Title: Finding Derivatives Using the Limit Definition
Class: Math 130 or Math 150
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Instructions to tutor: Read instructions and follow all steps for each problem exactly as given.
Keywords/Tags: Calculus, derivative, difference quotient, limit

Finding Derivatives Using the Limit Definition

Purpose:
This is intended to strengthen your ability to find derivatives using the limit definition.

Recall that an expression of the form $\frac{f(x)-f(a)}{x-a}$ or $\frac{f(x+h)-f(x)}{h}$ is called a **difference quotient**. For the definition of the derivative, we will focus mainly on the second of these two expressions. Before moving on to derivatives, let's get some practice working with the difference quotient.

The main difficulty is evaluating the expression $f(x+h)$, which seems to throw people off a bit.

Consider the function $f(x)=x^2-4x$. Let's evaluate this function at a few values.

$$f(2)=(2)^2-4(2)$$

$$f(0)=(0)^2-4(0)$$

$$f(-3)=(-3)^2-4(-3)$$

$$f(a)=(a)^2-4(a)$$

Note that we are just replacing the independent variable on each side of the equation with a particular value. So we should be able to do the same thing for $f(x+h)$: $f(x+h)=(x+h)^2-4(x+h)$

Now let's apply this to finding some difference quotients.

Example: Evaluate the difference quotient $\frac{f(x+h)-f(x)}{h}$ for the function $f(x)=x^2-4x$.

Now $\frac{f(x+h)-f(x)}{h}=\frac{[(x+h)^2-4(x+h)]-[x^2-4x]}{h}$.

Simplifying, $\frac{[(x+h)^2-4(x+h)]-[x^2-4x]}{h}=\frac{x^2+2xh+h^2-4x-4h-x^2+4x}{h}=\frac{2xh+h^2-4h}{h}$.

Note that we can reduce this fraction to obtain $\frac{2xh+h^2-4h}{h}=\frac{h(2x+h-4)}{h}=2x+h-4$.

PRACTICE EXAM 2 SOLUTIONS

1. Use the limit definition of the derivative to find $f'(x)$ for the following functions $f(x)$ (Sec. 2.2):

1b $f(x) = 3x^2 + 2x + 7$

$$f'(x) = \lim_{h\to 0} \frac{3(x+h)^2 + 2(x+h) + 7 - (3x^2 + 2x + 7)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{3h^2 + 6xh + 2h}{h} = 6x + 2$$

1c $f(x) = x^3$

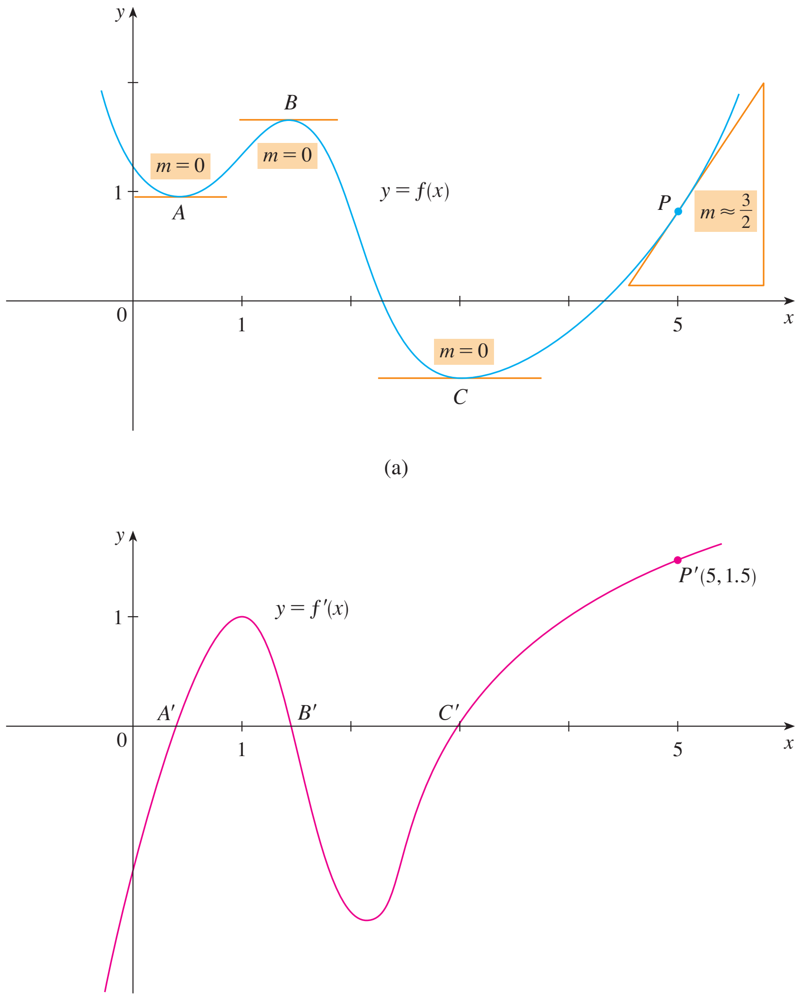
$$f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = 3x^2$$

1d $f(x) = \frac{1}{x+3}$

$$f'(x) = \lim_{h\to 0} \frac{1/(x+h+3) - 1/(x+3)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{-h/(x+3)(x+h+3)}{h} = \frac{-1}{(x+3)^2}$$



TEC Visual 2.8 shows an animation of Figure 2 for several functions.

(b)

FIGURE 2

V EXAMPLE 2

(a) If $f(x) = x^3 - x$, find a formula for $f'(x)$.

(b) Illustrate by comparing the graphs of f and f'.

SOLUTION

(a) When using Equation 2 to compute a derivative, we must remember that the variable is h and that x is temporarily regarded as a constant during the calculation of the limit.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2 - 1) = 3x^2 - 1$$

(b) We use a graphing device to graph f and f' in Figure 3. Notice that $f'(x) = 0$ when f has horizontal tangents and $f'(x)$ is positive when the tangents have positive slope. So these graphs serve as a check on our work in part (a).

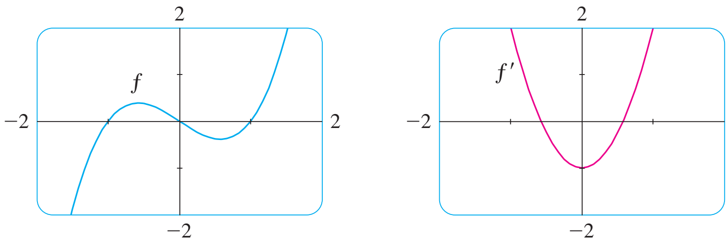


FIGURE 3

EXAMPLE 3 If $f(x) = \sqrt{x}$, find the derivative of f. State the domain of f'.

SOLUTION

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h\to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right)$$

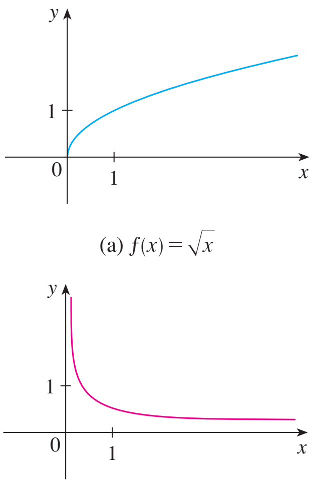
Here we rationalize the numerator.

$$= \lim_{h\to 0} \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

We see that $f'(x)$ exists if $x > 0$, so the domain of f' is $(0, \infty)$. This is smaller than the domain of f, which is $[0, \infty)$.

Let's check to see that the result of Example 3 is reasonable by looking at the graphs of f and f' in Figure 4. When x is close to 0, $\sqrt{x}$ is also close to 0, so $f'(x) = 1/\left(2\sqrt{x}\right)$ is very large and this corresponds to the steep tangent lines near $(0, 0)$ in Figure 4(a) and the large values of $f'(x)$ just to the right of 0 in Figure 4(b). When x is large, $f'(x)$ is very small and this corresponds to the flatter tangent lines at the far right of the graph of f and the horizontal asymptote of the graph of f'.



(a) $f(x) = \sqrt{x}$

(b) $f'(x) = \dfrac{1}{2\sqrt{x}}$

FIGURE 4

EXAMPLE 4 Find f' if $f(x) = \dfrac{1 - x}{2 + x}$.

SOLUTION

$$\frac{\dfrac{a}{b} - \dfrac{c}{d}}{e} = \frac{ad - bc}{bd} \cdot \frac{1}{e}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{1 - (x+h)}{2 + (x+h)} - \dfrac{1 - x}{2 + x}}{h}$$

$$= \lim_{h\to 0} \frac{(1 - x - h)(2 + x) - (1 - x)(2 + x + h)}{h(2 + x + h)(2 + x)}$$

$$= \lim_{h\to 0} \frac{(2 - x - 2h - x^2 - xh) - (2 - x + h - x^2 - xh)}{h(2 + x + h)(2 + x)}$$

$$= \lim_{h\to 0} \frac{-3h}{h(2 + x + h)(2 + x)} = \lim_{h\to 0} \frac{-3}{(2 + x + h)(2 + x)} = -\frac{3}{(2 + x)^2}$$

To see this try to compute the derivative at 0,

$$f'(0) = \lim_{x\to 0} \frac{|x| - |0|}{x - 0} = \lim_{x\to 0} \frac{|x|}{x} = \lim_{x\to 0} \operatorname{sign}(x).$$

We know this limit does not exist (see §7.2)

If you look at the graph of $f(x) = |x|$ then you see what is wrong: the graph has a corner at the origin and it is not clear which line, if any, deserves to be called the tangent to the graph at the origin.

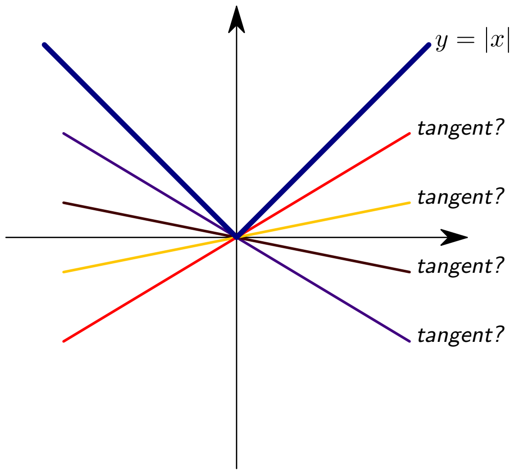


Figure 1. The graph of $y = |x|$ has no tangent at the origin.

2a)

4.2. A graph with a cusp. Another example of a function without a derivative at $x = 0$ is

$$f(x) = \sqrt{|x|}.$$

When you try to compute the derivative you get this limit

$$f'(0) = \lim_{x\to 0} \frac{\sqrt{|x|}}{x} = ?$$

The limit from the right is

$$\lim_{x\searrow 0} \frac{\sqrt{|x|}}{x} = \lim_{x\searrow 0} \frac{1}{\sqrt{x}},$$

which does not exist (it is "$+\infty$"). Likewise, the limit from the left also does not exist ('tis "$-\infty$). Nonetheless, a drawing for the graph of f suggests an obvious tangent to the graph at $x = 0$, namely, the y-axis. That observation does not give us a derivative, because the y-axis is vertical and hence has no slope.

4.3. A graph with absolutely no tangents, *anywhere.* The previous two examples were about functions which did not have a derivative at $x = 0$. In both examples the point $x = 0$ was the only point where the function failed to have a derivative. It is easy to give examples of functions which are not differentiable at more than one value of x, but here I would like to show you a function f which doesn't have a derivative ***anywhere in its domain.***

To keep things short I won't write a formula for the function, and merely show you a graph. In this graph you see a typical path of a Brownian motion, i.e. t is time, and $x(t)$ is the position of a particle which undergoes a Brownian motion – come to lecture for further explanation (see also the article on wikipedia). To see a similar graph check the Dow Jones or Nasdaq in the upper left hand corner of the web page at `http://finance.yahoo.com` in the afternoon on any weekday.

Example 2

2b)

$$f(x) = \begin{cases} -x & if \quad x < 0 \\ x^2 & if \quad x \geq 0 \end{cases}$$

Solution

Step 1 - Find the left hand derivative

The formula for computation of the left hand derivative is:

$$f'(a^-) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$$

Since $f(x) = -x$ and $x < 0$, hence we will put the values in the formula:

$$\lim_{x \to 0^-} \frac{-(0+h) - 0}{h}$$
$$= \frac{-h}{h} = -1$$

Step 2 - Find the right hand derivative

The formula for finding out the right hand derivative is given below:

$$f'(a^+) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

Since the second part of the function is $f(x) = x^2$ and $x \geq 0$, hence we will put the values in the formula:

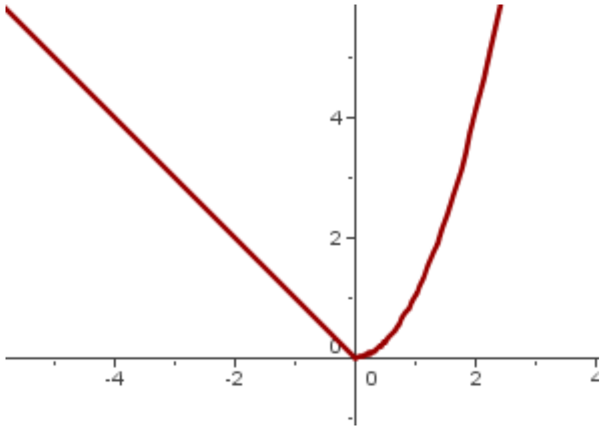
$$\lim_{x \to 0^+} \frac{(0+h)^2 - 0}{h}$$
$$= \lim_{x \to 0^+} \frac{h^2}{h} = 1$$
$$= \lim_{x \to 0^+} = 0$$

Step 3 - Compare the left and right hand derivatives

The left hand derivative is $f'_-(a) = -1$. Similarly, the right hand derivative is

$f'_+(a) = 0$. Since $f'_-(a) \neq f'_+(a)$, therefore we can say that the function is not differentiable at $x = 0$.

The function looks like this in the xy coordinate plane.



Example 2 - Graph of the piecewise function

Example 3

Check whether the function $f(x) = |x + 3|$ differentiable at $x = -3$.

Solution

This function is an absolute-value function. We will follow the following steps to determine if the function $f(x) = |x + 3|$ is differentiable at $x = -3$ or not.

Step 1 - Find the left hand derivative

To compute the left hand derivative, we use the following formula:

$$f'(a^-) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$$

$$f'(a^-) = \lim_{h \to 0^-} = \frac{|(-3+h) + 3 - f(-3)}{h}$$

$$f'(a^-) = \lim_{h \to 0^-} = \frac{|(-3+h) + 3| - 0}{h}$$

$$\lim_{h \to 0^-} = \frac{|h|}{h}$$

As this is a left hand derivative, so $h < 0$

$$\lim_{h \to 0^-} = \frac{-h}{h} = -1$$

Step 2 - Find the right hand derivative

The formula for finding out the right hand derivative is given below:

$$f'(a^+) = \lim_{h \to 0+} \frac{f(a+h) - f(a)}{h}$$

$$f'(a^+) = \lim_{h \to 0+} = \frac{|(-3+h)+3 - f(-3)}{h}$$

$$f'(a^-) = \lim_{h \to 0+} = \frac{|h|}{h}$$

$$\lim_{h \to 0-} = \frac{|h|}{h}$$

As this is a right hand derivative, so $h > 0$

$$\lim_{h \to 0^+} = \frac{h}{h} = 1$$

Step 3 - Compare the left and right hand derivatives

The left hand derivative is $f'_-(a) = -1$ and the right hand derivative is $f'_+(a) = 1$. Since $f'_-(a) \neq f'_+(a)$, therefore we can say that the function is not differentiable at $x = 3$.

MATH 1010E University Mathematics
Lecture Notes (week 4)
Martin Li

1 Derivatives of Piecewise Defined Functions

For piecewise defined functions, we often have to be very careful in computing the derivatives. The differentiation rules (product, quotient, chain rules) can only be applied if the function is defined by ONE formula in a neighborhood of the point where we evaluate the derivative. If we want to calculate the derivative at a point where two different formulas "meet", then we must use the definition of derivative as limit of difference quotient to correctly evaluate the derivative. Let us illustrate this by the following example.

Example 1.1 *Find the derivative $f'(x)$ at every $x \in \mathbb{R}$ for the piecewise defined function*

2d)

$$f(x) = \begin{cases} 5 - 2x & \text{when } x < 0, \\ x^2 - 2x + 5 & \text{when } x \geq 0. \end{cases}$$

Solution: We separate into 3 cases: $x < 0$, $x > 0$ and $x = 0$. For the first two cases, the function $f(x)$ is defined by a single formula, so we could just apply differentiation rules to differentiate the function.

$$f'(x) = (5 - 2x)' = -2 \qquad \text{for } x < 0,$$

$$f'(x) = (x^2 - 2x + 5)' = 2x - 2 \qquad \text{for } x > 0.$$

At $x = 0$, we have to use the definition of derivative as limit of difference quotient. First of all,

$$f(0) = 0^2 - 2(0) + 5 = 5.$$

Then we calculate the left-hand and right-hand limits:

$$\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{(5 - 2h) - 5}{h} = \lim_{h \to 0^-} -2 = -2,$$

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{(h^2 - 2h + 5) - 5}{h} = \lim_{h \to 0^+} (h - 2) = -2.$$

Since both of them exists and are equal, we have

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = -2.$$

Therefore, putting all of these together, we see that f is differentiable for every $x \in \mathbb{R}$ and

$$f'(x) = \begin{cases} -2 & \text{when } x \leq 0, \\ 2x - 2 & \text{when } x > 0. \end{cases}$$

Remark 1.2 *From the example above, we see that the derivative $f'(x)$ is still a continuous function (check this!). This is not always true for any function! (Have you seen a counterexample? See Homework 2)*

Example 1.3 *Consider the function defined by*

$$f(x) = \begin{cases} ax + b & \textit{when } x \leq -1, \\ ax^3 + x + 2b & \textit{when } x > -1, \end{cases}$$

for what value(s) of $a, b \in \mathbb{R}$ is the function f differentiable at every $x \in \mathbb{R}$?

Solution: First, it is easy to see that for ANY $a, b \in \mathbb{R}$, the function f is differentiable at every $x \neq -1$ since f is defined by a polynomial on $(-1, +\infty)$ and $(-\infty, -1)$. The only catch is at the point $x = -1$.

If f is differentiable at $x = -1$, it must also be continuous at $x = -1$. Therefore, we need

$$\lim_{x \to -1} f(x) = f(-1).$$

Now, $f(-1) = -a + b$ and the left-hand and right-hand limits are

$$\lim_{x \to -1^-} f(x) = a(-1)^3 + (-1) + 2b = -a + 2b - 1,$$

$$\lim_{x \to -1^+} f(x) = a(-1) + b = -a + b.$$

If f is continuous, then both of these limits must be the same and equal to $f(-1)$. Hence, we have

$$-a + b = -a + 2b - 1 \qquad \Rightarrow \qquad b = 1.$$

Now, we take $b = 1$. To find the value of a which make f differentiable at $x = -1$, we require the limit

$$\lim_{h \to 0} \frac{f(-1 + h) - f(-1)}{h}$$

$$\frac{f(x)-f(x_0)}{x-x_0}=f'(\xi_1) \quad (x\in(x_0-\delta,x_0),\xi_1\in(x,x_0)).$$

$$\frac{f(x)-f(x_0)}{x-x_0}=f'(\xi_2) \quad (x\in(x_0,x_0+\delta),\xi_2\in(x,x_0)).$$

Because $\lim\limits_{x\to x_0^-} f'(x)=\lim\limits_{x\to x_0^+} f'(x)$=A， so

$$f'_-(x_0)=\lim_{x\to x_0^-}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{x\to x_0^-}f'(\xi_1)=\lim_{\xi_1\to x_0^-}f'(\xi_1)=A,$$
$$f'_+(x_0)=\lim_{x\to x_0^+}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{x\to x_0^+}f'(\xi_1)=\lim_{\xi_2\to x_0^+}f'(\xi_2)=A,$$

so $f'(x_0)=f'_-(x_0)=f'_+(x_0)=A$.

Example 1[2] Discuss the derivative of $f(x)=\begin{cases}\sin x & x<0\\ x & x\ge 0\end{cases}$ at point $x=0$.

Solution: Because $f(x)$ is continuous at point $x=0$, and there are

$$\lim_{x\to 0^-}f'(x)=\lim_{x\to 0^-}\cos x=1 \qquad \lim_{x\to 0^+}f'(x)=\lim_{x\to 0^+}1=1$$

then, $f'(0)$ exist and $f'(0)=1$.

In theorem 1, it is worth noting that the existence of $\lim\limits_{x\to x_0^-}f'(x)$ and $\lim\limits_{x\to x_0^+}f'(x)$ can not guarantee the existence of the derivatives at the piecewise point.

Example 2[3] Discuss the derivative of function $f(x)=\begin{cases}x+1 & x<0\\ x & x\ge 0\end{cases}$ at point $x=0$.

Solution: Because $\lim\limits_{x\to 0^-}f(x)\ne\lim\limits_{x\to 0^+}f(x)$, so $f(x)$ is not continuous at point $x=0$ and are not be differentiable, but

$$\lim_{x\to 0^-}f'(x)=\lim_{x\to 0^+}f'(x)=1$$

So, theorem 1 is used when it satisfied the two conditions at the same time. According to Theorem 1, we can obtain the corollary 1 when the left and right limitations are not equal.

Corollary 1 The piecewise function $f(x)$ is guided within a $\overset{o}{U}(x_0)$ of the picewise-point x_0, if both $\lim\limits_{x\to x_0^-}f'(x)$ and $\lim\limits_{x\to x_0^+}f'(x)$ are present but not equal, then $f(x)$ is not directed at point x_0.

Example 3[4] Discuss the derivatives of function $f(x)=\begin{cases}x^2 & x\ge 0\\ -x & x<0\end{cases}$ at point $x=0$.

Since both of them exists and are equal, we have

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = -2.$$

Therefore, putting all of these together, we see that f is differentiable for every $x \in \mathbb{R}$ and

$$f'(x) = \begin{cases} -2 & \text{when } x \leq 0, \\ 2x - 2 & \text{when } x > 0. \end{cases}$$

Remark 1.2 *From the example above, we see that the derivative $f'(x)$ is still a continuous function (check this!). This is not always true for any function! (Have you seen a counterexample? See Homework 2)*

Example 1.3 *Consider the function defined by*

$$f(x) = \begin{cases} ax + b & \textit{when } x \leq -1, \\ ax^3 + x + 2b & \textit{when } x > -1, \end{cases}$$

for what value(s) of $a, b \in \mathbb{R}$ is the function f differentiable at every $x \in \mathbb{R}$?

Solution: First, it is easy to see that for ANY $a, b \in \mathbb{R}$, the function f is differentiable at every $x \neq -1$ since f is defined by a polynomial on $(-1, +\infty)$ and $(-\infty, -1)$. The only catch is at the point $x = -1$.

If f is differentiable at $x = -1$, it must also be continuous at $x = -1$. Therefore, we need

$$\lim_{x \to -1} f(x) = f(-1).$$

Now, $f(-1) = -a + b$ and the left-hand and right-hand limits are

$$\lim_{x \to -1^-} f(x) = a(-1)^3 + (-1) + 2b = -a + 2b - 1,$$

$$\lim_{x \to -1^+} f(x) = a(-1) + b = -a + b.$$

If f is continuous, then both of these limits must be the same and equal to $f(-1)$. Hence, we have

$$-a + b = -a + 2b - 1 \qquad \Rightarrow \qquad b = 1.$$

Now, we take $b = 1$. To find the value of a which make f differentiable at $x = -1$, we require the limit

$$\lim_{h \to 0} \frac{f(-1 + h) - f(-1)}{h}$$

to exists, which is equivalent to the statement that the left-hand and right-hand limits exist and are equal. The left hand limit is

$$\lim_{h\to 0^-} \frac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0^-} \frac{[a(-1+h)+1]-(-a+1)}{h} = \lim_{h\to 0^-} \frac{ah}{h} = a.$$

The right hand limit is

$$\begin{aligned}
\lim_{h\to 0^+} \frac{f(-1+h)-f(-1)}{h} &= \lim_{h\to 0^+} \frac{[a(-1+h)^3+(-1+h)+2]-(-a+1)}{h} \\
&= \lim_{h\to 0^+} \frac{a[(-1+h)^3+1]+h}{h} \\
&= \lim_{h\to 0^+} \frac{ah[(-1+h)^2-(-1+h)+1]+h}{h} \\
&= 3a+1.
\end{aligned}$$

Therefore, if we set them equal to each other, we obtain the condition

$$a = 3a+1 \qquad \Rightarrow \qquad a = -\frac{1}{2}.$$

In summary, we have $a = -1/2$ and $b = 1$ if f is differentiable at every $x \in \mathbb{R}$.

2 Differentiation Rules II: Product and Quotient Rules

Theorem 2.1 *If f and g are differentiable functions, then both their product fg and quotient f/g are differentiable and we have*

(1) Product Rule:

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x),$$

(2) Quotient Rule:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2},$$

provided that $g(x) \neq 0$.

Remark 2.2 *Observe that the differentiation rule $[kf(x)]' = kf'(x)$ where k is a constant is just a special case of product rule by taking $g(x) \equiv k$, which has $g'(x) = 0$.*

3)

A toy company wants to design a track for a toy car that starts out along a parabolic curve and then converts to a straight line (Figure 3.2.7). The function that describes the track is to have the form $f(x) = \begin{cases} \frac{1}{10}x^2 + bx + c, & \text{if } x < -10 \\ -\frac{1}{4}x + \frac{5}{2}, & \text{if } x \geq -10 \end{cases}$ where x and $f(x)$ are in inches. For the car to move smoothly along the track, the function $f(x)$ must be both continuous and differentiable at -10. Find values of b and c that make $f(x)$ both continuous and differentiable.

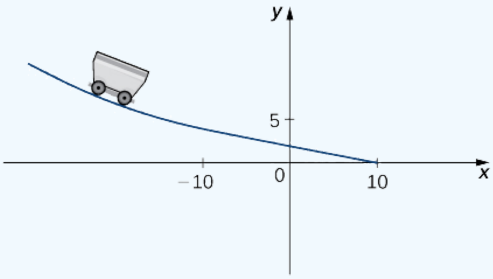


Figure 3.2.7: For the car to move smoothly along the track, the function must be both continuous and differentiable.

Solution

For the function to be continuous at $x = -10$, $\lim_{x\to 10^-} f(x) = f(-10)$. Thus, since

$$\lim_{x\to -10^-} f(x) = \frac{1}{10}(-10)^2 - 10b + c = 10 - 10b + c$$

and $f(-10) = 5$, we must have $10 - 10b + c = 5$. Equivalently, we have $c = 10b - 5$.

For the function to be differentiable at -10,

$$f'(10) = \lim_{x\to -10} \frac{f(x) - f(-10)}{x + 10}$$

must exist. Since $f(x)$ is defined using different rules on the right and the left, we must evaluate this limit from the right and the left and then set them equal to each other:

$$\begin{aligned} \lim_{x\to -10^-} \frac{f(x) - f(-10)}{x+10} &= \lim_{x\to -10^-} \frac{\frac{1}{10}x^2 + bx + c - 5}{x+10} \\ &= \lim_{x\to -10^-} \frac{\frac{1}{10}x^2 + bx + (10b-5) - 5}{x+10} && \text{Substitute } c = 10b - 5. \\ &= \lim_{x\to -10^-} \frac{x^2 - 100 + 10bx + 100b}{10(x+10)} \\ &= \lim_{x\to -10^-} \frac{(x+10)(x-10+10b)}{10(x+10)} && \text{Factor by grouping} \\ &= b - 2 \end{aligned}$$

We also have

$$\begin{aligned} \lim_{x\to -10^+} \frac{f(x) - f(-10)}{x+10} &= \lim_{x\to -10^+} \frac{-\frac{1}{4}x + \frac{5}{2} - 5}{x+10} \\ &= \lim_{x\to -10^+} \frac{-(x+10)}{4(x+10)} \\ &= -\frac{1}{4} \end{aligned}$$

This gives us $b - 2 = -\frac{1}{4}$. Thus $b = \frac{7}{4}$ and $c = 10(\frac{7}{4}) - 5 = \frac{25}{2}$.

3

4

Any function which is differentiable at a point x_0 must also be continuous at x_0. Since the left and right hand limits of f do not agree, your function is not continuous at 0. Therefore the derivative does not exist at 0 even though the derivative seems to be approaching the same value from both directions.

In more detail,

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{h^2+1-1}{h} = \lim_{h\to 0^+} h = 0.$$

But

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{h^2-1-1}{h} = \lim_{h\to 0^-} \frac{h^2-2}{h} = \infty.$$

Example 4.

Find the equation of the tangent line to the curve $y = x^3$ at $x_0 = 1$.

Solution.

First we find the derivative:

$$y' = f'(x) = \left(x^3\right)' = 3x^2.$$

The value of the derivative at the point of tangency is

$$f'(x_0) = 3 \cdot 1^2 = 3.$$

Calculate y_0 :

$$y_0 = (x_0)^3 = 1^3 = 1.$$

Substitute this in the equation of tangent:

$$y - 1 = 3(x - 1),$$

$$y - 1 = 3x - 3,$$

$$y = 3x - 2.$$

WORKED EXAMPLE 14: FINDING THE EQUATION OF A TANGENT TO A CURVE

QUESTION

Given $g(x) = (x+2)(2x+1)^2$, determine the equation of the tangent to the curve at $x = -1$.

SOLUTION

Step 1: Determine the y-coordinate of the point

$$\begin{aligned} g(x) &= (x+2)(2x+1)^2 \\ g(-1) &= (-1+2)[2(-1)+1]^2 \\ &= (1)(-1)^2 \\ &= 1 \end{aligned}$$

Therefore the tangent to the curve passes through the point $(-1;1)$.

Step 2: Expand and simplify the given function

$$\begin{aligned} g(x) &= (x+2)(2x+1)^2 \\ &= (x+2)(4x^2+4x+1) \\ &= 4x^3+4x^2+x+8x^2+8x+2 \\ &= 4x^3+12x^2+9x+2 \end{aligned}$$

Step 3: Find the derivative

$$\begin{aligned} g'(x) &= 4(3x^2)+12(2x)+9+0 \\ &= 12x^2+24x+9 \end{aligned}$$

Step 4: Calculate the gradient of the tangent

Substitute $x = -1$ into the equation for $g'(x)$:

$$\begin{aligned} g'(-1) &= 12(-1)^2+24(-1)+9 \\ \therefore m &= 12-24+9 \\ &= -3 \end{aligned}$$

Step 5: Determine the equation of the tangent

Substitute the gradient of the tangent and the coordinates of the point into the gradient-point form of the straight line equation.

$$\begin{aligned} y - y_1 &= m\left(x - x_1\right) \\ y - 1 &= -3\left(x - (-1)\right) \\ y &= -3x - 3 + 1 \\ y &= -3x - 2 \end{aligned}$$

Example 1.

Find the equation of the tangent to the curve $y = \sqrt{x}$ at the point $(1, 1)$ (Figure 4).

Solution.

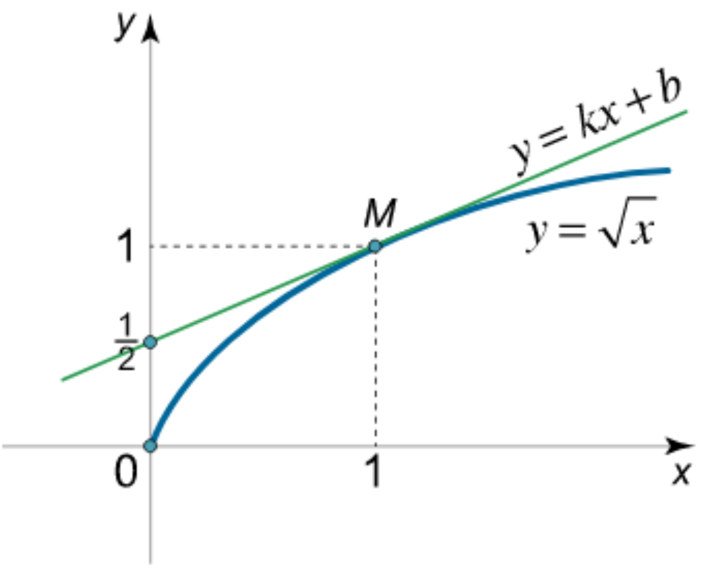


Figure 4.

$$y' = f'(x) = \left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}},$$

$$f'(x_0) = f'(1) = \frac{1}{2\sqrt{1}} = \frac{1}{2},$$

$$x_0 = 1,\ y_0 = 1,\ f'(x_0) = \frac{1}{2}$$

Substitute the 3 values into the equation of the tangent line:

$$y - y_0 = f'(x_0)(x - x_0).$$

This yields:

$$y - 1 = \frac{1}{2}(x - 1)\ \ \Rightarrow y - 1 = \frac{x}{2} - \frac{1}{2}\ \ \Rightarrow y = \frac{x}{2} - \frac{1}{2} + 1\ \ \Rightarrow y = \frac{x}{2} + \frac{1}{2}.$$

Answer:

$$y = \frac{x}{2} + \frac{1}{2}.$$

Example 15.

Write equations of the tangent and normal to the graph of the function $y = x\sqrt{x-1}$ at $x = 2$.

Solution.

Calculate the derivative for the given function:

$$y'(x) = \left(x\sqrt{x-1}\right)' = x'\sqrt{x-1} + x\left(\sqrt{x-1}\right)' = \sqrt{x-1} + \frac{x}{2\sqrt{x-1}}$$
$$= \frac{2(x-1)+x}{2\sqrt{x-1}} = \frac{3x-2}{2\sqrt{x-1}}.$$

At the point $x = 2$, the derivative is

$$y'(2) = \frac{3\cdot 2 - 2}{2\sqrt{2-1}} = 2.$$

The value of the function at this point is

$$y(2) = 2\cdot 1 = 2.$$

Find the equation of the tangent:

$$y - y_0 = y'(x_0)(x - x_0), \quad \Rightarrow y - 2 = 2(x-2), \quad \Rightarrow y - 2 = 2x - 4,$$
$$\Rightarrow y = 2x - 2.$$

and the equation of the normal at this point:

$$y - y_0 = -\frac{1}{y'(x_0)}(x - x_0), \quad \Rightarrow y - 2 = -\frac{1}{2}(x-2), \quad \Rightarrow y - 2 = -\frac{x}{2} + 1,$$
$$\Rightarrow y = -\frac{x}{2} + 3.$$

Example 2.

Find a point on the curve $y = x^2 - 2x - 3$ at which the tangent is parallel to the $x-$axis.

Solution.

Since the tangent is parallel to the $x-$axis, the derivative is equal to zero at this point. Hence,

$$y' = \left(x^2 - 2x - 3\right)' = 2x - 2 = 0.$$

Then we find that

$$x_0 = 1.$$

Example 5.

Find the equation of the tangent line to the curve $y = \ln x^2$ that is parallel to the straight line $y = x$.

Solution.

The derivative of the function is given by

$$y' = \left(\ln x^2\right)' = \frac{1}{x^2} \cdot 2x = \frac{2}{x}.$$

The slope of the tangent line must be equal to 1 as it follows from the equation of the straight line. This allows to find the tangency point:

$$\frac{2}{x} = 1, \Rightarrow x_0 = 2.$$

Calculate the value of the function at this point:

$$y_0 = y\left(2\right) = \ln 2^2 = \ln 4.$$

Now we can write the equation of the tangent line:

$$y - y_0 = f'\left(x_0\right)\left(x - x_0\right),$$

$$y - \ln 4 = 1 \cdot \left(x - 2\right),$$

$$y - \ln 4 = x - 2,$$

$$y = x + \ln 4 - 2.$$

Example 23.

A parabola is defined by the equation $y = x^2 + 2x + 3$. Write equations of the tangents to the parabola passing through the point $A\left(-1, 1\right)$.

Solution.

We transform the equation of the parabola to the form

$$y = x^2 + 2x + 3 = x^2 + 2x + 1 + 2 = \left(x+1\right)^2 + 2.$$

It can be seen that the graph of the parabola is obtained from the graph of the function $y = x^2$ by parallel shifting by 1 unit to the left and 2 units up (Figure 13).

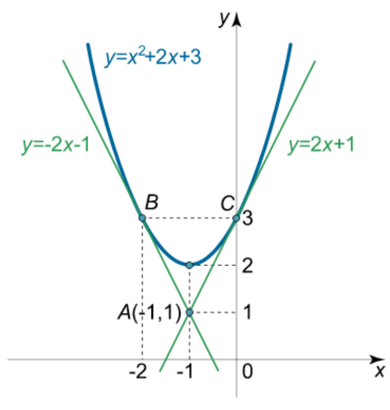


Figure 13.

Let us find the equations of two tangents to the parabola passing through the point $A\left(-1, 1\right)$. Each of these tangents is defined by the equation

$$y - y_A = k\left(x - x_A\right), \quad \Rightarrow y - 1 = k\left(x - (-1)\right), \quad \Rightarrow y - 1 = kx + k,$$
$$\Rightarrow y = kx + k + 1,$$

where k is the slope (k_1 for the first tangent and k_2 for the second). Thus, the problem reduces to finding of the slopes of the tangents k_1 and k_2. Take into account that at the points of tangency B and C the following condition holds:

$$\begin{cases} y = kx + k + 1 \\ y = x^2 + 2x + 3 \end{cases}, \quad \Rightarrow kx + k + 1 = x^2 + 2x + 3.$$

Also at the points of tangency B and C, the slope is equal to the derivative of the function $y = x^2 + 2x + 3$. Since

$$y' = \left(x^2 + 2x + 3\right)' = 2x + 2,$$

then we obtain another equation in the form

$$k = 2x + 2.$$

As a result, we have the system of two equations

$$\begin{cases} kx + k + 1 = x^2 + 2x + 3 \\ k = 2x + 2 \end{cases}$$

with two unknowns k and x. Solving this system, we find the values of k and x (i.e. the slopes of the tangents k_1, k_2 and x-coordinates of the points of tangency B and C):

$$\begin{cases} kx + k + 1 = x^2 + 2x + 3 \\ k = 2x + 2 \end{cases}, \quad \Rightarrow \left(2x + 2\right)x + 2x + 2 + 1 = x^2 + 2x + 3,$$
$$\Rightarrow 2x^2 + 2x + 2x + 3 = x^2 + 2x + 3, \quad \Rightarrow x^2 + 2x = 0, \quad \Rightarrow x_1 = -2, \ x_2 = 0.$$

The first solution $x_1 = -2$ corresponds to point B. The second solution $x_2 = 0$ is the coordinate of the point of tangency C. The slopes have the following values:

1. tangent $AB:\ x_1 = -2, k_1 = -2;$
2. tangent $AC:\ x_2 = 0, k_2 = 2.$

Then the equations of the tangents to the parabola are given by

1. tangent $AB:\ y = -2x - 1;$
2. tangent $AC:\ y = 2x + 3.$