18th lesson

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Theory

Theorem 1. Suppose that functions f and g have finite derivatives on some punctured neighbourhood of $a \in \mathbb{R}$ and the limit $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists. Suppose further that one of the following conditions hold:

1. $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$,
2. $\lim_{x\to a} |g(x)| = \infty$.

Then the limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists and

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Hints

$a^b = e^{b \ln a}$

Exercises

Find the limits

1. (a) $\lim\limits_{x\to 0} \dfrac{x^2}{\sin x}$
   (b) $\lim\limits_{x\to -\infty} \dfrac{3x-2}{e^{x^2}}$
   (c) $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$
   (d) $\lim\limits_{x\to \infty} \dfrac{3x^2+x+4}{x^2+8x}$
   (e) $\lim\limits_{x\to 0} \dfrac{e^x - 1 - x - \frac{x^2}{2}}{x^3}$
   (f) $\lim\limits_{x\to 3} \dfrac{x^2+x-12}{x^2-9}$
   (g) $\lim\limits_{x\to 0} \dfrac{\sin x}{x+1}$
   (h) $\lim\limits_{x\to 0} \dfrac{\sin 3x}{\tan 4x}$
   (i) $\lim\limits_{x\to \frac{\pi}{2}} (x - \dfrac{\pi}{2})\tan x$
   (j) $\lim\limits_{x\to 1} \dfrac{\sqrt{2-x}-x}{x-1}$
   (k) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2}$
   (l) $\lim\limits_{x\to \infty} \dfrac{e^x}{x}$
   (m) $\lim\limits_{x\to 2+} \dfrac{\ln(x-2)}{\ln(x^2-4)}$
   (n) $\lim\limits_{x\to 0} \dfrac{\sin^2 x}{1-\cos x}$
   (o) $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x^3}$

2. (a) $\lim_{x\to\infty} xe^{-x}$
   (b) $\lim_{x\to 0+} \cot(2x)\sin(6x)$
   (c) $\lim_{x\to\frac{\pi}{2}-} \tan x - \frac{1}{\cos x}$
   (d) $\lim_{x\to 1+} \frac{x}{x-1} - \frac{1}{\ln x}$
   (e) $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x$
   (f) $\lim_{x\to\infty} x^2 e^{-x}$
   (g) $\lim_{x\to 0+} x\ln x$
   (h) $\lim_{x\to\infty} x\sin\frac{1}{x}$
   (i) $\lim_{x\to 0} \frac{1}{\sin x} - \frac{1}{x}$
   (j) $\lim_{x\to\infty} x^{\frac{1}{x}}$

3. Find the mistake:

$$\lim_{x\to\infty} \frac{\ln x}{x^2} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}\cdot x^2 - \ln x\cdot 2x}{(x^2)^2} = \lim_{x\to\infty} \frac{x-2x\ln x}{x^4} = \lim_{x\to\infty} \frac{1-2\ln x}{x^3}$$

4. Find the mistake:

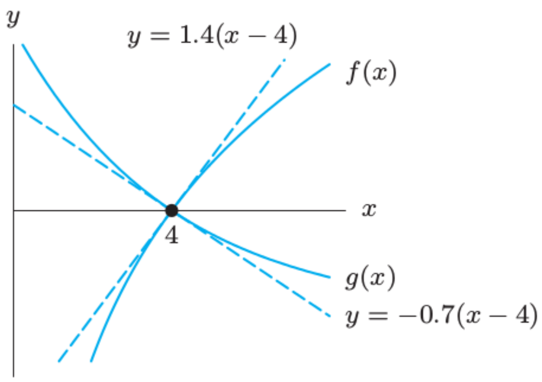
$$\lim_{x\to 0} \frac{\sin 2x}{5x-1} \overset{L'H}{=} \lim_{x\to 0} \frac{2\cos 2x}{5} = \frac{2}{5}$$

5. Try to compute the following limit with L'Hospital rule. What happened? Then sketch a graph with some program. What does it look like? Try to compute the limit with two policemen theorem.

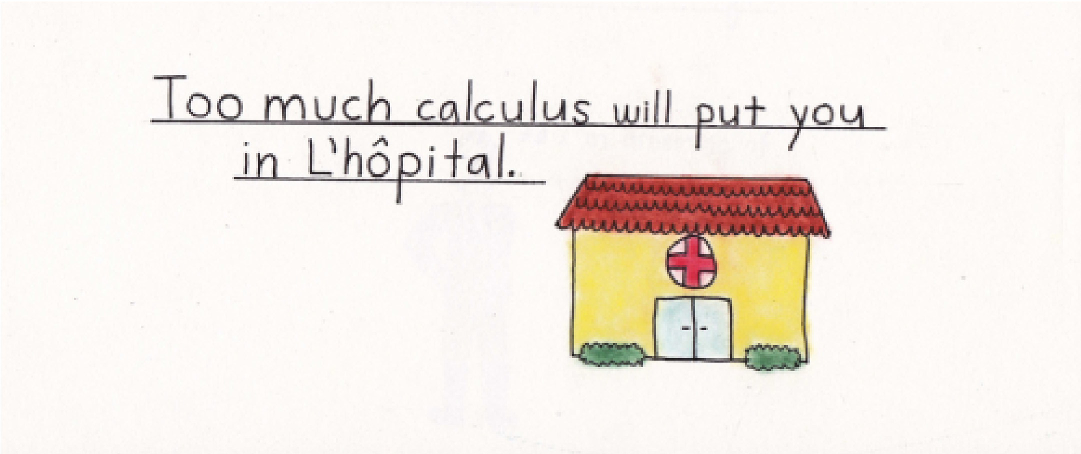
$$\lim_{x\to\infty} \frac{x^2+\sin x}{x^2}$$

6. Let us suppose that f and g has continuous derivatives. Find

$$\lim_{x\to 4} \frac{f(x)}{g(x)}.$$



Source 1: Calculus: Single and Multivariable 6th edition by Hughes-Hallett, Deborah, McCallum, William G., Gleason, Andr (2012)



Source 2: https://christinanotchris.tumblr.com/post/56160714523