\pm\sqrt{40}}{6} = \frac{7\pm\sqrt{10}}{3} \Rightarrow t = 3.39$ and $t \approx 1.30$. Perform the number line test and obtain that the distance changes from forwards to backwards at 1.30 and from backwards to forwards at 3.39. 1.3 second after the movement the object is about 7.27 meters from the starting point and 3.39 seconds after the start of the movement, the object is 2.58 meters from the starting point.

   Thus the distance increases from 0 to 7.27 meters in the first 1.3 seconds, then decreases from 7.27 to 2.58 meters from 1.3 to 3.39 seconds and then increases again after 3.39 seconds.

5. $p(t) = \frac{230t}{t^2+6t+9} \Rightarrow p'(t) = \frac{230(t^2+6t+9)-(2t+6)230t}{(t^2+6t+9)^2} = \frac{230(t^2+6t+9-2t^2-6t)}{(t^2+6t+9)^2} = \frac{230(9-t^2)}{(t^2+6t+9)^2} = \frac{230(3-t)(3+t)}{(t+3)^4} = \frac{230(3-t)}{(t+3)^3}$. Thus the critical points are ± 3. Perform the number line test and determine that there is a maximum at 3. There is no extreme value at -3 but this time is not relevant in the context of the problem anyway. Thus the concentration is maximal 3 hours after the medication is administered and the maximal percent concentration is $p(3) = \frac{115}{6} \approx 19.17$ %.

6. $R(x) = 15.22xe^{-.015x} \Rightarrow R'(x) = 15.22e^{-.015x} + 15.22xe^{-.015x}(-.015) = 15.22e^{-.015x}\left(1 - .015x\right)$. Since the term in front of parenthesis is never zero, the only critical point is $1 - .015x = 0 \Rightarrow x = \frac{200}{3} \approx 66.67$. Perform the number line test to show that the function is changing from increasing to decreasing at this point. Thus, there is a maximal value at this point. The maximum value is $R(66.67) \approx 373.27$. When interpreting this answer you can round the integer x-value assuming that the entire items are produced. So, the maximal revenue of \$373.27 is obtained when 67 items are sold.

10

2. $h(x) = x^2 e^{3x}$.

<u>*solution:*</u> We first have to find the critical points:

$$h'(x) = 3x^2 e^{3x} + 2xe^{3x} = x(3x+2)e^{3x}$$

So the critical points are $x = 0, x = -\frac{2}{3}$. Notice that the e^{3x} doesn't give us any critical points because it is defined everywhere and is never 0. To fill in our real line we plug in some test points from $(-\infty, -\frac{2}{3}), (-\frac{2}{3}, 0)$ and $(0, \infty)$:

$$h'(-1) = (-1)(-3+2)e^{-3} = e^{-3} > 0$$
$$h'\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}+2\right)e^{-3/2} = -\frac{1}{4}e^{-3/2} < 0$$
$$h'(1) = (1)(3+2)e^{3} = 5e^{3} > 0$$

Thus we have the following

		$-\frac{2}{3}$		0	
$h'(x)$	$+$	0	$-$	0	$+$
$h(x)$	$\nearrow$		$\searrow$		$\nearrow$

So by the first derivative test we can conclude that $h(x)$ has a local max at $x = -\frac{2}{3}$ and a local min at $x = 0$.

3. $f(x) = x^{1/3} e^{x/6}$.

12

<u>*solution:*</u> Notice that this function is defined everywhere, i.e. its domain is $\mathbb{R}$. We once again find the critical points:

$$\begin{aligned} f'(x) &= x^{1/3}\left(\frac{1}{6}e^{x/6}\right) + \frac{1}{3}x^{-2/3}e^{x/6} \\ &= e^{x/6}\left(\frac{x^{1/3}}{6} + \frac{1}{3x^{2/3}}\right) \\ &= e^{x/6}\left(\frac{x}{6x^{2/3}} + \frac{2}{6x^{2/3}}\right) \\ &= e^{x/6}\left(\frac{x+2}{6x^{2/3}}\right) \end{aligned}$$

So $f'(x) = 0$ at $x = 2$ and is undefined at $x = 0$, and thus we have two critical points. To fill in our real line we plug in some test points from $(-\infty, 0), (0, 2)$ and $(2, \infty)$:

$$f'(-3) = e^{-3/6}\left(\frac{-1}{6(-3)^{2/3}}\right) < 0$$

Notice that the denominator is always positive (why?).

$$f'(-1) = e^{-1/6}\left(\frac{1}{6(-1)^{2/3}}\right) > 0$$

$$f'(1) = e^{1/6}\left(\frac{3}{6(1)^{2/3}}\right) > 0$$

Thus we have the following

		-2		0	
$f'(x)$	$-$	0	$+$	UN	$+$
$f(x)$	$\searrow$		$\nearrow$		$\nearrow$

So there is a local min at $x = -2$ and at $x = 0$ there is neither a max nor a min (notice that there is not a saddle point there because the derivative doesn't exist there). It is important to note that $x = 0$ is still a critical point.

THIS IS A CRUCIAL ISSUE

In the previous example, there was a critical point at $x = 0$ since f' was undefined at $x = 0$, but 0 was not in the domain of f.

$$\boxed{\text{If } x = c \text{ is not in the domain, you cannot have a critical point at } x = c.}$$

Example: Let $f(x) = \frac{1}{x}$. Then $f'(x) = -\frac{1}{x^2}$. f' is undefined at $x = 0$, but there is no critical point there since f is also undefined at $x = 0$.

Absolute Extrema

In ecomonics applications, we are generally uninterested in local maxes and mins. Instead we want to know when our function is maximal or minimal on its entire domain.

Definition 9.4 *Let f be defined on an interval I (possibly all of $\mathbb{R}$) containing c. Then f is said to have an* **absolute maximum** *or* **global maximum** *on I if $f(c) \geq f(x)$ for every x in I.*

There is an analogous statement for absolute/global minimums.

The next theorem is both deep and extremely important.

Determine at which hour is the concentration maximal and what the maximal percent concentration is.

6. A company determines that its revenue function is $R(x) = 15.22xe^{-.015x}$. Determine the production level which produces the maximal revenue and find that maximal revenue.

Solutions.

1. (a) The function is increasing on (-1, 1) and decreasing on $(-\infty, -1)$ and $(1, \infty)$. The critical points are $x = 1$ and $x = -1$ and the function has extreme values at ± 1. At 1, f has a maximum value $f(1) = 2$ and at -1 f has a minimum value $f(-1) = -2$.

   (b) The function is increasing on $(-\infty, -3)$ and $(-3, 2)$ and decreasing on $(2, \infty)$. The critical points are $x = -3$ and $x = 2$ but since the function is not define at them, there are no extreme values.

   (c) The function is increasing on $(-\infty, -2)$, (-2,-1.2) and $(0, 1)$. The function is decreasing on (-1.2, 0) and $(1, \infty)$. The critical points are -2, -1.2, 0 and 1. At -1.2 and 1, there are maximum values of $f(-1.2) = 2.8$ and $f(1) = 1$. At -2 and 0 there are no extreme values: at -2 the derivative does not change the sign and at 0 the function is not continuous because $\lim_{x\to 0} = -2$ and $f(0) = 2$. So the function never reaches the lowest value of -2 which it is approaching when $x \to 0$.

   (d) The function is increasing on $(1, \infty)$ and decreasing on $(-\infty, -1)$ and $(-1, 1)$. The critical points are -1 and 1. At 1, there is a minimum values of $f(1) = 0$. At -1 there is no extreme value since the derivative does not change the sign.

2. For increasing/decreasing intervals, look for the intervals on which the derivative is positive/ negative. (a) The derivative on the first graph is positive on $(-\infty, -2)$ and $(0, 2.5)$ so the function is increasing then. The derivative is negative on (-2, 0) and $(2.5, \infty)$ so the function is decreasing then. -2, 0 and 2.5 are critical points. At -2 and 2.5 the function is changing from increasing to decreasing so at these points there are maximum values. At 0 the function is changing from decreasing to increasing so there is a minimum value at 0.

   (b) The derivative on the first graph is positive on $(-5, -1)$ and $(2, \infty)$ so the function is increasing then. The derivative is negative on $(-\infty, -5)$ and (-1, 2) so the function is decreasing then. -5, -1 and 2 are critical points. At -1 the function is changing from increasing to decreasing so there is a maximum value at -1. At -5 and 2 the function is changing from decreasing to increasing so there are minimum values at those points.

3. (a) $f(x) = \frac{1}{3}x^3 + x^2 - 15x + 3 \Rightarrow f'(x) = x^2 + 2x - 15 = (x + 5)(x - 3)$. Setting the derivative to zero gives you the critical points $x = -5$ and $x = 3$. Perform the number line test for the derivative sign and obtain that f is increasing for $(-\infty, -5)$ and $(3, \infty)$, and decreasing for $(-5, 3)$. At 3 there is a relative minimum of -24) and at -5 a relative maximum of $\frac{184}{3}$ ≈61.33.

   (b) $f(x) = 6\sqrt[3]{(x-2)^2} \Rightarrow f'(x) = 6\frac{2}{3}(x-2)^{-1/3} = \frac{4}{\sqrt[3]{x-2}}$. f' is not defined at 2 and it is never zero so 2 is the only critical point. The derivative is changing the sign at 2 from negative to positive. Thus there is a minimum of $f(2) = 0$ at $x = 2$. f is increasing on $(2, \infty)$ and decreasing on $(-\infty, 2)$.

Now we have a mathematical problem, to maximize the function $V(x) = 4x^3 - 46x2 + 120x$, so we use existing calculus techniques, computing $V'(x) = 12x^2 - 92x + 120$ to find the critical points.

- Set $V'(x) = 0$ and solve by factoring or using the quadratic formula:

  $$V'(x) = 12x^2 - 92x + 120 = 4(3x - 5)(x - 6) = 0 \Rightarrow x = \frac{5}{3} \text{ or } x = 6$$

  so $x = \frac{5}{3}$ and $x = 6$ are critical points of V.

- $V'(x)$ is a polynomial so it is defined everywhere and there are no critical points resulting from an undefined derivative.

- What are the endpoints for x in this problem? A square cannot have a negative length, so $x \geq 0$. We cannot remove more than half of the width, so $8 - 2x \geq 0 \Rightarrow x \leq 4$. Together, these two inequalities say that $0 \leq x \leq 4$, so the endpoints are $x = 0$ and $x = 4$. (Note that the value $x = 6$ is not in this interval, so $x = 6$ cannot maximize the volume and we do not consider it further.)

The maximum volume must occur at the critical point $x = \frac{5}{3}$ or at one of the endpoints ($x = 0$ and $x = 4$): $V(0) = 0$, $V(\frac{5}{3}) = \frac{2450}{27} \approx 90.74$ cubic inches, and $V(4) = 0$, so the maximum volume of the box occurs when we remove a $\frac{5}{3}$-inch by $\frac{5}{3}$-inch square from each corner, resulting in a box $\frac{5}{3}$ inches high, $8 - 2(\frac{5}{3}) = \frac{14}{3}$ inches wide and $15 - 2(\frac{5}{3}) = \frac{35}{3}$ inches long. ◀

Practice 1. If you start with 7-inch by 15-inch pieces of tin, what size square should you remove from each corner so the box will have as large a volume as possible? [Hint: $12x^2 - 88x + 105 = (2x - 3)(6x - 35)$]

We were fortunate in the previous Example and Practice problem because the functions we created to describe the volume were functions of only one variable. In other situations, the function we get will have more than one variable, and we will need to use additional information to rewrite our function as a function of a single variable. Typically, the constraints will contain the additional information we need.

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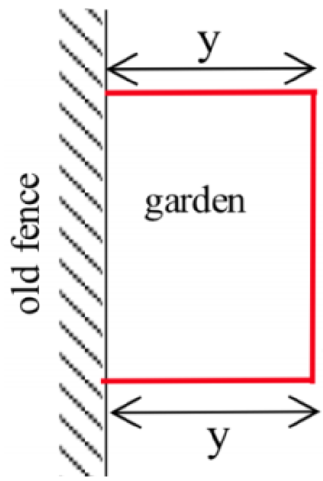
Example 2. We want to fence a rectangular area in our backyard for a garden. One side of the garden is along the edge of the yard, which is already fenced, so we only need to build a new fence along the other three sides of the rectangle (see margin). If a neighbor gives us 80 feet of fencing left over from a home-improvement project, what dimensions should the garden have in order to enclose the largest possible area using all of the available material?

Solution. As a first step toward understanding the problem, we draw a diagram or picture of the situation. Next, we identify the variables:



3

in this case, the length (call it x) and width (call it y) of the garden. The margin figure shows a labeled diagram, which we can use to write a formula for the function that we want to maximize:

$$A = \text{area of the rectangle} = (\text{length})(\text{width}) = x \cdot y$$

Unfortunately, our function A involves two variables, x and y, so we need to find a relationship between them (an equation containing both x and y) that we can solve for wither x or y. The constraint says that we have 80 feet of fencing available, so $x + 2y = 80 \Rightarrow y = 40 - \frac{x}{2}$. Then:

$$A = x \cdot y = x\left(40 - \frac{x}{2}\right) = 40x - \frac{x^2}{2}$$

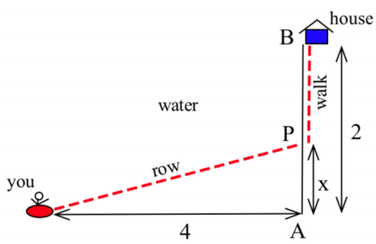
which is a function of a single variable (x). We want to maximize A.

$A'(x) = 40 - x$ so the only way $A'(x) = 0$ is to have $x = 40$, and $A(x)$ is differentiable for all x so the only critical number (other than the endpoints) is $x = 40$. Finally, $0 \le x \le 80$ (why?) so we also need to check $x = 0$ and $x = 80$: the maximum area must occur at $x = 0$, $x = 40$ or $x = 80$.

$$A(0) = 40(0) - \frac{0^2}{2} = 0 \text{ square feet}$$
$$A(40) = 40(40) - \frac{40^2}{2} = 800 \text{ square feet}$$
$$A(80) = 40(80) - \frac{80^2}{2} = 0 \text{ square feet}$$

so the largest rectangular garden has an area of 800 square feet, with dimensions $x = 40$ feet by $y = 40 - \frac{40}{2} = 20$ feet. ◀

Practice 2. Suppose you decide to create the rectangular garden in a **corner** of your yard. Then two sides of the garden are bounded by the existing fence, so you only need to use the available 80 feet of fencing to enclose the other two sides. What are the dimensions of the new garden of largest area? What are the dimensions if you have F feet of new fencing available?



Example 3. You need to reach home as quickly as possible, but you are in a rowboat on a lake 4 miles from shore and your home is 2 miles up the shore (see margin). If you can row at 3 miles per hour and walk at 5 miles per hour, toward which point on the shore should you row? What if your home is 7 miles up the coast?

Solution. The margin figure shows a labeled diagram with the variable x representing the distance along the shore from point A, the nearest point on the shore to your boat, to point P, the point you row toward.

2. **How many shirts should be produced to maximize profit?**

   <u>SOLUTION:</u> To maximize the $\mathbb{P}(x)$, we need to find the critical numbers:

   $\mathbb{P}'(x) = 21 - 0.3x^{\frac{1}{2}} = 0 \Rightarrow \sqrt{x} = \frac{21}{0.3} = 70$. Therefore, $\boxed{x = 4900}$ is the only critical number.

   Moreover, $\mathbb{P}''(x) = -0.15x^{-\frac{1}{2}} \Rightarrow \mathbb{P}''(4900) = -0.15\ 4900^{-\frac{1}{2}} = -\frac{0.15}{70} < 0$. By the second derivative test, R has a local maximum at $x = 4900$, which is an absolute maximum since it is the only critical number.

3. **At what price will the shirts be sold?**

   <u>SOLUTION:</u> The best price to maximize the profit is then: $\boxed{p(4900) = 30 - 0.2\sqrt{4900} = \$16}$.

4. **What is her resulting profit?**

   <u>SOLUTION:</u> The corresponding profit is $\boxed{\mathbb{P}(4900) = \$33800}$

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Problem 5. Farmers can get 2 dollars per bushel for their potatoes on July 1, and after that, the price drops by 2 cents per bushel per extra day. On July 1, a farmer had 80 bushels of potatoes in the field and estimates that the crop is increasing at the rate of 1 bushel per day. When should the farmer harvest the potatoes to maximize his revenue?

- <u>SOLUTION:</u>

  Let **x**= the number of extra days after July 1.

  Let **R**= the revenue = quantity × price. Then:

  $$\begin{cases} \text{price} & = \$2 - x\cdot(\$0.02) = 2 - 0.02x \text{ dollars,} \\ \text{quantity} & = \text{number of bushels} = 80 + x\cdot 1 \text{ bushel per day)} = 80 + x \text{ bushels.} \end{cases}$$

  Hence $\boxed{R(x) = (80+x)(2-0.02x) = 160 + 0.4x - 0.02x^2}$ to maximize ! Lets find the critical numbers:

  $R'(x) = 0.4 - 0.04x = 0 \Rightarrow x = \frac{0.4}{0.04} = 10$ is the only critical number.

  Moreover, $R''(x) = -0.04 \Rightarrow R''(10) = -0.04 < 0$. By the second derivative test, R has a local maximum at $\boxed{x = 10}$, which is an absolute maximum since it is the only critical number.

  The farmer should harvest the potatoes 10 extra days after July 1, so on $\boxed{\text{July 11}}$.

Problem 6. A landscape architect plans to enclose a 3000 square foot rectangular region in a botanical garden, She will use shrubs costing $\$25$ per foot along three sides and fencing costing $\$10$ per foot along the fourth side, Find the minimum total cost.

- <u>SOLUTION:</u> If the rectangular region has dimensions x and y, then its area is $A = xy = 3000ft^2$. So $\boxed{y = \frac{3000}{x}}$.

  If **y** is the side with fencing costing $10 per foot, then the cost for this side is **$ 10 y**.

  The cost for the three other sides, where shrubs costing $15 is used, is then **$ 15 (2x+y)**.

  Therefore the total cost is: $C(x) = 10y + 15(2x+y) = 30x + 25y$.

  Since $y = \frac{3000}{x}$, then $C(x) = 30x + 25\frac{3000}{x}$ that we wish to minimize.

  Since $C'(x) = 30 - 25\frac{3000}{x^2}$, then $C'(x) = 0$ for $x^2 = 25\frac{3000}{30} = 2500$. Therefore, since x is positive, we have only one critical number in the domain which is $\boxed{x = 50ft}$.

  Since $C''(x) = 25\frac{1500}{x^3}$, we have $C''(50) > 0$. Thus, by the 2^{nd} *derivative test*, C has a local minimum at $x = 50$, and therefore an absolute minimum because we have only one critical number in the domain. Hence, the minimum cost is $C(50) = \$4500$, with the dimensions $x = 50$ ft and $y = \frac{3000}{50} = 60$ ft.