16th lesson

https://www2.karlin.mff.cuni.cz/~kuncova/en/teachMat1.php
kunck6am@natur.cuni.cz

Theory

Definition 1. Let f be a function and $a \in \mathbb{R}$. If the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

exists, then it is called the *derivative* of the function f at a point a. It is denoted by $f'(a)$.

Theorem 2. Suppose that the function f has a finite derivative at a point $a \in \mathbb{R}$. Then f is continuous at a.

Theorem 3. Suppose that a function f is continuous from the right at $a \in \mathbb{R}$ and the limit $\lim_{x \to a+} f'(x)$ exists. Then the derivative $f'(a)$ exists and

$$f'_+(a) = \lim_{x \to a+} f'(x).$$

Exercises

Intro: Write a dating ad for a function and its derivation.

Find the derivatives using definition or theorem.

1. (a) $x^2 - 4x$
   (b) $3x^2 + 2x + 7$
   (c) x^3
   (d) $\dfrac{1}{x+3}$
   (e) $x^3 - x$
   (f) $\sqrt{x}$
   (g) $\dfrac{1-x}{2+x}$
2. (a) $\sqrt{|x|}$ at $x = 0$
   (b) $f(x) = \begin{cases} -x, & x < 0 \\ x^2, & x \geq 0. \end{cases}$
   (c) $|x+3|$ at $x = -3$
   (d) $f(x) = \begin{cases} 5 - 2x, & x < 0 \\ x^2 - 2x + 5, & x \geq 0. \end{cases}$
   (e) $f(x) = \begin{cases} \sin x, & x < 0 \\ x, & x \geq 0. \end{cases}$ at $x = 0$
   (f) $f(x) = \begin{cases} x+1, & x < 0 \\ x, & x \geq 0. \end{cases}$ at $x = 0$
3. Find such $a, b \in \mathbb{R}$ such that the following function is differentiable at every $x \in \mathbb{R}$:
   $$f(x) = \begin{cases} ax + b, & x \leq -1 \\ ax^3 + x + 2b, & x > -1. \end{cases}$$

4. You need to design a track for a toy car which starts as a parabola and continue as a line with the following parameters:

$$f(x) = \begin{cases} \frac{1}{10}x^2 + bx + c, & x < -10 \\ -\frac{1}{4}x + \frac{5}{2}, & x \geq -10. \end{cases}$$

Find $b, c \in \mathbb{R}$ such that the track is continuous and differentiable.

Source and a picture: `https://math.libretexts.org/Bookshelves/Calculus/Book%3A_Calculus_(OpenStax)/03%3A_Derivatives/3.2%3A_The_Derivative_as_a_Function`

5. Help:

Given the function:

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x \geq 0 \\ x^2 - 1 & \text{if } x < 0 \end{cases}$$

Question: are we justified to say that the derivative at $f(0)$ exists? If so, what is $f'(0)$? And how do we justify it?

Of course I do realize that the function isn't continuous at $x = 0$ but still since the slope near $x = 0$ seems equal near $0+$ and $0-$ I wondered why we can't say that $f'(0) = 0$

What I tried is this:

$$f'_+(0) = \lim_{h \to 0+} \frac{(x+h)^2 + 1 - (x^2+1)}{h} = \lim_{h \to 0+} \frac{(0+h)^2 + 1 - (0^2+1)}{h} = \lim_{h \to 0+} \frac{h^2}{h} = h = 0$$
$$f'_-(0) = \lim_{h \to 0-} \frac{(x+h)^2 + 1 - (x^2+1)}{h} = \lim_{h \to 0-} \frac{(0+h)^2 + 1 - (0^2+1)}{h} = \lim_{h \to 0-} \frac{h^2}{h} = h = 0$$

My conclusion is that since both the right and left limit using the definition of the derivative exist and generate the same answer the limit exists such that $f'(0) = 0$.

Apparently this is not true, so what is my mistake?

Source: `https://math.stackexchange.com/questions/1532014/how-to-apply-the-definition-of-a-derivative-with-a-piecewise-function`