15th lesson

https://www2.karlin.mff.cuni.cz/~kuncova/en/teachMat1.php
kunck6am@natur.cuni.cz

Theory

Definition 1. Let f be a function and $a \in \mathbb{R}$. If the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

exists, then it is called the *derivative* of the function f at a point a. It is denoted by $f'(a)$.

Theorem 2 (Arithmetics of derivatives)**.** Let $a \in \mathbb{R}$ and f a g be functions defined on some neigbourhood of a point a. Let us suppose that $f'(a) \in \mathbb{R}^*$ and $g'(a) \in \mathbb{R}^*$ exist.

(a) Then

$$(f \pm g)'(a) = f'(a) \pm g'(a),$$

(b) If f or g is continuous at a, then

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a),$$

(c) If g is continuous at a and $g(a) \neq 0$, then

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2},$$

if the right sides are well defined.

Theorem 3 (Derivative of a compound function)**.** Let us suppose that the function f has a derivative at $y_0 \in \mathbb{R}$, the function g has derivative at $x_0 \in \mathbb{R}$, $y_0 = g(x_0)$ and g is continuous at x_0. Then

$$(f \circ g)'(x_0) = f'(y_0)g'(x_0) = f'(g(x_0))\, g'(x_0),$$

if the right side is well defined.

Hints

$a^b = e^{b \ln a}$

Exercises

Find the derivatives (find also the domains of f and f'):

1. (a) $6x$

   Solution: $(6x)' = 6 \cdot 1$, $x \in \mathbb{R}$

   (b) $x^3 + 2x - \sin x + 2$

   Solution: $(x^3 + 2x - \sin x + 2)' = 3x^2 + 2 - \cos x + 0$, $x \in \mathbb{R}$

   (c) $-2\cos x + 4e^x + \frac{1}{3}x^7$

   Solution: $(-2\cos x + 4e^x + \frac{1}{3}x^7)' = -2 \cdot (-\sin x) + 4e^x + \frac{1}{3} \cdot 7x^6$, $x \in \mathbb{R}$

   (d) $\sqrt{x} + \frac{2}{\sqrt{x}}$

   Solution: $(\sqrt{x} + \frac{2}{\sqrt{x}})' = (x^{1/2} + 2x^{-1/2})' = \frac{1}{2}x^{-1/2} - 2\frac{1}{2}x^{-3/2} = \frac{1}{2\sqrt{x}} - \frac{1}{\sqrt{x^3}}$, $x > 0$

   (e) $\sqrt[3]{x} - \sqrt[4]{x^7}$

   Solution: $(\sqrt[3]{x} - \sqrt[4]{x^7})' = (x^{1/3} - x^{7/4})' = 1/3x^{-2/3} - 7/4x^{3/4} = \frac{1}{3\sqrt[3]{x^2}} - \frac{7}{4}\sqrt[4]{x^3}$ $x > 0$

   (f) $\frac{1}{x} + \frac{2}{x^2} + \frac{3}{x^3}$

   Solution:

   $$\left(\frac{1}{x} + \frac{2}{x^2} + \frac{3}{x^3}\right)' = \left(x^{-1} + 2x^{-2} + 3x^{-3}\right)' = -x^{-2} - 4x^{-3} - 9x^{-4}$$

   $x \neq 0$

   (g) $\ln x + \frac{\cos x}{\pi}$

   Solution: $(\ln x + \frac{\cos x}{\pi})' = \frac{1}{x} - \frac{1}{\pi}\sin x$

   $x > 0$

   (h) $\cot x + \tan x$

   Solution: $(\cot x + \tan x)' = -\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x}$

   $x \neq k\frac{\pi}{2}$, $k \in \mathbb{Z}$

   (i) $\arcsin x - 3\operatorname{arccot} x$

   Solution: $(\arcsin x - 3\operatorname{arccot} x)' = \frac{1}{\sqrt{1-x^2}} + \frac{3}{1+x^2}$

   $x \in (-1, 1)$

   (j) $2\arctan x + \arccos x$

   Solution: $(2\arctan x + \arccos x)' = \frac{2}{1+x^2} + \frac{-1}{\sqrt{1-x^2}}$ $x \in \mathbb{R}$

2. (a) xe^x

   Solution:

$$(xe^x) = x'e^x + x(e^x)' = e^x + xe^x$$

   $x \in \mathbb{R}$

   (b) $\dfrac{1+x-x^2}{1-x+x^2}$

   Solution:

$$\left(\frac{1+x-x^2}{1-x+x^2}\right)' = \frac{(1+x-x^2)'(1-x+x^2)-(1+x-x^2)(1-x+x^2)'}{(1-x+x^2)^2}$$
$$= \frac{(1-2x)(1-x+x^2)-(1+x-x^2)(-1+2x)}{(1-x+x^2)^2}$$

   $x \in \mathbb{R}$

   (c) $x^2e^x \sin x$

   Solution:

$$(x^2e^x\sin x)' = (x^2)'e^x\sin x + x^2(e^x\sin x)' = 2xe^x\sin x + x^2((e^x)'\sin x + e^x(\sin x)')$$
$$= 2xe^x\sin x + x^2(e^x\sin x + e^x\cos x)$$

   $x \in \mathbb{R}$

   (d) $\dfrac{3x-2}{x^2+1}$

   Solution:

$$\left(\frac{3x-2}{x^2+1}\right)' = \frac{3(1+x^2)-(3x-2)2x}{(1+x^2)^2}$$

   $x \in \mathbb{R}$

   (e) $e^x(x^2-2x+2)$

   Solution: Let us use the arithmetics of derivatives:

$$(e^x(x^2-2x+2))' \overset{AD}{=} e^{x\prime}(x^2-2x+2) + e^x(x^2-2x+2)' =$$
$$e^x(x^2-2x+2) + e^x(2x-2) = e^x \cdot x^2$$

   Conditions: e^x is continuous on $\mathbb{R}$.

   (f) $\dfrac{1}{\ln x}$

   Solution:

$$\left(\frac{1}{\ln x}\right)' = \frac{0-\frac{1}{x}}{(\ln x)^2} = \frac{-1}{x\ln^2 x}$$

   $x > 0$, $x \neq 1$

3. (a) $\operatorname{arcctg} 2x$

   Solution: $(\operatorname{arcctg} 2x)' = \frac{-1}{1+(2x)^2} \cdot 2$ $x \in \mathbb{R}$

(b) $(3x^2-2x+10)^{10}$
Solution: $(3x^2-2x+10)^{10} = 10(3x^2-2x+10)^9(6x-2)$, $x \in \mathbb{R}$

(c) $\sqrt{x} - \arctan \sqrt{x}$
Solution:

$$(\sqrt{x} - \arctan \sqrt{x})' = \frac{1}{2\sqrt{x}} - \frac{1}{1+(\sqrt{x})^2}\frac{1}{2\sqrt{x}}$$

$x > 0$

(d) $\ln^3 x^2$ **Solution:**

$$(\ln^3 x^2)' = 3(\ln^2 x^2) \cdot \frac{1}{x^2} \cdot 2x$$

$x > 0$

(e) $\sqrt{4-x^2}$
Solution:

$$\sqrt{4-x^2}' = \frac{1}{2\sqrt{4-x^2}}(-2x)$$

$x \in (-2, 2)$

(f) $\ln(\sin x)$ **Solution:**

$$(\ln(\sin x))' = \frac{1}{\sin x}\cos x$$

$\sin x > 0$, hence $x \in (0, \pi) + k\pi$, $k \in \mathbb{Z}$

(g) $\ln\ln(x-3) + \arcsin \dfrac{x-5}{2}$
Solution:

$$\left(\ln\ln(x-3) + \arcsin \frac{x-5}{2}\right)' = \frac{1}{\ln(x-3)}\frac{1}{x-3} + \frac{1}{\sqrt{1-\left(\frac{x-5}{2}\right)^2}}\frac{1}{2}$$

$x > 3$, $x - 3 > 1$, hence $x > 4$. Further $3 < x < 7$. Together we have $4 < x < 7$

(h) x^x
Solution: At first we use the hint:

$$x^x = e^{x \ln x}.$$

At first we decompose the function: $f(y) = e^y$, $f(y)' = e^y$ a $g(x) = x \ln x$ and $g'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x}$ (derivative of a product, condition: x continuous on$\mathbb{R}$). Finally we have:

$$(e^{x\ln x})' = e^{x \ln x}(\ln x + 1) = x^x(\ln x + 1)$$

Because of the logarithm domain, we have $x > 0$. Conditions: x and $\ln x$ are continusou, their product is continuous too.

(i) $x^{(\sin x)}$

Solution:

$$(x^{\sin x})' = (e^{\sin x \ln x})' = e^{\sin x \ln x} \cdot (\cos x \ln x + \sin x \cdot \frac{1}{x})$$

$x > 0$

(j) $\sin(\sin(\sin x))$

Solution:

$$(\sin(\sin(\sin x)))' \overset{SD}{=} \cos(\sin(\sin x)) \cdot (\sin(\sin x))' \overset{SD}{=}$$
$$\cos(\sin(\sin x)) \cdot \cos(\sin x) \cdot (\sin x)' \overset{SD}{=} \cos(\sin(\sin x)) \cdot \cos(\sin x) \cdot (\cos x)$$

Conditions: all functions are continuous on $\mathbb{R}$.

(k) $\ln(\ln^2(\ln^3 x))$

Solution: We need to apply the derivative of compounded function serveral times. At first: $f(y) = \ln y$, $f(y)' = \frac{1}{y}$, $g(x) = \ln^2(\ln^3 x)$:

$$(\ln(\ln^2(\ln^3 x)))' \overset{SD}{=} \frac{1}{\ln^2(\ln^3 x)} \cdot (\ln^2(\ln^3 x))'$$

Now we need to derive $(\ln^2(\ln^3 x))$: outer function $f(y) = y^2$, $f'(y) = 2y$ inner function $g(x) = \ln(\ln^3 x)$. Together:

$$(\ln^2(\ln^3 x))' \overset{SD}{=} 2\ln(\ln^3 x) \cdot (\ln(\ln^3 x))'.$$

Further, outer function $f(y) = \ln y$, $f'(y) = \frac{1}{y}$ inner function $g(x) = \ln^3 x$:

$$(\ln(\ln^3 x))' \overset{SD}{=} \frac{1}{\ln^3 x} \cdot (\ln^3 x)'$$

Finally, outer function $f(y) = y^3$, $f'(y) = 3y^2$ inner function $g(x) = \ln x$, $g'(x) = \frac{1}{x}$, together

$$(\ln^3 x)' = 3\ln^2 x \cdot \frac{1}{x}$$

Result:

$$(\ln(\ln^2(\ln^3 x)))' \overset{SD}{=} \frac{1}{\ln^2(\ln^3 x)} \cdot (\ln^2(\ln^3 x))' \overset{SD}{=} \frac{1}{\ln^2(\ln^3 x)} \cdot 2\ln(\ln^3 x) \cdot (\ln(\ln^3 x))' \overset{SD}{=}$$
$$\frac{1}{\ln^2(\ln^3 x)} \cdot 2\ln(\ln^3 x) \cdot \frac{1}{\ln^3 x} \cdot (\ln^3 x)' \overset{SD}{=} \frac{1}{\ln^2(\ln^3 x)} \cdot 2\ln(\ln^3 x) \cdot \frac{1}{\ln^3 x} \cdot 3\ln^2 x \cdot (\ln x)' \overset{SD}{=}$$
$$\frac{1}{\ln^2(\ln^3 x)} \cdot 2\ln(\ln^3 x) \cdot \frac{1}{\ln^3 x} \cdot 3\ln^2 x \cdot \frac{1}{x} = \frac{6}{x\ln(\ln^3 x)\ln x}$$

Domain:

$$\ln^2(\ln^3 x) > 0$$

$$|\ln^3 x| > 1$$
$$\ln x > 1$$
$$x > e.$$

Conditions: all polynomials and logarithm are continuous on (e, ∞), their composition and product is also continuous.

(l) $\dfrac{\sin^2 x}{\sin x^2}$

Solution: We have the derivative of a fraction and also compounded function:

$$\left(\frac{\sin^2 x}{\sin x^2}\right)' \overset{AD}{=} \frac{(\sin^2 x)' \sin x^2 - \sin^2 x(\sin x^2)'}{(\sin x^2)^2}$$

Compounded functions: at first – $\sin^2 x$, outer $f(y) = y^2$, $f'(y) = 2y$ and inner $g(x) = \sin x$, $g'(x) = \cos x$, sin is a continuous function $(\sin^2 x)' \overset{SD}{=} 2 \sin x \cdot \cos x$.
at second – $\sin x^2$, outer $f(y) = \sin y$, $f'(y) = \cos y$, inner $g(x) = x^2$, $g'(x) = 2x$. (Conditions: polynomials are continuous.) Together: $(\sin x^2)' \overset{SD}{=} \cos x^2 \cdot 2x$

Finally we obtain:

$$\left(\frac{\sin^2 x}{\sin x^2}\right)' \overset{AD}{=} \frac{(\sin^2 x)' \sin x^2 - \sin^2 x(\sin x^2)'}{(\sin x^2)^2} \overset{SD}{=} \frac{2 \sin x \cos x \sin x^2 - \sin^2 x \cos x^2 2x}{(\sin x^2)^2}$$

Conditions: $x^2 \neq k\pi$.

(m) $2^{\tan \frac{1}{x}}$

Solution: At first we rewrite:

$$2^{\tan \frac{1}{x}} = e^{\tan \frac{1}{x} \cdot \ln 2}$$

Let us note that $\ln 2$ is just a constant.
Now we derive a composed function. Outer function: $f(y) = e^y$, $f'(y) = e^y$, inner $g(x) = \ln 2 \tan \frac{1}{x}$:

$$\left(e^{\tan \frac{1}{x} \cdot \ln 2}\right)' = e^{\tan \frac{1}{x} \cdot \ln 2} \cdot \left(\ln 2 \tan \frac{1}{x}\right)'$$

Again composed function, outer: $f(y) = \tan y$, $f'(y) = \frac{1}{\cos^2 y}$ inner $g(x) = \frac{1}{x}$, $g'(x) = -\frac{1}{x^2}$, $\frac{1}{x}$ is continuouso on $\mathbb{R} \setminus \{0\}$, together $\left(\tan \frac{1}{x}\right)' = \frac{1}{\cos^2 \frac{1}{x}} \cdot \frac{-1}{x^2}$
Hence we have

$$\left(e^{\tan \frac{1}{x} \cdot \ln 2}\right)' = e^{\tan \frac{1}{x} \cdot \ln 2} \cdot \left(\tan \frac{1}{x}\right)' = 2^{\tan \frac{1}{x}} \cdot \ln 2 \frac{1}{\cos^2 \frac{1}{x}} \cdot \frac{-1}{x^2}$$

Conditions: $\frac{1}{x} \neq 0$ and (tan function gives) $\frac{1}{x} \neq \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$. The function $\tan \frac{1}{x}$ is continuous on these intervals.

(n) $\dfrac{1}{\sqrt{2}} \operatorname{arccotg} \dfrac{\sqrt{2}}{x}$

Solution: Composed function, outer: $f(y) = \operatorname{arcctg} y$, $f'(y) = \frac{-1}{1+y^2}$ and inner: $g(x) = \frac{\sqrt{2}}{x}$, $g'(x) = \sqrt{2}\frac{-1}{x^2}$, g is continuous on $(-\infty, 0) \cup (0, \infty)$. Further

$$\left(\frac{1}{\sqrt{2}} \operatorname{arcctg} \frac{\sqrt{2}}{x} \right)' \overset{SD}{=} \frac{1}{\sqrt{2}} \frac{-1}{1 + \frac{2}{x^2}} \sqrt{2} \cdot \frac{-1}{x^2} = \frac{1}{2 + x^2}$$

Conditions: $x \neq 0$.

(o) $\dfrac{x^p(1-x)^q}{1+x}, \qquad p, q > 0$

Solution: Derivative of fraction, product and of composed function:

$$\left(\frac{x^p(1-x)^q}{1+x} \right)' \overset{AD}{=} \frac{[px^{p-1}(1-x)^q + x^p q(1-x)^{q-1}(-1)](1+x) - x^p(1-x)^q \cdot 1}{(1+x)^2}$$

Conditions: All functions are continuous and defined on $\mathbb{R} \setminus \{-1\}$.