31.2. L'Hôpital's rule

L'HÔPITAL'S RULE. If the limit

$$\lim \frac{f(x)}{g(x)}$$

is of indeterminate type $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)},$$

provided this last limit exists. Here, lim stands for $\lim\limits_{x\to a}$, $\lim\limits_{x\to a^\pm}$, or $\lim\limits_{x\to\pm\infty}$.

The pronunciation is lō-pē-täl. Evidently, this result is actually due to the mathematician Bernoulli rather than to l'Hôpital. The verification of l'Hôpital's rule (omitted) depends on the mean value theorem.

31.2.1 Example Find $\lim\limits_{x\to 0} \dfrac{x^2}{\sin x}$. 4a

Solution As observed above, this limit is of indeterminate type $\frac{0}{0}$, so l'Hôpital's rule applies. We have

$$\lim_{x\to 0} \frac{x^2}{\sin x} \left(\frac{0}{0}\right) \stackrel{\text{l'H}}{=} \lim_{x\to 0} \frac{2x}{\cos x} = \frac{0}{1} = 0,$$

where we have first used l'Hôpital's rule and then the substitution rule. □

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The solution of the previous example shows the notation we use to indicate the type of an indeterminate limit and the subsequent use of l'Hôpital's rule.

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31.2.2 Example Find $\displaystyle\lim_{x\to-\infty}\frac{3x-2}{e^{x^2}}$.

Solution We have

$$\lim_{x\to-\infty}\frac{3x-2}{e^{x^2}}\ \left(\frac{-\infty}{\infty}\right) \overset{\text{l'H}}{=} \lim_{x\to-\infty}\frac{3}{e^{x^2}(2x)} \qquad \left(\frac{3}{\text{large neg.}}\right)$$
$$= 0.$$

□

31.3. Common mistakes

Here are two pitfalls to avoid:

- L'Hôpital's rule should not be used if the limit is not indeterminate (of the appropriate type). For instance, the following limit is *not* indeterminate; in fact, the substitution rule applies to give the limit:

$$\lim_{x\to 0}\frac{\sin x}{x+1}=\frac{0}{1}=0.$$

An application of l'Hôpital's rule gives the wrong answer:

$$\lim_{x\to 0}\frac{\sin x}{x+1}\overset{\text{l'H}}{=}\lim_{x\to 0}\frac{\cos x}{1}=\frac{1}{1}=1 \qquad \text{(wrong)}.$$

- Although l'Hôpital's rule involves a quotient $f(x)/g(x)$ as well as derivatives, the quotient rule of differentiation is not involved. The expression in l'Hôpital's rule is

$$\frac{f'(x)}{g'(x)} \qquad \text{and not} \qquad \left(\frac{f(x)}{g(x)}\right)'.$$

31.4. Examples

1c **31.4.1 Example** Find $\displaystyle\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$.

Solution We have

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\ \left(\frac{0}{0}\right) \overset{\text{l'H}}{=} \lim_{\theta\to 0}\frac{\cos\theta}{1} = \frac{1}{1} = 1.$$

(In 19, we had to work pretty hard to determine this important limit. It is tempting to go back and replace that argument with this much easier one, but unfortunately we used this limit to derive the formula for the derivative of $\sin\theta$, which is used here in the application of l'Hôpital's rule, so that would make for a circular argument.) □

Sometimes repeated use of l'Hôpital's rule is called for:

1d **31.4.2 Example** Find $\displaystyle\lim_{x\to\infty}\frac{3x^2+x+4}{5x^2+8x}$.

Solution We have

$$\lim_{x\to\infty}\frac{3x^2+x+4}{5x^2+8x}\ \left(\frac{\infty}{\infty}\right) \overset{\text{l'H}}{=} \lim_{x\to\infty}\frac{6x+1}{10x+8}\ \left(\frac{\infty}{\infty}\right) \overset{\text{l'H}}{=} \lim_{x\to\infty}\frac{6}{10} = \frac{6}{10} = \frac{3}{5}.$$

$\square$

For the limit at infinity of a rational function (i.e., polynomial over polynomial) as in the preceding example, we also have the method of dividing numerator and denominator by the highest power of the variable in the denominator (see 12). That method is probably preferable to using l'Hôpital's rule repeatedly, especially if the degrees of the polynomials are large. Sometimes though, we have no alternate approach:

1e **31.4.3 Example** Find $\lim_{x \to 0} \dfrac{e^x - 1 - x - x^2/2}{x^3}$.

Solution We have

$$\begin{aligned}\lim_{x \to 0} \frac{e^x - 1 - x - x^2/2}{x^3} \left(\frac{0}{0}\right) &\overset{\text{l'H}}{=} \lim_{x \to 0} \frac{e^x - 1 - x}{3x^2} \left(\frac{0}{0}\right) \\ &\overset{\text{l'H}}{=} \lim_{x \to 0} \frac{e^x - 1}{6x} \left(\frac{0}{0}\right) \\ &\overset{\text{l'H}}{=} \lim_{x \to 0} \frac{e^x}{6} = \frac{e^0}{6} = \frac{1}{6}.\end{aligned}$$

$\square$

There are other indeterminate types, to which we now turn. The strategy for each is to transform the limit into either type $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ and then use l'Hôpital's rule.

Math 112 (71) Fall 2010 | **Examples 1 - 9 (L'Hopital's Rule) Problems & Solutions**

Example 1

Evaluate the limit $\lim\limits_{x\to 3}\dfrac{x^2+x-12}{x^2-9}$ using

(a) algebraic manipulation (factor and cancel)

Solution

$$\lim_{x\to 3}\frac{x^2+x-12}{x^2-9}=\lim_{x\to 3}\frac{(x-3)(x+4)}{(x-3)(x+3)}=\lim_{x\to 3}\frac{x+4}{x+3}=\frac{7}{6}$$

(b) L'Hopital's Rule

Solution

Since direct substitution gives $\frac{0}{0}$ we can use L'Hopital's Rule to give

$$\lim_{x\to 3}\frac{x^2+x-12}{x^2-9}\overset{H}{=}\lim_{x\to 3}\frac{2x+1}{2x}=\frac{7}{6}$$

Example 2

Evaluate the limit $\lim\limits_{x\to 0}\dfrac{\sin 3x}{\tan 4x}$ using

(a) the basic trigonometric limit $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$ together with appropriate changes of variables

Solution

Write the limit as

$$\lim_{x\to 0}\frac{\sin 3x}{\tan 4x}=\left(\lim_{x\to 0}\frac{\sin 3x}{x}\right)\left(\lim_{x\to 0}\frac{x\cos 4x}{\sin 4x}\right)$$

In the first limit let $u=3x$ and in the second let $v=4x$. Then the limit is

$$\begin{aligned}\lim_{x\to 0}\frac{\sin 3x}{\tan 4x}&=\left(\lim_{u\to 0}\frac{3\sin u}{u}\right)\left(\lim_{v\to 0}\frac{v\cos v}{4\sin v}\right)\\&=\tfrac{3}{4}\left(\lim_{u\to 0}\frac{\sin u}{u}\right)\left(\lim_{v\to 0}\frac{v}{\sin v}\right)\left(\lim_{v\to 0}\cos v\right)=\tfrac{3}{4}\end{aligned}$$

(b) L'Hopital's Rule

Solution

Since direct substitution gives $\frac{0}{0}$ we can use L'Hopital's Rule to give

$$\lim_{x\to 0}\frac{\sin 3x}{\tan 4x}\overset{H}{=}\lim_{x\to 0}\frac{3\cos 3x}{4\sec^2 4x}=\tfrac{3}{4}$$

The solution of the previous example shows the notation we use to indicate the type of an indeterminate limit and the subsequent use of l'Hôpital's rule.

1b **31.2.2 Example** Find $\lim\limits_{x\to-\infty} \dfrac{3x-2}{e^{x^2}}$.

Solution We have

$$\lim_{x\to-\infty} \frac{3x-2}{e^{x^2}} \left(\frac{-\infty}{\infty}\right) \stackrel{\text{l'H}}{=} \lim_{x\to-\infty} \frac{3}{e^{x^2}(2x)} \qquad \left(\frac{3}{\text{large neg.}}\right)$$
$$= 0.$$

$\square$

31.3. Common mistakes

Here are two pitfalls to avoid:

1a

- L'Hôpital's rule should not be used if the limit is not indeterminate (of the appropriate type). For instance, the following limit is *not* indeterminate; in fact, the substitution rule applies to give the limit:

$$\lim_{x\to 0} \frac{\sin x}{x+1} = \frac{0}{1} = 0.$$

An application of l'Hôpital's rule gives the wrong answer:

$$\lim_{x\to 0} \frac{\sin x}{x+1} \stackrel{\text{l'H}}{=} \lim_{x\to 0} \frac{\cos x}{1} = \frac{1}{1} = 1 \qquad \text{(wrong)}.$$

Example 1

1 f Evaluate the limit $\lim\limits_{x\to 3} \dfrac{x^2+x-12}{x^2-9}$ using

(a) algebraic manipulation (factor and cancel)

Solution

$$\lim_{x\to 3} \frac{x^2+x-12}{x^2-9} = \lim_{x\to 3} \frac{(x-3)(x+4)}{(x-3)(x+3)} = \lim_{x\to 3} \frac{x+4}{x+3} = \frac{7}{6}$$

(b) L'Hopital's Rule

Solution

Since direct substitution gives $\frac{0}{0}$ we can use L'Hopital's Rule to give

$$\lim_{x\to 3} \frac{x^2+x-12}{x^2-9} \overset{H}{=} \lim_{x\to 3} \frac{2x+1}{2x} = \frac{7}{6}$$

Example 2

1h Evaluate the limit $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\tan 4x}$ using

(a) the basic trigonometric limit $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$ together with appropriate changes of variables

Solution

Write the limit as

$$\lim_{x\to 0} \frac{\sin 3x}{\tan 4x} = \left(\lim_{x\to 0} \frac{\sin 3x}{x}\right)\left(\lim_{x\to 0} \frac{x\cos 4x}{\sin 4x}\right)$$

In the first limit let $u = 3x$ and in the second let $v = 4x$. Then the limit is

$$\begin{aligned}\lim_{x\to 0} \frac{\sin 3x}{\tan 4x} &= \left(\lim_{u\to 0} \frac{3\sin u}{u}\right)\left(\lim_{v\to 0} \frac{v\cos v}{4\sin v}\right)\\ &= \tfrac{3}{4}\left(\lim_{u\to 0} \frac{\sin u}{u}\right)\left(\lim_{v\to 0} \frac{v}{\sin v}\right)\left(\lim_{v\to 0} \cos v\right) = \tfrac{3}{4}\end{aligned}$$

(b) L'Hopital's Rule

Solution

Since direct substitution gives $\frac{0}{0}$ we can use L'Hopital's Rule to give

$$\lim_{x\to 0} \frac{\sin 3x}{\tan 4x} \overset{H}{=} \lim_{x\to 0} \frac{3\cos 3x}{4\sec^2 4x} = \tfrac{3}{4}$$

Example 3

1i

Evaluate the limit $\lim\limits_{x\to\frac{\pi}{2}} \left(x - \frac{\pi}{2}\right)\tan x$ using L'Hopital's Rule.

Solution

Write the limit as

$$\lim_{x\to\frac{\pi}{2}} \left(x - \tfrac{\pi}{2}\right)\tan x = \lim_{x\to\frac{\pi}{2}} \frac{x - \frac{\pi}{2}}{\cot x}$$

Then direct substitution gives $\frac{0}{0}$ so we can use L'Hopital's Rule to give

$$\lim_{x\to\frac{\pi}{2}} \left(x - \tfrac{\pi}{2}\right)\tan x \overset{\text{H}}{=} \lim_{x\to\frac{\pi}{2}} \frac{1}{(-\csc^2 x)} = -1$$

Example 4

1j

Evaluate the limit $\lim\limits_{x\to 1} \dfrac{\sqrt{2-x} - x}{x-1}$ using L'Hopital's Rule.

Solution

Since direct substitution gives $\frac{0}{0}$ use L'Hopital's Rule to give

$$\lim_{x\to 1} \frac{\sqrt{2-x} - x}{x-1} \overset{\text{H}}{=} \lim_{x\to 1} \frac{-\dfrac{1}{\sqrt{2-x}} - 1}{1} = -\frac{3}{2}$$

Note that this result can also be obtained by rationalizing the numerator by multiplying top and bottom by the root conjugate $\sqrt{2-x} + x$.

Example 5

1k

Evaluate the limit $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2}$ using L'Hopital's Rule.

Solution

Since direct substitution gives $\frac{0}{0}$ use L'Hopital's Rule to give

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} \overset{\text{H}}{=} \lim_{x\to 0} \frac{\sin x}{2x}$$

Again direct substitution gives $\frac{0}{0}$ so use L'Hopital's Rule a second time to give

$$\lim_{x\to 0} \frac{1-\cos x}{x^2} \overset{\text{H}}{=} \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$$

Example 6

Evaluate the limits at infinity

(a) $\displaystyle\lim_{x\to\infty} \frac{e^x}{x}$

1e

Solution

Direct "substitution" gives $\frac{\infty}{\infty}$ so we can use L'Hopital's Rule to give

$$\lim_{x\to\infty} \frac{e^x}{x} \overset{H}{=} \lim_{x\to\infty} \frac{e^x}{1} = \infty$$

(b) $\displaystyle\lim_{x\to\infty} x^2 e^{-x}$

Solution

Write the limit as

$$\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x}$$

Then direct "substitution" gives $\frac{\infty}{\infty}$ so we can use L'Hopital's Rule to give

$$\lim_{x\to\infty} x^2 e^{-x} \overset{H}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{H}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$$

(c) $\displaystyle\lim_{x\to\infty} \left(\sqrt{x^2+x+1} - x\right)$

Solution

Direct "substitution" gives the indeterminate form $\infty - \infty$. Write the limit as

$$\lim_{x\to\infty} \left(\sqrt{x^2+x+1} - x\right) = \lim_{x\to\infty} x\left(\sqrt{1+\frac{1}{x}+\frac{1}{x^2}} - 1\right) = \lim_{x\to\infty} \frac{\left(\sqrt{1+\frac{1}{x}+\frac{1}{x^2}} - 1\right)}{\frac{1}{x}}$$

Now direct "substitution" gives $\frac{\infty}{\infty}$ so we can use L'Hopital's Rule to give

$$\lim_{x\to\infty} \left(\sqrt{x^2+x+1} - x\right) \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{2}\left(1+\frac{1}{x}+\frac{1}{x^2}\right)^{-1/2}\left(-\frac{1}{x^2}-\frac{2}{x^3}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{1+\frac{2}{x}}{2\left(1+\frac{1}{x}+\frac{1}{x^2}\right)^{1/2}} = \frac{1}{2}$$

Example 7

Evaluate the limit $\displaystyle\lim_{x\to 2^+} \frac{\ln(x-2)}{\ln(x^2-4)}$ using L'Hopital's Rule.

1m

Solution

Direct substitution gives $\frac{\infty}{\infty}$ so we can use L'Hopital's Rule

$$\lim_{x\to 2^+} \frac{\ln(x-2)}{\ln(x^2-4)} \overset{H}{=} \lim_{x\to 2^+} \frac{\frac{1}{x-2}}{\frac{2x}{x^2-4}} = \lim_{x\to 2^+} \frac{x^2-4}{2x(x-2)} = \lim_{x\to 2^+} \frac{x^2-4}{2x^2-4x}$$

$$\overset{H}{=} \lim_{x\to 2^+} \frac{2x}{4x-4} = 1$$

MATH 1010E University Mathematics
Lecture Notes (week 8)
Martin Li

1 L'Hospital's Rule

Another useful application of mean value theorems is *L'Hospital's Rule*. It helps us to evaluate limits of "*indeterminate forms*" such as $\frac{0}{0}$. Let's look at the following example. Recall that we have proved in week 3 (using the sandwich theorem and a geometric argument)

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

We say that the limit above has indeterminate form $\frac{0}{0}$ since both the numerator and denominator goes to 0 as $x \to 0$. Roughly speaking, L'Hospital's rule says that under such situation, we can differentiate the numerator and denominator first and then take the limit. The result, if exists, should be equal to the original limit. For example,

$$\lim_{x\to 0} \frac{(\sin x)'}{(x)'} = \lim_{x\to 0} \frac{\cos x}{1} = 1,$$

which is equal to the limit before we differentiate!

Theorem 1.1 (L'Hospital's Rule) *Let $f, g : (a, b) \to \mathbb{R}$ be differentiable functions in (a, b) and fix an $x_0 \in (a, b)$. Assume that*

(i) $f(x_0) = 0 = g(x_0)$.

(ii) $\lim_{x\to x_0} \frac{f'(x)}{g'(x)} = L$ *(i.e. the limit exists and is finite).*

Then, we have

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)} = L.$$

Example 1.2 *Consider the limit*

$$\lim_{x\to 0} \frac{\sin^2 x}{1 - \cos x},$$

this is a limit of indeterminate form $\frac{0}{0}$. Therefore, we can apply L'Hospital's Rule to obtain

$$\lim_{x\to 0} \frac{\sin^2 x}{1 - \cos x} = \lim_{x\to 0} \frac{(\sin^2 x)'}{(1 - \cos x)'},$$

1h

if the limit on the right hand side exists. Since the right hand side is the same as

$$\lim_{x\to 0}\frac{2\sin x\cos x}{\sin x} = \lim_{x\to 0}(2\cos x) = 2.$$

Therefore, we conclude that $\lim_{x\to 0}\frac{\sin^2 x}{1-\cos x} = 2$*.*

Exercise: Calculate the limit in Example 1.2 without using L'Hospital's Rule (hint: $\sin^2 x = 1-\cos^2 x$).

Sometimes we have to apply L'Hospital's Rule a few times before we can evaluate the limit directly. This is illustrated by the following two examples.

1σ

Example 1.3 *Consider the limit*

$$\lim_{x\to 0}\frac{x-\sin x}{x^3},$$

this is of the form "$\frac{0}{0}$*". Therefore, by L'Hospital's rule*

$$\lim_{x\to 0}\frac{x-\sin x}{x^3} = \lim_{x\to 0}\frac{1-\cos x}{3x^2},$$

if the right hand side exists. The right hand side is still in the form "$\frac{0}{0}$*", therefore we can apply L'Hospital's Rule again*

$$\lim_{x\to 0}\frac{1-\cos x}{3x^2} = \lim_{x\to 0}\frac{\sin x}{6x},$$

if the right hand side exists. But now the right hand side can be evaluated:

$$\lim_{x\to 0}\frac{\sin x}{6x} = \frac{1}{6}\lim_{x\to 0}\frac{\sin x}{x} = \frac{1}{6}.$$

As a result, if we trace backwards, we conclude that the original limit exists and

$$\lim_{x\to 0}\frac{x-\sin x}{x^3} = \frac{1}{6}.$$

Example 1.4 *Consider the limit*

$$\lim_{x\to 0}\frac{e^x-x-1}{1-\cosh x}.$$

Applying L'Hospital's Rule twice, we can argue as in Example 1.3 that

$$\lim_{x\to 0}\frac{e^x-x-1}{1-\cosh x} = \lim_{x\to 0}\frac{e^x-1}{-\sinh x} = \lim_{x\to 0}\frac{e^x}{-\cosh x} = \frac{1}{-1} = -1.$$

31.5. Indeterminate product

2a

Type $\infty \cdot 0$. The limit

$$\lim_{x\to\infty} xe^{-x} \quad (\infty \cdot 0)$$

cannot be determined by using inspection. The first factor going to ∞ is trying to make the expression large, while the second factor going to 0 is trying to make the expression small. There is a struggle going on. We say that this limit is indeterminate of type $\infty \cdot 0$.

The strategy for handling this type is to rewrite the product as a quotient and then use l'Hôpital's rule.

31.5.1 Example Find $\lim\limits_{x\to\infty} xe^{-x}$.

Solution As noted above, this limit is indeterminate of type $\infty \cdot 0$. We rewrite the expression as a fraction and then use l'Hôpital's rule:

$$\begin{aligned} \lim_{x\to\infty} xe^{-x} &= \lim_{x\to\infty} \frac{x}{e^x} \quad \left(\frac{\infty}{\infty}\right) \\ &\overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{1}{e^x} \quad \left(\frac{1}{\text{large pos.}}\right) \\ &= 0. \end{aligned}$$

□

2b

31.5.2 Example Find $\lim\limits_{x\to 0^+} (\cot 2x)(\sin 6x)$.

2b *Solution* First

$$\lim_{x\to 0^+} \cot 2x = \lim_{x\to 0^+} \frac{\cos 2x}{\sin 2x} \quad \left(\frac{\text{about } 1}{\text{small pos.}}\right)$$
$$= \infty.$$

Therefore, the given limit is indeterminate of type $\infty \cdot 0$. We rewrite as a fraction and then use l'Hôpital's rule:

$$\lim_{x\to 0^+} (\cot 2x)(\sin 6x) = \lim_{x\to 0^+} \frac{\sin 6x}{\tan 2x} \quad \left(\frac{0}{0}\right)$$
$$\overset{\text{l'H}}{=} \lim_{x\to 0^+} \frac{6\cos 6x}{2\sec^2 2x}$$
$$= \frac{6}{2} = 3.$$

$\square$

31.6. Indeterminate difference

Type $\infty - \infty$. The substitution rule cannot be used on the limit

$$\lim_{x\to \frac{\pi}{2}^-} (\tan x - \sec x) \quad (\infty - \infty)$$

since $\tan \pi/2$ is undefined. Nor can one determine this limit by using inspection. The first term going to infinity is trying to make the expression large and positive, while the second term going to negative infinity is trying to make the expression large and negative. There is a struggle going on. We say that this limit is indeterminate of type $\infty - \infty$.

The strategy for handling this type is to combine the terms into a single fraction and then use l'Hôpital's rule.

2e

31.6.1 Example Find $\lim_{x \to \frac{\pi}{2}^-} (\tan x - \sec x)$.

Solution As observed above, this limit is indeterminate of type $\infty - \infty$. We combine the terms and then use l'Hôpital's rule:

$$\begin{aligned}
\lim_{x \to \frac{\pi}{2}^-} (\tan x - \sec x) &= \lim_{x \to \frac{\pi}{2}^-} \left(\frac{\sin x}{\cos x} - \frac{1}{\cos x} \right) \\
&= \lim_{x \to \frac{\pi}{2}^-} \frac{\sin x - 1}{\cos x} \qquad \left(\frac{0}{0} \right) \\
&\overset{\text{l'H}}{=} \lim_{x \to \frac{\pi}{2}^-} \frac{\cos x}{-\sin x} \\
&= \frac{0}{-1} = 0.
\end{aligned}$$

□

2el

31.6.2 Example Find $\lim_{x \to 1^+} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$.

Solution The limit is indeterminate of type $\infty - \infty$. We combine the terms and then use

2d)

l'Hôpital's rule:

$$\begin{aligned}\lim_{x\to 1^+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right) &= \lim_{x\to 1^+}\frac{x\ln x - x + 1}{(x-1)\ln x} \qquad \left(\frac{0}{0}\right)\\ &\stackrel{\text{l'H}}{=} \lim_{x\to 1^+}\frac{\ln x + x(1/x) - 1}{\ln x + (x-1)(1/x)}\\ &= \lim_{x\to 1^+}\frac{\ln x}{\ln x + 1 - 1/x} \qquad \left(\frac{0}{0}\right)\\ &\stackrel{\text{l'H}}{=} \lim_{x\to 1^+}\frac{1/x}{1/x + 1/x^2}\\ &= \frac{1}{2}.\end{aligned}$$

□

31.7. Indeterminate powers

Type ∞^0. The limit

$$\lim_{x\to\infty} x^{1/x} \quad (\infty^0)$$

cannot be determined by using inspection. The base going to infinity is trying to make the expression large, while the exponent going to 0 is trying to make the expression equal to 1. There is a struggle going on. We say that this limit is indeterminate of type ∞^0.

The strategy for handling this type (as well as the types 1^∞ and 0^0 yet to be introduced) is to first find the limit of the natural logarithm of the expression (ultimately using l'Hôpital's rule) and then use an inverse property of logarithms to get the original limit.

31.7.1 Example Find $\lim_{x\to\infty} x^{1/x}$.

Solution As was noted above, this limit is of indeterminate type ∞^0. We first find the limit of the natural logarithm of the given expression:

$$\begin{aligned}\lim_{x\to\infty} \ln x^{1/x} &= \lim_{x\to\infty} (1/x)\ln x \quad (0\cdot\infty)\\ &= \lim_{x\to\infty} \frac{\ln x}{x} \quad \left(\frac{\infty}{\infty}\right)\\ &\overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{1/x}{1}\\ &= 0.\end{aligned}$$

Therefore,

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\ln x^{1/x}} = e^0 = 1,$$

where we have used the inverse property of logarithms $y = e^{\ln y}$ and then the previous computation. (The next to the last equality also uses continuity of the exponential function.) □

Type 1^∞. The limit

2e)

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \quad (1^\infty)$$

cannot be determined by using inspection. The base going to 1 is trying to make the expression equal to 1, while the exponent going to infinity is trying to make the expression go to ∞ (raising a number greater than 1 to ever higher powers produces ever larger results). We say that this limit is indeterminate of type 1^∞.

2e

31.7.2 Example Find $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

Solution As was noted above, this limit is indeterminate of type 1^∞. We first find the limit of the natural logarithm of the given expression:

$$\begin{aligned}
\lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^x &= \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)\\
&= \lim_{x\to\infty}\frac{\ln(1+1/x)}{x^{-1}} \qquad \left(\frac{0}{0}\right)\\
&\overset{\text{l'H}}{=} \lim_{x\to\infty}\frac{\frac{1}{1+1/x}(-x^{-2})}{-x^{-2}}\\
&= \lim_{x\to\infty}\frac{1}{1+1/x} = 1.
\end{aligned}$$

Therefore,

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{\ln\left(1+\frac{1}{x}\right)^x} = e^1 = e.$$

□

Type 0^0. The limit

$$\lim_{x\to 0}(\tan x)^x \qquad (0^0)$$

cannot be determined by using inspection. The base going to 0 is trying to make the expression small, while the exponent going to 0 is trying to make the expression equal to 1. We say that this limit is indeterminate of type 0^0.

Example 6

Evaluate the limits at infinity

(a) $\lim\limits_{x\to\infty} \dfrac{e^x}{x}$

1e

Solution

Direct "substitution" gives $\frac{\infty}{\infty}$ so we can use L'Hopital's Rule to give

$$\lim_{x\to\infty} \frac{e^x}{x} \overset{H}{=} \lim_{x\to\infty} \frac{e^x}{1} = \infty$$

(b) $\lim\limits_{x\to\infty} x^2 e^{-x}$

2f)

Solution

Write the limit as

$$\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x}$$

Then direct "substitution" gives $\frac{\infty}{\infty}$ so we can use L'Hopital's Rule to give

$$\lim_{x\to\infty} x^2 e^{-x} \overset{H}{=} \lim_{x\to\infty} \frac{2x}{e^x} \overset{H}{=} \lim_{x\to\infty} \frac{2}{e^x} = 0$$

(c) $\lim\limits_{x\to\infty} \left(\sqrt{x^2+x+1} - x\right)$

Solution

Direct "substitution" gives the indeterminate form $\infty - \infty$. Write the limit as

$$\lim_{x\to\infty} \left(\sqrt{x^2+x+1} - x\right) = \lim_{x\to\infty} x\left(\sqrt{1+\frac{1}{x}+\frac{1}{x^2}} - 1\right) = \lim_{x\to\infty} \frac{\left(\sqrt{1+\frac{1}{x}+\frac{1}{x^2}} - 1\right)}{\frac{1}{x}}$$

Now direct "substitution" gives $\frac{\infty}{\infty}$ so we can use L'Hopital's Rule to give

$$\lim_{x\to\infty} \left(\sqrt{x^2+x+1} - x\right) \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{2}\left(1+\frac{1}{x}+\frac{1}{x^2}\right)^{-1/2}\left(-\frac{1}{x^2}-\frac{2}{x^3}\right)}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{1+\frac{2}{x}}{2\left(1+\frac{1}{x}+\frac{1}{x^2}\right)^{1/2}} = \frac{1}{2}$$

Example 7

Evaluate the limit $\lim\limits_{x\to 2^+} \dfrac{\ln(x-2)}{\ln(x^2-4)}$ using L'Hopital's Rule.

1m

Solution

Direct substitution gives $\frac{\infty}{\infty}$ so we can use L'Hopital's Rule

$$\lim_{x\to 2^+} \frac{\ln(x-2)}{\ln(x^2-4)} \overset{H}{=} \lim_{x\to 2^+} \frac{\frac{1}{x-2}}{\frac{2x}{x^2-4}} = \lim_{x\to 2^+} \frac{x^2-4}{2x(x-2)} = \lim_{x\to 2^+} \frac{x^2-4}{2x^2-4x}$$

$$\overset{H}{=} \lim_{x\to 2^+} \frac{2x}{4x-4} = 1$$

$$\frac{\infty}{\infty} = \frac{1/0}{1/0} = \frac{0}{0}.,$$

$$0^0 = \exp(0 \ln 0) = \exp(0 \cdot (-\infty)) = \exp(-\frac{0}{0}).$$

We should emphasize that the "calculations" above are just formal. They indicate the general idea of transforming the limits rather than actual arithmetic of numbers. Using these ideas, we can actually handle all the determinate forms in (2.1) by the L'Hospital's Rule. We have

Theorem 2.2 (L'Hospital's Rule) *The same conclusion holds if we replace (i) by*

$$\lim_{x \to x_0} f(x) = \pm\infty = \lim_{x \to x_0} g(x).$$

Remark 2.3 *The theorem also holds in the case $x_0 = \pm\infty$ and for one-sided limits as well.*

We postpone the proof of Theorem 2.2 until the end of this section but we will first look at a few applications.

Example 2.4 *Consider the one-side limit*

$$\lim_{x \to 0^+} x \ln x.$$

This is of the form $0 \cdot (-\infty)$. However, we can rewrite it as

$$x \ln x = \frac{\ln x}{1/x},$$

which is of the form $\frac{-\infty}{\infty}$ as $x \to 0^+$. Therefore, we can apply Theorem 2.2 to conclude that

$$\lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0.$$

Therefore, we have $\lim_{x \to 0^+} x \ln x = 0$. In words, this means that as $x \to 0^+$, the linear function x is going to 0 faster than the logarithm function $\ln x$ going to $-\infty$.

Example 2.5 *Sometimes we have to apply L'Hospital's Rule a few times. For example,*

$$\lim_{x \to +\infty} \frac{x^2}{e^x} = \lim_{x \to +\infty} \frac{2x}{e^x} = \lim_{x \to +\infty} \frac{2}{e^x} = 0.$$

Similarly, we can prove that

$$\lim_{x \to +\infty} \frac{x^k}{e^x} = 0, \qquad \textit{for any } k.$$

In other words, as $x \to +\infty$, the exponential function e^x is going to ∞ faster than any polynomial of x.

The following example shows that L'Hospital's Rule may not always work:

$$\lim_{x \to \infty} \frac{\sinh x}{\cosh x} = \lim_{x \to \infty} \frac{\cosh x}{\sinh x} = \lim_{x \to \infty} \frac{\sinh x}{\cosh x},$$

which gets back to the original limit we want to evaluate! So L'Hospital's Rule leads us nowhere in such situation. For this example, we have to do some cancellations first,

$$\lim_{x \to \infty} \frac{\sinh x}{\cosh x} = \lim_{x \to \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x \to \infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = 1.$$

3 Some tricky examples of L'Hospital's Rule

Sometimes it is not very obvious how we should transform a limit into a "standard" indeterminate form.

Example 3.1 *Evaluate that limit*

20

$$\lim_{x \to \infty} x \sin \frac{1}{x}.$$

We can choose to transform it to either

$$x \sin \frac{1}{x} = \frac{\sin(1/x)}{1/x} \qquad \text{or} \qquad x \sin \frac{1}{x} = \frac{x}{1/\sin(1/x)}.$$

The first one has the form "$\frac{0}{0}$" and the second one has the form "$\frac{\infty}{\infty}$" as $x \to \infty$. Therefore, we can apply L'Hospital's Rule to both cases. For the first case, we have

$$\lim_{x \to \infty} \frac{\sin(1/x)}{1/x} = \lim_{x \to \infty} \frac{-\frac{1}{x^2} \cos \frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to \infty} \cos \frac{1}{x} = 1.$$

However, for the second case, we have

$$\lim_{x \to \infty} \frac{x}{1/\sin(1/x)} = \lim_{x \to \infty} \frac{1}{\frac{1}{x^2} \frac{\cos(1/x)}{\sin^2(1/x)}},$$

which doesn't seem to simplify after L'Hospital's Rule. Therefore, sometimes we have to choose a good way to transform the limit before we apply the L'Hospital's Rule. A general rule of thumb here is that the expression should get simpler after taking the derivatives.

Example 3.2 *Evaluate the limit*

2i)

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right).$$

This limit has the indeterminate form "$\infty-\infty$", which we haven't mentioned. There is in fact no general way to evaluate limits of such forms. But for this particular example, we can transform it as

$$\frac{1}{\sin x}-\frac{1}{x}=\frac{x-\sin x}{x\sin x},$$

which has the standard indeterminate form "$\frac{0}{0}$". Therefore, we can apply L'Hospital's Rule a few times to get

$$\lim_{x\to 0}\frac{x-\sin x}{x\sin x}=\lim_{x\to 0}\frac{1-\cos x}{\sin x+x\cos x}=\lim_{x\to 0}\frac{\sin x}{2\cos x-x\sin x}=0.$$

Example 3.3 *Evaluate the limit*

2j

$$\lim_{x\to\infty} x^{\frac{1}{x}}.$$

Recall that if $a>0, b$ are *real* numbers, we define $a^b := \exp(b\ln a)$. Therefore,

$$\lim_{x\to\infty} x^{\frac{1}{x}}=\lim_{x\to\infty}\exp\left(\frac{1}{x}\ln x\right)=\exp\left(\lim_{x\to\infty}\frac{\ln x}{x}\right)=\exp\left(\lim_{x\to\infty}\frac{1/x}{1}\right)=e^0=1.$$

Note that we can move the limit into the function "exp" since the exponential function "exp" is continuous.

We end this section with a proof of Theorem 2.2.

Proof of Theorem 2.2: The idea is that if $f(x_0)=\pm\infty=g(x_0)$, then we have $\frac{1}{f(x_0)}=0=\frac{1}{g(x_0)}$. Therefore, we can apply L'Hospital's Rule to conclude that

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\lim_{x\to x_0}\frac{1/g(x)}{1/f(x)}=\lim_{x\to x_0}\frac{-g'(x)/g(x)^2}{-f'(x)/f(x)^2}.$$

2

Solution: L'Hopital's Rule gives $\lim\limits_{x\to\infty} \dfrac{\ln x}{x^2} = \lim\limits_{x\to\infty} \dfrac{(1/x)\cdot x^2 - \ln x \cdot 2x}{(x^2)^2}$

$= \lim\limits_{x\to\infty} \dfrac{x - 2x\ln x}{x^4} = \lim\limits_{x\to\infty} \dfrac{1 - 2\ln x}{x^3} = ???$ (The solver is stuck because this expression is similar to but more complicated than the original problem.)

The Error: The solver used the quotient rule rather than L'Hopital's Rule.

A Correct Solution: Using L'Hopital's Rule correctly gives $\lim\limits_{x\to\infty} \dfrac{\ln x}{x^2} = \lim\limits_{x\to\infty} \dfrac{1/x}{2x}$

$= \lim\limits_{x\to\infty} \dfrac{1}{2x^2} = 0.$

Problem #5: Find the following limits: (a) $\lim\limits_{x\to 0} \dfrac{\sin 2x}{5x-1}$; (b) $\lim\limits_{x\to\infty} \dfrac{x\ln x}{2^x}$.

Solution: Since both problems involve quotients, we can use L'Hopital's Rule for each:

For (a), differentiating top and bottom gives $\lim\limits_{x\to 0} \dfrac{2\cos 2x}{5} = \dfrac{2\cdot\cos 0}{5} = 2/5$.

For (b), differentiating top and bottom gives $\lim\limits_{x\to\infty} \dfrac{x\cdot 1/x}{(\ln 2)2^x} = \lim\limits_{x\to\infty} \dfrac{1}{(\ln 2)2^x} = 0$ since the denominator blows up.

3

The Error--(a): L'Hopital's Rule can only be used on expressions that have the indeterminate form ∞/∞ or 0/0. But plugging in 0 gives $\dfrac{\sin(2\cdot 0)}{5(0)-1} = \dfrac{0}{-1}$, so L'Hopital's Rule does not apply.

A Correct Solution: $\lim\limits_{x\to 0} \dfrac{\sin 2x}{5x-1} = \dfrac{\sin(2\cdot 0)}{5(0)-1} = \dfrac{0}{-1} = 0$.

The Error--(b): This expression *does* have the indeterminate form ∞/∞, so it is correct to use L'Hopital's Rule. However, the derivative of the numerator requires the product rule.

A Correct Solution: $\lim\limits_{x\to\infty} \dfrac{x\ln x}{2^x} = \lim\limits_{x\to\infty} \dfrac{1\cdot\ln x + x\cdot 1/x}{(\ln 2)2^x} = \lim\limits_{x\to\infty} \dfrac{\ln x + 1}{(\ln 2)2^x}$. As this is still of the form ∞/∞, we must apply L'Hopital's Rule again:

2. (a) $\lim_{x\to\infty} xe^{-x}$
   (b) $\lim_{x\to 0+} \cot(2x)\sin(6x)$
   (c) $\lim_{x\to\frac{\pi}{2}-} \tan x - \frac{1}{\cos x}$
   (d) $\lim_{x\to 1+} \frac{x}{x-1} - \frac{1}{\ln x}$
   (e) $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$
   (f) $\lim_{x\to\infty} x^2e^{-x}$
   (g) $\lim_{x\to 0} x\ln x$
   (h) $\lim_{x\to\infty} x\sin\frac{1}{x}$
   (i) $\lim_{x\to 0} \frac{1}{\sin x} - \frac{1}{x}$
   (j) $\lim_{x\to\infty} x^{\frac{1}{x}}$

3. Try to compute the following limit with L'Hospital rule. What happened? Then sketch a graph with some program. What does it look like? Try to compute the limit with two policemen theorem.

$$\lim_{x\to\infty}\frac{x^2+\sin x}{x^2}$$

With LH:

$$\lim_{x\to\infty}\frac{x^2+\sin x}{x^2} \overset{L'H}{\underset{\frac{\cdot}{\infty}}{=}} \lim_{x\to\infty}\frac{2x+\cos x}{2x} \overset{L'H}{\underset{\frac{\cdot}{\infty}}{=}} \lim_{x\to\infty}\frac{2-\sin x}{2}$$

$\to \nexists$ $\to$ wrong use of LH

Without:

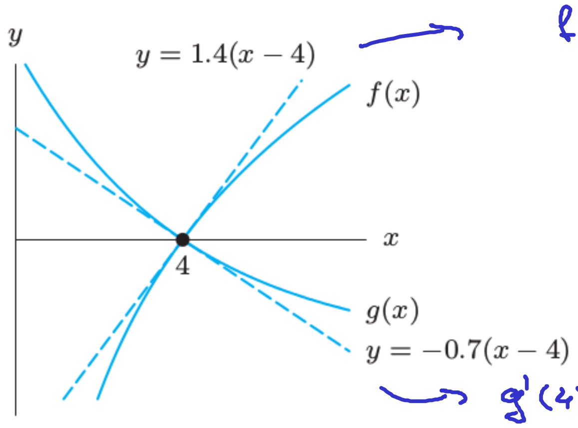
$$\lim_{x\to\infty} 1+\frac{\sin x}{x^2} = 1+0 = 0$$

$$\lim_{x\to\infty}\frac{\sin x}{x^2}=0 \text{ since}$$

$$\underset{\substack{\downarrow\\0}}{\frac{-1}{x^2}} \le \frac{\sin x}{x^2} \le \underset{\substack{\downarrow\\0}}{\frac{1}{x^2}}$$

6. Find

$$\lim_{x\to 4} \frac{f(x)}{g(x)}.$$



$f'(4) = 1{,}4$

$$\frac{f'(4)}{g'(4)} = \frac{1{,}4}{-0{,}7} = \underline{\underline{-2}}$$

$$\lim_{x\to 4} \frac{f}{g} \overset{l'H}{=} \lim_{x\to 4} \frac{f'(x)}{g'(x)} = \underline{\underline{-2}}$$

$g'(4) = -0{,}7$

Source 1: Calculus: Single and Multivariable 6th edition by Hughes-Hallett, Deborah, McCallum, William G., Gleason, Andr (2012)