Title: Finding Derivatives Using the Limit Definition
Class: Math 130 or Math 150
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Instructions to tutor: Read instructions and follow all steps for each problem exactly as given.
Keywords/Tags: Calculus, derivative, difference quotient, limit

Finding Derivatives Using the Limit Definition

Purpose:
This is intended to strengthen your ability to find derivatives using the limit definition.

Recall that an expression of the form $\dfrac{f(x)-f(a)}{x-a}$ or $\dfrac{f(x+h)-f(x)}{h}$ is called a **difference quotient**. For the definition of the derivative, we will focus mainly on the second of these two expressions. Before moving on to derivatives, let's get some practice working with the difference quotient.

The main difficulty is evaluating the expression $f(x+h)$, which seems to throw people off a bit.

Consider the function $f(x)=x^2-4x$. Let's evaluate this function at a few values.

$$f(2)=(2)^2-4(2)$$

$$f(0)=(0)^2-4(0)$$

$$f(-3)=(-3)^2-4(-3)$$

$$f(a)=(a)^2-4(a)$$

Note that we are just replacing the independent variable on each side of the equation with a particular value. So we should be able to do the same thing for $f(x+h)$: $f(x+h)=(x+h)^2-4(x+h)$

Now let's apply this to finding some difference quotients.

Example: Evaluate the difference quotient $\dfrac{f(x+h)-f(x)}{h}$ for the function $f(x)=x^2-4x$.

Now $\dfrac{f(x+h)-f(x)}{h}=\dfrac{[(x+h)^2-4(x+h)]-[x^2-4x]}{h}$.

Simplifying, $\dfrac{[(x+h)^2-4(x+h)]-[x^2-4x]}{h}=\dfrac{x^2+2xh+h^2-4x-4h-x^2+4x}{h}=\dfrac{2xh+h^2-4h}{h}$.

Note that we can reduce this fraction to obtain $\dfrac{2xh+h^2-4h}{h}=\dfrac{h(2x+h-4)}{h}=2x+h-4$.

PRACTICE EXAM 2 SOLUTIONS

1. Use the limit definition of the derivative to find $f'(x)$ for the following functions $f(x)$ (Sec. 2.2):

1b $f(x) = 3x^2 + 2x + 7$

$$f'(x) = \lim_{h\to 0} \frac{3(x+h)^2 + 2(x+h) + 7 - (3x^2 + 2x + 7)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{3h^2 + 6xh + 2h}{h} = 6x + 2$$

1c $f(x) = x^3$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

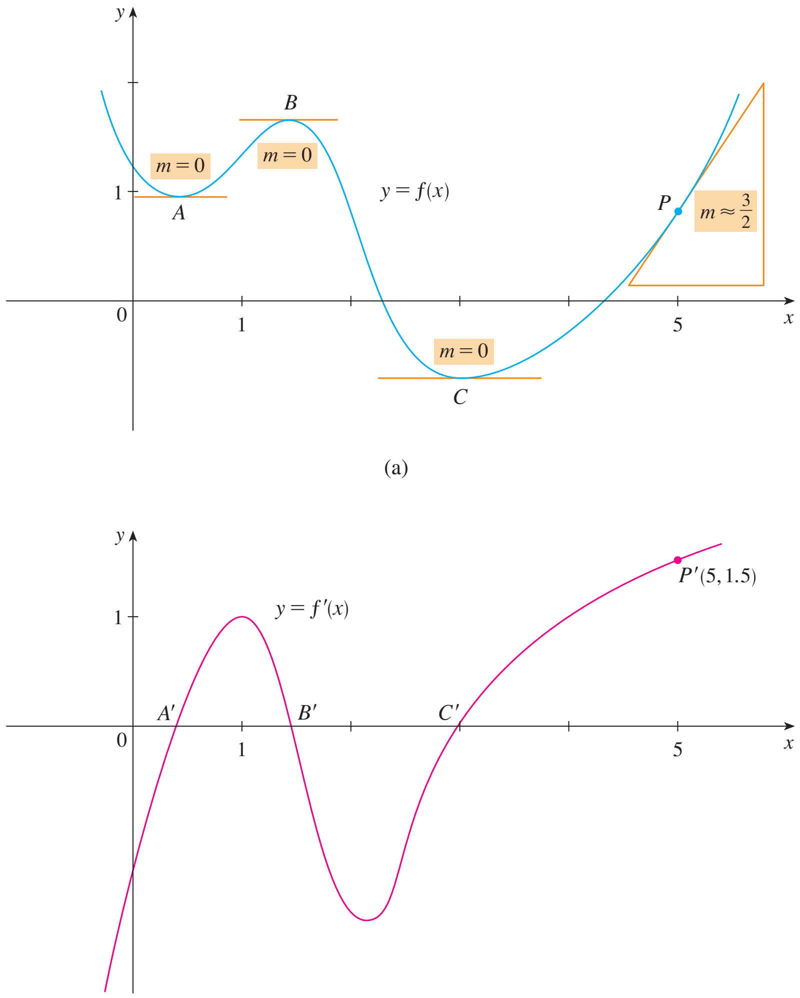
$$f'(x) = \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = 3x^2$$

1d $f(x) = \frac{1}{x+3}$

$$f'(x) = \lim_{h\to 0} \frac{1/(x+h+3) - 1/(x+3)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{-h/(x+3)(x+h+3)}{h} = \frac{-1}{(x+3)^2}$$

TEC Visual 2.8 shows an animation of Figure 2 for several functions.



(a)

(b)

FIGURE 2

EXAMPLE 2

(a) If $f(x) = x^3 - x$, find a formula for $f'(x)$.
(b) Illustrate by comparing the graphs of f and f'.

SOLUTION

(a) When using Equation 2 to compute a derivative, we must remember that the variable is h and that x is temporarily regarded as a constant during the calculation of the limit.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2 - 1) = 3x^2 - 1$$

(b) We use a graphing device to graph f and f' in Figure 3. Notice that $f'(x) = 0$ when f has horizontal tangents and $f'(x)$ is positive when the tangents have positive slope. So these graphs serve as a check on our work in part (a).

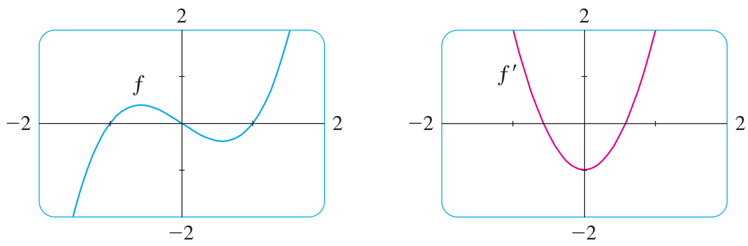


FIGURE 3

EXAMPLE 3 If $f(x) = \sqrt{x}$, find the derivative of f. State the domain of f'.

SOLUTION

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Here we rationalize the numerator.

$$= \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right)$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

We see that $f'(x)$ exists if $x > 0$, so the domain of f' is $(0, \infty)$. This is smaller than the domain of f, which is $[0, \infty)$.

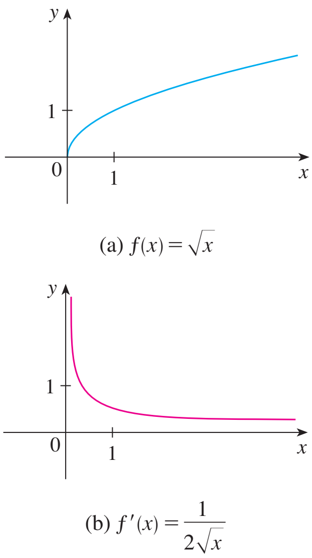


FIGURE 4

Let's check to see that the result of Example 3 is reasonable by looking at the graphs of f and f' in Figure 4. When x is close to 0, $\sqrt{x}$ is also close to 0, so $f'(x) = 1/\left(2\sqrt{x}\right)$ is very large and this corresponds to the steep tangent lines near $(0, 0)$ in Figure 4(a) and the large values of $f'(x)$ just to the right of 0 in Figure 4(b). When x is large, $f'(x)$ is very small and this corresponds to the flatter tangent lines at the far right of the graph of f and the horizontal asymptote of the graph of f'.

EXAMPLE 4 Find f' if $f(x) = \dfrac{1 - x}{2 + x}$.

SOLUTION

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\dfrac{1 - (x+h)}{2 + (x+h)} - \dfrac{1 - x}{2 + x}}{h}$$

$$\frac{\dfrac{a}{b} - \dfrac{c}{d}}{e} = \frac{ad - bc}{bd} \cdot \frac{1}{e}$$

$$= \lim_{h \to 0} \frac{(1 - x - h)(2 + x) - (1 - x)(2 + x + h)}{h(2 + x + h)(2 + x)}$$

$$= \lim_{h \to 0} \frac{(2 - x - 2h - x^2 - xh) - (2 - x + h - x^2 - xh)}{h(2 + x + h)(2 + x)}$$

$$= \lim_{h \to 0} \frac{-3h}{h(2 + x + h)(2 + x)} = \lim_{h \to 0} \frac{-3}{(2 + x + h)(2 + x)} = -\frac{3}{(2 + x)^2}$$

To see this try to compute the derivative at 0,

$$f'(0) = \lim_{x\to 0} \frac{|x| - |0|}{x - 0} = \lim_{x\to 0} \frac{|x|}{x} = \lim_{x\to 0} \operatorname{sign}(x).$$

We know this limit does not exist (see §7.2)

If you look at the graph of $f(x) = |x|$ then you see what is wrong: the graph has a corner at the origin and it is not clear which line, if any, deserves to be called the tangent to the graph at the origin.

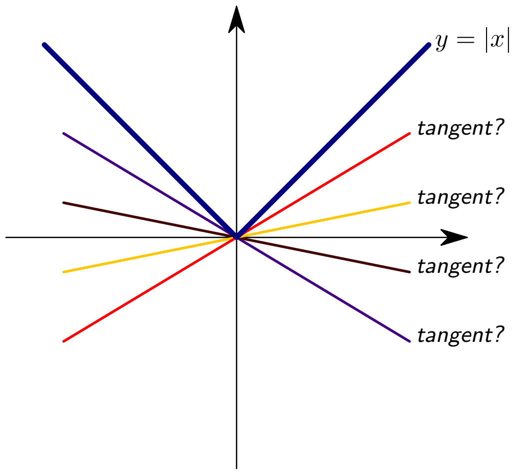


Figure 1. The graph of $y = |x|$ has no tangent at the origin.

4.2. A graph with a cusp. Another example of a function without a derivative at $x = 0$ is

$$f(x) = \sqrt{|x|}.$$

When you try to compute the derivative you get this limit

$$f'(0) = \lim_{x\to 0} \frac{\sqrt{|x|}}{x} = ?$$

The limit from the right is

$$\lim_{x\searrow 0} \frac{\sqrt{|x|}}{x} = \lim_{x\searrow 0} \frac{1}{\sqrt{x}},$$

which does not exist (it is "$+\infty$"). Likewise, the limit from the left also does not exist ('tis "$-\infty$). Nonetheless, a drawing for the graph of f suggests an obvious tangent to the graph at $x = 0$, namely, the y-axis. That observation does not give us a derivative, because the y-axis is vertical and hence has no slope.

4.3. A graph with absolutely no tangents, *anywhere.* The previous two examples were about functions which did not have a derivative at $x = 0$. In both examples the point $x = 0$ was the only point where the function failed to have a derivative. It is easy to give examples of functions which are not differentiable at more than one value of x, but here I would like to show you a function f which doesn't have a derivative ***anywhere in its domain.***

To keep things short I won't write a formula for the function, and merely show you a graph. In this graph you see a typical path of a Brownian motion, i.e. t is time, and $x(t)$ is the position of a particle which undergoes a Brownian motion – come to lecture for further explanation (see also the article on wikipedia). To see a similar graph check the Dow Jones or Nasdaq in the upper left hand corner of the web page at http://finance.yahoo.com in the afternoon on any weekday.

Example 2

2b)

$$f(x) = \begin{cases} -x & if \quad x < 0 \\ x^2 & if \quad x \geq 0 \end{cases}$$

Solution

Step 1 - Find the left hand derivative

The formula for computation of the left hand derivative is:

$$f'(a^-) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$$

Since $f(x) = -x$ and $x < 0$, hence we will put the values in the formula:

$$\lim_{x \to 0^-} \frac{-(0+h) - 0}{h}$$

$$= \frac{-h}{h} = -1$$

Step 2 - Find the right hand derivative

The formula for finding out the right hand derivative is given below:

$$f'(a^+) = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

Since the second part of the function is $f(x) = x^2$ and $x \geq 0$, hence we will put the values in the formula:

$$\lim_{x \to 0^+} \frac{(0+h)^2 - 0}{h}$$

$$= \lim_{x \to 0^+} \frac{h^2}{h} = 1$$

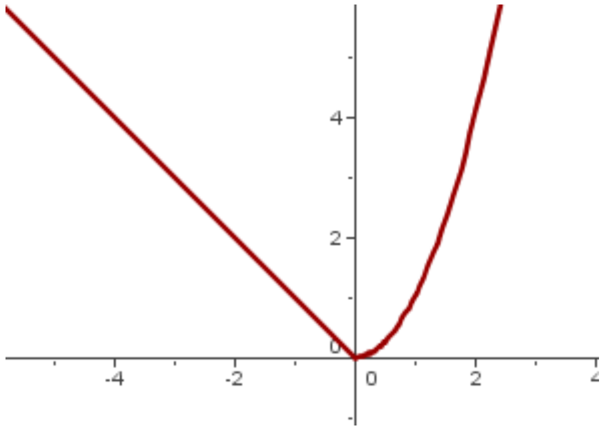
$$= \lim_{x \to 0^+} = 0$$

Step 3 - Compare the left and right hand derivatives

The left hand derivative is $f'_-(a) = -1$. Similarly, the right hand derivative is

$f'_+(a) = 0$. Since $f'_-(a) \neq f'_+(a)$, therefore we can say that the function is not differentiable at $x = 0$.

The function looks like this in the xy coordinate plane.



Example 2 - Graph of the piecewise function

Example 3

Check whether the function $f(x) = |x+3|$ differentiable at $x = -3$.

Solution

This function is an absolute-value function. We will follow the following steps to determine if the function $f(x) = |x+3|$ is differentiable at $x = -3$ or not.

Step 1 - Find the left hand derivative

To compute the left hand derivative, we use the following formula:

$$f'(a^-) = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h}$$

$$f'(a^-) = \lim_{h \to 0^-} = \frac{|(-3+h)+3 - f(-3)}{h}$$

$$f'(a^-) = \lim_{h \to 0^-} = \frac{|(-3+h)+3| - 0}{h}$$

$$\lim_{h \to 0^-} = \frac{|h|}{h}$$

As this is a left hand derivative, so $h < 0$

$$\lim_{h \to 0^-} = \frac{-h}{h} = -1$$

Step 2 - Find the right hand derivative

The formula for finding out the right hand derivative is given below:

$$f'(a^+) = \lim_{h \to 0+} \frac{f(a+h) - f(a)}{h}$$

$$f'(a^+) = \lim_{h \to 0+} = \frac{|(-3+h) + 3 - f(-3)}{h}$$

$$f'(a^-) = \lim_{h \to 0+} = \frac{|h|}{h}$$

$$\lim_{h \to 0-} = \frac{|h|}{h}$$

As this is a right hand derivative, so $h > 0$

$$\lim_{h \to 0^+} = \frac{h}{h} = 1$$

Step 3 - Compare the left and right hand derivatives

The left hand derivative is $f'_{-}(a) = -1$ and the right hand derivative is $f'_{+}(a) = 1$. Since $f'_{-}(a) \neq f'_{+}(a)$, therefore we can say that the function is not differentiable at $x = 3$.

MATH 1010E University Mathematics
Lecture Notes (week 4)
Martin Li

1 Derivatives of Piecewise Defined Functions

For piecewise defined functions, we often have to be very careful in computing the derivatives. The differentiation rules (product, quotient, chain rules) can only be applied if the function is defined by ONE formula in a neighborhood of the point where we evaluate the derivative. If we want to calculate the derivative at a point where two different formulas "meet", then we must use the definition of derivative as limit of difference quotient to correctly evaluate the derivative. Let us illustrate this by the following example.

Example 1.1 *Find the derivative $f'(x)$ at every $x \in \mathbb{R}$ for the piecewise defined function*

$$f(x) = \begin{cases} 5 - 2x & \text{when } x < 0, \\ x^2 - 2x + 5 & \text{when } x \geq 0. \end{cases}$$

Solution: We separate into 3 cases: $x < 0$, $x > 0$ and $x = 0$. For the first two cases, the function $f(x)$ is defined by a single formula, so we could just apply differentiation rules to differentiate the function.

$$f'(x) = (5 - 2x)' = -2 \qquad \text{for } x < 0,$$

$$f'(x) = (x^2 - 2x + 5)' = 2x - 2 \qquad \text{for } x > 0.$$

At $x = 0$, we have to use the definition of derivative as limit of difference quotient. First of all,

$$f(0) = 0^2 - 2(0) + 5 = 5.$$

Then we calculate the left-hand and right-hand limits:

$$\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} \frac{(5 - 2h) - 5}{h} = \lim_{h \to 0^-} -2 = -2,$$

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} \frac{(h^2 - 2h + 5) - 5}{h} = \lim_{h \to 0^+} (h - 2) = -2.$$

Since both of them exists and are equal, we have

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = -2.$$

Therefore, putting all of these together, we see that f is differentiable for every $x \in \mathbb{R}$ and

$$f'(x) = \begin{cases} -2 & \text{when } x \leq 0, \\ 2x - 2 & \text{when } x > 0. \end{cases}$$

Remark 1.2 *From the example above, we see that the derivative $f'(x)$ is still a continuous function (check this!). This is not always true for any function! (Have you seen a counterexample? See Homework 2)*

Example 1.3 *Consider the function defined by*

$$f(x) = \begin{cases} ax + b & \textit{when } x \leq -1, \\ ax^3 + x + 2b & \textit{when } x > -1, \end{cases}$$

for what value(s) of $a, b \in \mathbb{R}$ is the function f differentiable at every $x \in \mathbb{R}$?

Solution: First, it is easy to see that for ANY $a, b \in \mathbb{R}$, the function f is differentiable at every $x \neq -1$ since f is defined by a polynomial on $(-1, +\infty)$ and $(-\infty, -1)$. The only catch is at the point $x = -1$.

If f is differentiable at $x = -1$, it must also be continuous at $x = -1$. Therefore, we need

$$\lim_{x\to -1} f(x) = f(-1).$$

Now, $f(-1) = -a + b$ and the left-hand and right-hand limits are

$$\lim_{x\to -1^-} f(x) = a(-1)^3 + (-1) + 2b = -a + 2b - 1,$$

$$\lim_{x\to -1^+} f(x) = a(-1) + b = -a + b.$$

If f is continuous, then both of these limits must be the same and equal to $f(-1)$. Hence, we have

$$-a + b = -a + 2b - 1 \qquad \Rightarrow \qquad b = 1.$$

Now, we take $b = 1$. To find the value of a which make f differentiable at $x = -1$, we require the limit

$$\lim_{h\to 0} \frac{f(-1 + h) - f(-1)}{h}$$

$$\frac{f(x)-f(x_0)}{x-x_0}=f'(\xi_1) \quad (x\in(x_0-\delta,x_0),\xi_1\in(x,x_0)).$$

$$\frac{f(x)-f(x_0)}{x-x_0}=f'(\xi_2) \quad (x\in(x_0,x_0+\delta),\xi_2\in(x,x_0)).$$

Because $\lim\limits_{x\to x_0^-} f'(x)=\lim\limits_{x\to x_0^+} f'(x)$=A， so

$$f'_-(x_0)=\lim_{x\to x_0^-}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{x\to x_0^-} f'(\xi_1)=\lim_{\xi_1\to x_0^-} f'(\xi_1)=A,$$

$$f'_+(x_0)=\lim_{x\to x_0^+}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{x\to x_0^+} f'(\xi_1)=\lim_{\xi_2\to x_0^+} f'(\xi_2)=A,$$

so $f'(x_0)=f'_-(x_0)=f'_+(x_0)=A$.

Example 1[2] Discuss the derivative of $f(x)=\begin{cases}\sin x & x<0\\ x & x\geq 0\end{cases}$ at point $x=0$.

Solution: Because $f(x)$ is continuous at point $x=0$, and there are

$$\lim_{x\to 0^-} f'(x)=\lim_{x\to 0^-}\cos x=1 \qquad \lim_{x\to 0^+} f'(x)=\lim_{x\to 0^+} 1=1$$

then, $f'(0)$ exist and $f'(0)=1$.

In theorem 1, it is worth noting that the existence of $\lim\limits_{x\to x_0^-} f'(x)$ and $\lim\limits_{x\to x_0^+} f'(x)$ can not guarantee the existence of the derivatives at the piecewise point.

Example 2[3] Discuss the derivative of function $f(x)=\begin{cases}x+1 & x<0\\ x & x\geq 0\end{cases}$ at point $x=0$.

Solution: Because $\lim\limits_{x\to 0^-} f(x)\neq\lim\limits_{x\to 0^+} f(x)$, so $f(x)$ is not continuous at point $x=0$ and are not be differentiable, but

$$\lim_{x\to 0^-} f'(x)=\lim_{x\to 0^+} f'(x)=1$$

So, theorem 1 is used when it satisfied the two conditions at the same time. According to Theorem 1, we can obtain the corollary 1 when the left and right limitations are not equal.

Corollary 1 The piecewise function $f(x)$ is guided within a $\overset{o}{U}(x_0)$ of the picewise-point x_0, if both $\lim\limits_{x\to x_0^-} f'(x)$ and $\lim\limits_{x\to x_0^+} f'(x)$ are present but not equal, then $f(x)$ is not directed at point x_0.

Example 3[4] Discuss the derivatives of function $f(x)=\begin{cases}x^2 & x\geq 0\\ -x & x<0\end{cases}$ at point $x=0$.

Since both of them exists and are equal, we have

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = -2.$$

Therefore, putting all of these together, we see that f is differentiable for every $x \in \mathbb{R}$ and

$$f'(x) = \begin{cases} -2 & \text{when } x \leq 0, \\ 2x - 2 & \text{when } x > 0. \end{cases}$$

Remark 1.2 *From the example above, we see that the derivative $f'(x)$ is still a continuous function (check this!). This is not always true for any function! (Have you seen a counterexample? See Homework 2)*

Example 1.3 *Consider the function defined by*

3

$$f(x) = \begin{cases} ax + b & \textit{when } x \leq -1, \\ ax^3 + x + 2b & \textit{when } x > -1, \end{cases}$$

for what value(s) of $a, b \in \mathbb{R}$ is the function f differentiable at every $x \in \mathbb{R}$?

Solution: First, it is easy to see that for ANY $a, b \in \mathbb{R}$, the function f is differentiable at every $x \neq -1$ since f is defined by a polynomial on $(-1, +\infty)$ and $(-\infty, -1)$. The only catch is at the point $x = -1$.

If f is differentiable at $x = -1$, it must also be continuous at $x = -1$. Therefore, we need

$$\lim_{x \to -1} f(x) = f(-1).$$

Now, $f(-1) = -a + b$ and the left-hand and right-hand limits are

$$\lim_{x \to -1^-} f(x) = a(-1)^3 + (-1) + 2b = -a + 2b - 1,$$

$$\lim_{x \to -1^+} f(x) = a(-1) + b = -a + b.$$

If f is continuous, then both of these limits must be the same and equal to $f(-1)$. Hence, we have

$$-a + b = -a + 2b - 1 \qquad \Rightarrow \qquad b = 1.$$

Now, we take $b = 1$. To find the value of a which make f differentiable at $x = -1$, we require the limit

$$\lim_{h \to 0} \frac{f(-1 + h) - f(-1)}{h}$$

to exists, which is equivalent to the statement that the left-hand and right-hand limits exist and are equal. The left hand limit is

$$\lim_{h\to 0^-} \frac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0^-} \frac{[a(-1+h)+1]-(-a+1)}{h} = \lim_{h\to 0^-} \frac{ah}{h} = a.$$

The right hand limit is

$$\begin{aligned}
\lim_{h\to 0^+} \frac{f(-1+h)-f(-1)}{h} &= \lim_{h\to 0^+} \frac{[a(-1+h)^3+(-1+h)+2]-(-a+1)}{h} \\
&= \lim_{h\to 0^+} \frac{a[(-1+h)^3+1]+h}{h} \\
&= \lim_{h\to 0^+} \frac{ah[(-1+h)^2-(-1+h)+1]+h}{h} \\
&= 3a+1.
\end{aligned}$$

Therefore, if we set them equal to each other, we obtain the condition

$$a = 3a+1 \qquad \Rightarrow \qquad a = -\frac{1}{2}.$$

In summary, we have $a = -1/2$ and $b = 1$ if f is differentiable at every $x \in \mathbb{R}$.

2 Differentiation Rules II: Product and Quotient Rules

Theorem 2.1 *If f and g are differentiable functions, then both their product fg and quotient f/g are differentiable and we have*

(1) Product Rule:

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x),$$

(2) Quotient Rule:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2},$$

provided that $g(x) \neq 0$.

Remark 2.2 *Observe that the differentiation rule $[kf(x)]' = kf'(x)$ where k is a constant is just a special case of product rule by taking $g(x) \equiv k$, which has $g'(x) = 0$.*

3)

A toy company wants to design a track for a toy car that starts out along a parabolic curve and then converts to a straight line (Figure $3.2.7$). The function that describes the track is to have the form $f(x)=\begin{cases}\frac{1}{10}x^2+bx+c, & \text{if } x<-10\\ -\frac{1}{4}x+\frac{5}{2}, & \text{if } x\geq -10\end{cases}$ where x and $f(x)$ are in inches. For the car to move smoothly along the track, the function $f(x)$ must be both continuous and differentiable at -10. Find values of b and c that make $f(x)$ both continuous and differentiable.

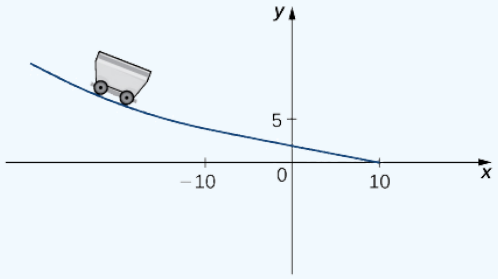


Figure $3.2.7$: For the car to move smoothly along the track, the function must be both continuous and differentiable.

Solution

For the function to be continuous at $x=-10$, $\lim\limits_{x\to 10^-} f(x)=f(-10)$. Thus, since

$$\lim_{x\to -10^-} f(x)=\frac{1}{10}(-10)^2-10b+c=10-10b+c$$

and $f(-10)=5$, we must have $10-10b+c=5$. Equivalently, we have $c=10b-5$.

For the function to be differentiable at -10,

$$f'(10)=\lim_{x\to -10}\frac{f(x)-f(-10)}{x+10}$$

must exist. Since $f(x)$ is defined using different rules on the right and the left, we must evaluate this limit from the right and the left and then set them equal to each other:

$$\begin{aligned}\lim_{x\to -10^-}\frac{f(x)-f(-10)}{x+10} &= \lim_{x\to -10^-}\frac{\frac{1}{10}x^2+bx+c-5}{x+10}\\ &= \lim_{x\to -10^-}\frac{\frac{1}{10}x^2+bx+(10b-5)-5}{x+10} && \text{Substitute } c=10b-5.\\ &= \lim_{x\to -10^-}\frac{x^2-100+10bx+100b}{10(x+10)}\\ &= \lim_{x\to -10^-}\frac{(x+10)(x-10+10b)}{10(x+10)} && \text{Factor by grouping}\\ &= b-2\end{aligned}$$

We also have

$$\begin{aligned}\lim_{x\to -10^+}\frac{f(x)-f(-10)}{x+10} &= \lim_{x\to -10^+}\frac{-\frac{1}{4}x+\frac{5}{2}-5}{x+10}\\ &= \lim_{x\to -10^+}\frac{-(x+10)}{4(x+10)}\\ &= -\frac{1}{4}\end{aligned}$$

This gives us $b-2=-\frac{1}{4}$. Thus $b=\frac{7}{4}$ and $c=10(\frac{7}{4})-5=\frac{25}{2}$.

4

Any function which is differentiable at a point x_0 must also be continuous at x_0. Since the left and right hand limits of f do not agree, your function is not continuous at 0. Therefore the derivative does not exist at 0 even though the derivative seems to be approaching the same value from both directions.

In more detail,

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{h^2+1-1}{h} = \lim_{h\to 0^+} h = 0.$$

But

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{h^2-1-1}{h} = \lim_{h\to 0^-} \frac{h^2-2}{h} = \infty.$$