15th lesson

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Theory

Definition 1. Let f be a function and $a \in \mathbb{R}$. If the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

exists, then it is called the *derivative* of the function f at a point a. It is denoted by $f'(a)$.

Theorem 2 (Arithmetics of derivatives)**.** Let $a \in \mathbb{R}$ and f a g be functions defined on some neigbourhood of a point a. Let us suppose that $f'(a) \in \mathbb{R}^*$ and $g'(a) \in \mathbb{R}^*$ exist.

(a) Then
$$(f \pm g)'(a) = f'(a) \pm g'(a),$$

(b) If f or g is continuous at a, then
$$(fg)'(a) = f'(a)g(a) + f(a)g'(a),$$

(c) If g is continuous at a and $g(a) \neq 0$, then
$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2},$$

if the right sides are well defined.

Theorem 3 (Derivative of a compound function)**.** Let us suppose that the function f has a derivative at $y_0 \in \mathbb{R}$, the function g has derivative at $x_0 \in \mathbb{R}$, $y_0 = g(x_0)$ and g is continuous at x_0. Then

$$(f \circ g)'(x_0) = f'(y_0)g'(x_0) = f'(g(x_0))\, g'(x_0),$$

if the right side is well defined.

Hints

$a^b = e^{b \ln a}$

Exercises

Find the derivatives (find also the domains of f and f'):

1. (a) $6x$
   (b) $x^3 + 2x - \sin x + 2$
   (c) $-2\cos x + 4e^x + \frac{1}{3}x^7$
   (d) $\sqrt{x} + \frac{2}{\sqrt{x}}$
   (e) $\sqrt[3]{x} - \sqrt[4]{x^7}$
   (f) $\frac{1}{x} + \frac{2}{x^2} + \frac{3}{x^3}$
   (g) $\ln x + \frac{\cos x}{\pi}$
   (h) $\cot x + \tan x$
   (i) $\arcsin x - 3\operatorname{arccot} x$
   (j) $2\arctan x + \arccos x$

2. (a) xe^x
   (b) $\frac{1 + x - x^2}{1 - x + x^2}$
   (c) $x^2 e^x \sin x$
   (d) $\frac{3x - 2}{x^2 + 1}$
   (e) $e^x(x^2 - 2x + 2)$
   (f) $\frac{1}{\ln x}$

3. (a) $\operatorname{arcctg} 2x$
   (b) $(3x^2 - 2x + 10)^{10}$
   (c) $\sqrt{x} - \arctan \sqrt{x}$
   (d) $\ln^3 x^2$
   (e) $\sqrt{4 - x^2}$
   (f) $\ln(\sin x)$
   (g) $\ln\ln(x - 3) + \arcsin \frac{x - 5}{2}$
   (h) x^x
   (i) $x^{(\sin x)}$
   (j) $\sin(\sin(\sin x))$
   (k) $\ln(\ln^2(\ln^3 x))$
   (l) $\frac{\sin^2 x}{\sin x^2}$
   (m) $2^{\tan \frac{1}{x}}$
   (n) $\frac{1}{\sqrt{2}}\operatorname{arccotg} \frac{\sqrt{2}}{x}$
   (o) $\frac{x^p(1 - x)^q}{1 + x}, \qquad p, q > 0$