10th lesson

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Theory

Theorem 1 (Squeeze theorem)**.** Let I be an interval having the point a as a limit point. Let g, f, and h be function defined on I, except possibly at a itself. Suppose that for every x in I not equal to a, we have

$$g(x) \leq f(x) \leq h(x)$$

and also suppose that

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L.$$

Then

$$\lim_{x \to a} f(x) = L.$$

Facts

1. $\beta > 0$, $a > 1$: $\lim_{x \to +\infty} \frac{x^\beta}{a^x} = 0$.
2. $\alpha > 0$, $\beta > 0$: $\lim_{x \to +\infty} \frac{\ln^\alpha x}{x^\beta} = 0$.

Exercises

Set x

1. Find limits:

   (a) $\lim_{x \to 5} 10x + 7 = 57$

   (b) $\lim_{x \to 1} (3x - 1)^{10} = 2^{10}$

   (c) $\lim_{x \to -1} \dfrac{3x - 4}{8x^2 + 2x - 2} = \dfrac{-7}{4}$

   (d) $\lim_{x \to \pi} \dfrac{\tan x}{x} = 0$

   (e) $\lim_{x \to \pi} x \cos x = -\pi$

   (f) $\lim_{x \to \infty} 4 - \dfrac{3}{x^2} = 4$

   (g) $\lim_{x \to 3} \ln(2x + 6) = \ln(12)$

   (h) $\lim_{x \to \infty} \sqrt{x} + \text{arccot } x = \infty$

   (i) $\lim_{x \to 0+} \dfrac{-\sin x}{\ln x} = 0$

∞

2. Find limits:

   (a) $\lim_{x \to \infty} \dfrac{-2x + 3}{3x^2 + 1}$

   (b) $\lim_{x \to \infty} \dfrac{-2x^2 + 3}{3x^2 + 1}$

(c) $\lim_{x\to\infty} \frac{-2x^3+3}{3x^2+1}$

(d) $\lim_{x\to\infty} \frac{1}{x^2-x-1}$

(e) $\lim_{x\to\infty} \frac{x}{\sqrt{3x^2+2}}$

(f) $\lim_{x\to\infty} \frac{x-\cos x}{x}$

(g) $\lim_{x\to 0} x^2 \sin \frac{1}{x}$

Example 2 Comparing Limits at Infinity

Find the limit as x approaches ∞ for each function.

a. $f(x) = \dfrac{-2x + 3}{3x^2 + 1}$

b. $f(x) = \dfrac{-2x^2 + 3}{3x^2 + 1}$

c. $f(x) = \dfrac{-2x^3 + 3}{3x^2 + 1}$

Solution

In each case, begin by dividing both the numerator and denominator by

$$x^2$$

the highest-powered term in the denominator.

2a

a. $$\lim_{x\to\infty} \frac{-2x + 3}{3x^2 + 1} = \lim_{x\to\infty} \frac{-\frac{2}{x} + \frac{3}{x^2}}{3 + \frac{1}{x^2}}$$

$$= \frac{-0 + 0}{3 + 0}$$

$$= 0$$

2b

b. $$\lim_{x\to\infty} \frac{-2x^2 + 3}{3x^2 + 1} = \lim_{x\to\infty} \frac{-2 + \frac{3}{x^2}}{3 + \frac{1}{x^2}}$$

$$= \frac{-2 + 0}{3 + 0}$$

$$= -\frac{2}{3}$$

2c

c. $$\lim_{x\to\infty} \frac{-2x^3 + 3}{3x^2 + 1} = \lim_{x\to\infty} \frac{-2x + \frac{3}{x^2}}{3 + \frac{1}{x^2}} = \frac{-2\cdot\infty + 0}{3 + 0} = -\infty$$

In this case, you can conclude that the limit does not exist because the numerator decreases without bound as the denominator approaches 3.

✓CHECKPOINT Now try Exercise 19.

In Example 2, observe that when the degree of the numerator is less than the degree of the denominator, as in part (a), the limit is 0. When the degrees of the numerator and denominator are equal, as in part (b), the limit is the ratio of the coefficients of the highest-powered terms. When the degree of the numerator is greater than the degree of the denominator, as in part (c), the limit does not exist.

This result seems reasonable when you realize that for large values of x, the highest-powered term of a polynomial is the most "influential" term. That is, a polynomial tends to behave as its highest-powered term behaves as x approaches positive or negative infinity.

Explore the Concept

Use a graphing utility to complete the table below to verify that

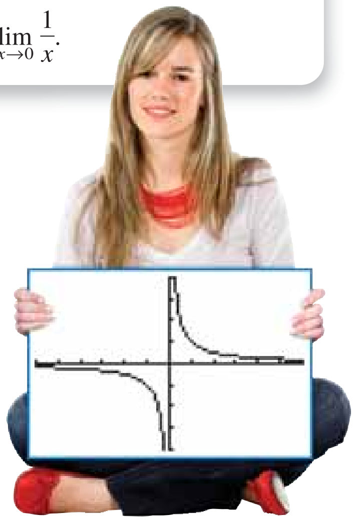
$$\lim_{x\to\infty} \frac{1}{x} = 0.$$

x	10^0	10^1	10^2
$\frac{1}{x}$			

x	10^3	10^4	10^5
$\frac{1}{x}$			

Make a conjecture about

$$\lim_{x\to 0} \frac{1}{x}.$$



Activity
Have students use these observations from Example 2 to predict the following limits.

a. $\lim_{x\to\infty} \dfrac{5x(x - 3)}{2x}$

b. $\lim_{x\to\infty} \dfrac{4x^3 - 5x}{8x^4 + 3x^2 - 2}$

c. $\lim_{x\to\infty} \dfrac{-6x^2 + 1}{3x^2 + x - 2}$

Then ask several students to verify the predictions algebraically, several other students to verify the predictions numerically, and several more students to verify the predictions graphically. Lead a discussion comparing the results.

2d

EXAMPLE 4. Evaluate limit (not evoke graphs)

$$\lim_{x\to\infty} \frac{1}{x^2 - x - 1}$$

Attention: indeterminacy $\infty - \infty$

$$= \lim_{x\to\infty} \frac{1}{x^2\left(1 - \frac{1}{x} - \frac{1}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{1}{x^2} \lim_{x\to\infty} \frac{1}{1 - \frac{1}{x} - \frac{1}{x^2}} = 0 \cdot 1 = 0$$

Example 5: Evaluate the limit $\lim_{x\to\infty}(5x^3+6x^2+x+1)/(3x^3+x^2-x+7)$. Dividing numerator and denominator by the highest power in the denominator, we have

$$\lim_{x\to\infty}\frac{5x^3+6x^2+x+1}{3x^3+x^2-x+7}=\lim_{x\to\infty}\frac{5+\frac{6}{x}+\frac{1}{x^2}+\frac{1}{x^3}}{3+\frac{1}{x}-\frac{1}{x^2}+\frac{7}{x^3}}=\frac{5}{3}$$

The examples involve rational functions $R(x)=P(x)/Q(x)$, i.e. quotients of polynomials. Three typical situations are illustrated. In Example 3 the degree of the numerator is less than the degree of the denominator, and the limit is 0. In Example 4 the degree of the numerator is greater than the degree of the denominator, and the limit is ∞. In Example 5 the degrees of the numerator and denominator are the same, and the limit is the quotient of the coefficients of the highest power terms. These three cases are often codified as rules:

Dominant Term Rule: For the limit $\lim_{x\to\infty}P(x)/Q(x)$, where $P(x)$ is a polynomial of degree n and $Q(x)$ is a polynomial of degree m,

1. If $n<m$, the limit is 0,
2. If $n>m$, the limit is $\pm\infty$,
3. If $n=m$, the limit is the quotient of the coefficients of the highest powers.

Our advice is to ignore this rule as just so much clutter. Memorizing more rules just obscures the technique illustrated in the three examples. The technique applies to more than just limits of rational functions, hence warrants your attention.

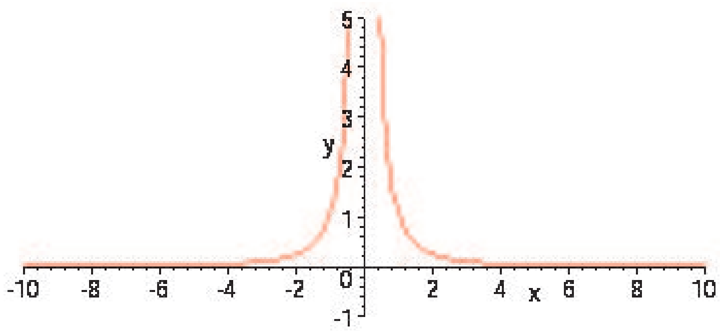
Example 6: As an example involving a non-rational function, evaluate $\lim_{x\to\infty}x/\sqrt{3x^2+2}$. In this example, thinking in the dominant term style, we suspect that the denominator will behave very much like the function $\sqrt{3x^2}=\sqrt{3}x$. Thus we guess that the limit is $1/\sqrt{3}$. This is indeed the case as the following computation shows:

2e

$$\begin{aligned}\lim_{x\to\infty}\frac{x}{\sqrt{3x^2+2}}&=\lim_{x\to\infty}\frac{x}{\sqrt{x^2(3+\frac{2}{x^2})}}\\&=\lim_{x\to\infty}\frac{1}{\sqrt{3+\frac{2}{x^2}}}=\frac{1}{\sqrt{3}}\end{aligned}$$

Note that the *Dominant Term Rule* does not apply directly to this example, but the technique underlying it does. Before concluding this section, we give a few examples of infinite limits:

Example 7: Evaluate $\lim_{x\to 0}1/x^2$. The limit does not exist, of course, since it is of the form "$\frac{1}{0}$". But let us analyse the right-hand and left-hand limits at 0. Clearly $\lim_{x\to 0^+}(1/x^2)=\infty$ as does the left-hand limit (the function is an even function). In this case the right-hand and left-hand limits do not differ, so we can also write $\lim_{x\to 0}(1/x^2)=\infty$. Although the limit DNE, the notation signals additional information about how the function $1/x^2$ behaves in the vicinity of 0. The y-axis is a vertical asymptote, and the x-axis is a horizontal asymptote.



Example 8: $\lim_{x\to\pi/2}\tan x=\lim_{x\to\pi/2}\frac{\sin x}{\cos x}$ does not exist (it is of the form "$\frac{1}{0}$"). In this case $\lim_{x\to\pi/2^+}\tan x=-\infty$ and $\lim_{x\to\pi/2^-}\tan x=\infty$ (see the graph below). The lines $x=\pi/2+n\pi$, n any integer, are vertical asymptotes.

Example 4

Evaluate

$$\lim_{x\to\infty} \frac{x-\cos(x)}{x}$$

We are eventually going to use the Squeeze Theorem on this example. There are a couple of ways to approach this; the part of the function being squeezed will be different in each case, but the end result is the same.

1. We first rewrite the function by dividing the numerator through by x, and then use limit laws to split into two separate limits:

$$\lim_{x\to\infty} \frac{x-\cos(x)}{x} = \lim_{x\to\infty}\left(1-\frac{\cos(x)}{x}\right) = \lim_{x\to\infty} 1 - \lim_{x\to\infty}\frac{\cos(x)}{x} = 1 - \lim_{x\to\infty}\frac{\cos(x)}{x}$$

   We now use the Squeeze Theorem on the remaining limit:

   We know that

$$-1 \leqslant \cos(x) \leqslant 1$$

   Since $x \to +\infty$, x is positive, dividing this inequality through by x won't change the inequalities:

$$-\frac{1}{x} \leqslant \frac{\cos(x)}{x} \leqslant \frac{1}{x}$$

   So,

$$\lim_{x\to\infty} -\frac{1}{x} \leqslant \lim_{x\to\infty}\frac{\cos(x)}{x} \leqslant \lim_{x\to\infty}\frac{1}{x}$$

   The outer limits are both 0, so by the Squeeze Theorem,

$$\lim_{x\to\infty}\frac{\cos(x)}{x} = 0,$$

   and thus

$$\lim_{x\to\infty} \frac{x-\cos(x)}{x} = 1-0 = 1$$

2. In this second approach, we start into the Squeeze Theorem right away:

$$-1 \leqslant \cos(x) \leqslant 1$$

   So

$$1 \geqslant -\cos(x) \geqslant -1,$$

   which is the same as

$$-1 \leqslant -\cos(x) \leqslant 1.$$

   Adding x to all sides gives us

$$x-1 \leqslant x-\cos(x) \leqslant x+1,$$

and then dividing through by x gives us

$$\frac{x-1}{x} \leqslant \frac{x-\cos(x)}{x} \leqslant \frac{x+1}{x}$$

(since $x \to +\infty$, x is positive, so dividing through by x won't change the inequalities). Now we can use the Squeeze Theorem to say that

$$\lim_{x\to\infty} \frac{x-1}{x} \leqslant \lim_{x\to\infty} \frac{x-\cos(x)}{x} \leqslant \lim_{x\to\infty} \frac{x+1}{x}$$

Both outside limits involve rational functions with the same degree in both numerator and denominator, so the limit as $x \to \infty$ is simply the ratio of the leading coefficients, which in both of these is $\frac{1}{1} = 1$. Since the outside limits go to the same value, then, by the Squeeze Theorem,

$$\lim_{x\to\infty} \frac{x-\cos(x)}{x} = 1$$

Example 5

Evaluate

$$\lim_{x\to-\infty} \frac{5x^2}{x+3}$$

Note: In this case we can't use the theorem we talked about in class for the limit of a rational function since that theorem only applied in cases where $x \to +\infty$, **not** when $x \to -\infty$. However, we can still use the method of dividing through by a power of x. Now, we don't always want to divide through by the highest power from either numerator or denominator (in this case, if we divided the numerator and denominator through by x^2, we'd end up with a numerator going to 0); here, we'll instead divide everything through be the highest power in the denominator:

$$\lim_{x\to-\infty} \frac{5x^2}{x+3} = \lim_{x\to-\infty} \frac{\frac{5x^2}{x}}{\frac{x+3}{x}} = \lim_{x\to-\infty} \frac{5x}{1+\frac{3}{x}}$$

Now $\frac{3}{x} \to 0$ as $x \to -\infty$, so the denominator is going to 1 (which is positive). The numerator is going to $-\infty$ since we have a positive constant times x, so the entire function is going to be negative: $\frac{(+)(-)}{(+)} = (-)$. Thus,

$$\lim_{x\to-\infty} \frac{5x^2}{x+3} = -\infty$$

Example 6

Evaluate

$$\lim_{x\to\infty} e^{3-x^2}$$

Similarly, for a number a in the domain of the given trigonometric function

$$\lim_{x \to a} \tan x = \tan a, \qquad \lim_{x \to a} \cot x = \cot a, \tag{3}$$

$$\lim_{x \to a} \sec x = \sec a, \qquad \lim_{x \to a} \csc x = \csc a. \tag{4}$$

EXAMPLE 1 Using (1) and (2)

From (1) and (2) we have

$$\lim_{x \to 0} \sin x = \sin 0 = 0 \quad \text{and} \quad \lim_{x \to 0} \cos x = \cos 0 = 1. \tag{5}$$ ■

We will draw on the results in (5) in the following discussion on computing other trigonometric limits. But first, we consider a theorem that is particularly useful when working with trigonometric limits.

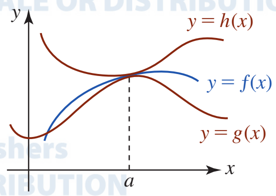


FIGURE 2.4.1 Graph of f squeezed between the graphs g and h

Squeeze Theorem The next theorem has many names: **Squeeze Theorem**, **Pinching Theorem**, **Sandwiching Theorem**, **Squeeze Play Theorem**, and **Flyswatter Theorem** are just a few of them. As shown in FIGURE 2.4.1, if the graph of $f(x)$ is "squeezed" between the graphs of two other functions $g(x)$ and $h(x)$ for all x close to a, and if the functions g and h have a common limit L as $x \to a$, it stands to reason that f also approaches L as $x \to a$. The proof of Theorem 2.4.1 is given in the *Appendix*.

Theorem 2.4.1 Squeeze Theorem

Suppose f, g, and h are functions for which $g(x) \le f(x) \le h(x)$ for all x in an open interval that contains a number a, except possibly at a itself. If

$$\lim_{x \to a} g(x) = L \quad \text{and} \quad \lim_{x \to a} h(x) = L,$$

then $\lim_{x \to a} f(x) = L$.

◀ A colleague from Russia said this result was called the **Two Soldiers Theorem** when he was in school. Think about it.

Before applying Theorem 2.4.1, let us consider a trigonometric limit that does not exist.

EXAMPLE 2 A Limit That Does Not Exist

The limit $\lim_{x \to 0} \sin(1/x)$ does not exist. The function $f(x) = \sin(1/x)$ is odd but is not periodic. The graph f oscillates between -1 and 1 as $x \to 0$:

$$\sin\frac{1}{x} = \pm 1 \quad \text{for} \quad \frac{1}{x} = \frac{\pi}{2} + n\pi, \quad n = 0, \pm 1, \pm 2, \ldots.$$

For example, $\sin(1/x) = 1$ for $n = 500$ or $x \approx 0.00064$, and $\sin(1/x) = -1$ for $n = 501$ or $x \approx 0.00063$. This means that near the origin the graph of f becomes so compressed that it appears to be one continuous smear of color. See FIGURE 2.4.2. ■

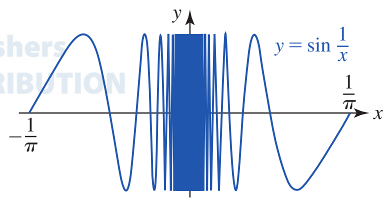


FIGURE 2.4.2 Graph of function in Example 2

EXAMPLE 3 Using the Squeeze Theorem

Find the limit $\lim_{x \to 0} x^2 \sin\frac{1}{x}$.

Solution First observe that

$$\lim_{x \to 0} x^2 \sin\frac{1}{x} \ne \left(\lim_{x \to 0} x^2\right)\left(\lim_{x \to 0} \sin\frac{1}{x}\right)$$

because we have just seen in Example 2 that $\lim_{x \to 0} \sin(1/x)$ does not exist. But for $x \ne 0$ we have $-1 \le \sin(1/x) \le 1$. Therefore,

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2.$$

Now if we make the identifications $g(x) = -x^2$ and $h(x) = x^2$, it follows from (1) of Section 2.2 that $\lim_{x\to 0} g(x) = 0$ and $\lim_{x\to 0} h(x) = 0$. Hence, from the Squeeze Theorem we conclude that

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0.$$

In FIGURE 2.4.3 note the small scale on the x- and y-axes.

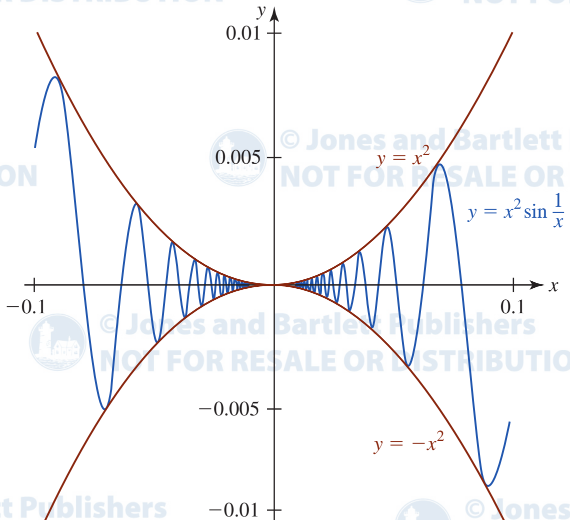


FIGURE 2.4.3 Graph of function in Example 3 ■

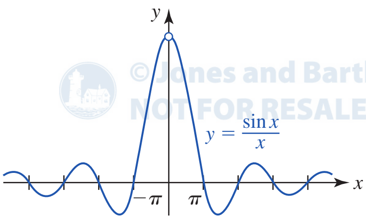


FIGURE 2.4.4 Graph of $f(x) = (\sin x)/x$

An Important Trigonometric Limit Although the function $f(x) = (\sin x)/x$ is not defined at $x = 0$, the numerical table in Example 7 of Section 2.1 and the graph in FIGURE 2.4.4 suggests that $\lim_{x\to 0} (\sin x)/x$ exists. We are now able to prove this conjecture using the Squeeze Theorem.

Consider a circle centered at the origin O with radius 1. As shown in FIGURE 2.4.5(a), let the shaded region OPR be a sector of the circle with central angle t such that $0 < t < \pi/2$. We see from parts (b), (c), and (d) of Figure 2.4.5 that

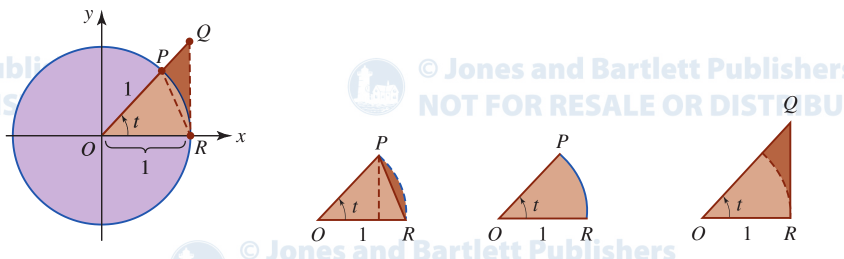
$$\text{area of } \triangle OPR \le \text{area of sector } OPR \le \text{area of } \triangle OQR. \qquad (6)$$

From Figure 2.4.5(b) the height of $\triangle OPR$ is $\overline{OP}\sin t = 1 \cdot \sin t = \sin t$, and so

$$\text{area of } \triangle OPR = \frac{1}{2}\overline{OR}\cdot(\text{height}) = \frac{1}{2}\cdot 1 \cdot \sin t = \frac{1}{2}\sin t. \qquad (7)$$

From Figure 2.4.5(d), $\overline{QR}/\overline{OR} = \tan t$ or $\overline{QR} = \tan t$, so that

$$\text{area of } \triangle OQR = \frac{1}{2}\overline{OR}\cdot\overline{QR} = \frac{1}{2}\cdot 1 \cdot \tan t = \frac{1}{2}\tan t. \qquad (8)$$



(a) Unit circle (b) Triangle OPR (c) Sector OPR (d) Right triangle OQR

FIGURE 2.4.5 Unit circle along with two triangles and a circular sector

(h) $\lim\limits_{x\to\infty} \dfrac{2^x+3^x}{2^{x+1}+3^{x+1}}$

Solution:

Let us factor out the greatest term:

$$\lim_{x\to\infty} \frac{2^x+3^x}{2^{x+1}+3^{x+1}} = \lim_{x\to\infty} \frac{3^x}{3^{x+1}} \cdot \lim_{x\to\infty} \frac{\left(\frac{2}{3}\right)^x+1}{\left(\frac{2}{3}\right)^{x+1}+1} = \frac{1}{3}\cdot\frac{0+1}{0+1} = \frac{1}{3}.$$

(i) $\lim\limits_{x\to\infty} \dfrac{1^x+2^x+3^x+4^x+5^x}{5,0001^x}$

Solution: We can factor out $5,0001^x$ or just split into five fractions and apply the Arithemics limit theorem:

$$\lim_{x\to+\infty} \frac{1^x+2^x+3^x+4^x+5^x}{5,0001^x} = \lim_{x\to+\infty} \left(\frac{1}{5,0001}\right)^x + \left(\frac{2}{5,0001}\right)^x +$$

$$+\left(\frac{3}{5,0001}\right)^x + \left(\frac{4}{5,0001}\right)^x + \left(\frac{5}{5,0001}\right)^x = 0+0+0+0+0 = 0,$$

(j) $\lim\limits_{x\to\infty} \dfrac{\ln x + x^3 + \frac{1}{x} + e^x + 5^x}{\ln_{10} x + x^4 + 5^x + x^3 + 4^x}$

Solution:

We factor out 5^x:

$$\lim_{x\to\infty} \frac{5^x}{5^x} \frac{\frac{\ln x}{5^x} + \frac{x^3}{5^x} + \frac{\frac{1}{x}}{5^x} + \frac{e^x}{5^x} + \frac{5^x}{5^x}}{\frac{\ln_{10} x}{5^x} + \frac{x^4}{5^x} + \frac{5^x}{5^x} + \frac{x^3}{5^x} + \frac{4^x}{5^x}} \stackrel{AL}{=} \lim_{x\to\infty} \frac{0+0+0+0+1}{0+0+1+0+0} = 1$$

(k) $\lim\limits_{x\to\infty} \dfrac{\sin x}{x}$

Solution: Since $|\sin x| \leq 1$ and $\lim_{x\to\infty} \frac{1}{x} = 0$, by the Squeeze theorem we obtain

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0$$

(l) $\lim\limits_{x\to\infty} e^{-x} \cos x$

Solution: Since $|\cos x| \leq 1$ and $\lim_{x\to\infty} e^{-x} = 0$, by the Squeeze theorem we obtain

$$\lim_{x\to\infty} e^{-x} \cos x = 0$$

(m) $\lim\limits_{x\to\infty} \dfrac{x+\sin x}{x-\sin x}$

Solution: Let us factor out the greatest term - x and then apply the Squeeze theorem.

$$\lim_{x\to\infty} \frac{x+\sin x}{x-\sin x} = \lim_{x\to\infty} \frac{x}{x} \cdot \frac{1+\frac{\sin x}{x}}{1-\frac{\sin x}{x}} = \frac{1+0}{1-0} = 1$$

(n) $\lim\limits_{x\to 0+} x \cos\left(\dfrac{x+3}{\sqrt{x}-1}\right)$

Solution: Since cos is bounded function and $\lim x \to 0{+}x = 0$, we apply the Squeeze theorem:

$$\lim_{x\to 0+} x \cos\left(\frac{x+3}{\sqrt{x}-1}\right) = 0$$

(o) $\lim\limits_{x\to\infty} \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

Solution: Let us factor out the greatest term - e^x and then apply the Squeeze theorem.

$$\lim_{x\to\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} \lim_{x\to\infty} \frac{1+e^{-2x}}{1-e^{-2x}} = \frac{1+0}{1+0} = 1$$

(p) $\lim\limits_{x\to\infty} e^x \cos x$

Solution: This limit does not exists. Sketch a (really oscillating graph).

(q) $\lim\limits_{x\to\infty} \dfrac{x}{\sin x}$

Solution: This limit does not exists. The function $\sin x = 0$ for $x = k\pi$, $k \in \mathbb{Z}$. Hence there is no neigbourhood of ∞, such that fraction is well defined. Hence the limit does not make sense.

0

3. Find limits:

(a) $\lim_{x\to 1} \frac{x-1}{x^2+x-2}$

(b) $\lim_{x\to -3} \frac{x^2+x-6}{x+3}$

(c) $\lim_{x\to 2} \frac{x^2+3x-4}{x^2-4x+4}$

(d) $\lim_{x\to 0} \frac{1}{\sin x}$

(e) $\lim_{x\to -2} \frac{-4}{x+2}$

(f) $\lim_{x\to 4} \frac{3}{(4-x)^3}$

(g) $\lim_{x\to 3} \frac{2x}{x-3}$

(h) $\lim_{x\to 4} \frac{x^2}{x^2-4}$

(i) $\lim_{x\to -3} \frac{x^2-2x-3}{x^2+6x+9}$

(j) $\lim_{x\to -\infty} \frac{1}{e^x}$

(k) $\lim_{x\to 0} \frac{|2x|}{x}$

Of course we must add to (3) the all-important requirement that the limit of the denominator is not 0, that is, $q(a) \neq 0$.

EXAMPLE 7 Using (2) and (3)

Evaluate $\displaystyle\lim_{x\to -1} \frac{3x-4}{8x^2+2x-2}$.

Solution $f(x) = \dfrac{3x-4}{8x^2+2x-2}$ is a rational function and so if we identify the polynomials $p(x) = 3x - 4$ and $q(x) = 8x^2 + 2x - 2$, then from (2),

$$\lim_{x\to -1} p(x) = p(-1) = -7 \qquad \text{and} \qquad \lim_{x\to -1} q(x) = q(-1) = 4.$$

Since $q(-1) \neq 0$ it follows from (3) that

$$\lim_{x\to -1} \frac{3x-4}{8x^2+2x-2} = \frac{p(-1)}{q(-1)} = \frac{-7}{4} = -\frac{7}{4}.$$ ■

You should not get the impression that we can *always* find a limit of a function by substituting the number *a directly into the function.*

EXAMPLE 8 Using Theorem 2.2.3

3a

Evaluate $\displaystyle\lim_{x\to 1} \frac{x-1}{x^2+x-2}$.

Solution The function in this limit is rational, but if we substitute $x = 1$ into the function we see that this limit has the indeterminate form 0/0. However, by simplifying *first*, we can then apply Theorem 2.2.3(*iii*):

$$\begin{aligned}\lim_{x\to 1} \frac{x-1}{x^2+x-2} &= \lim_{x\to 1} \frac{x-1}{(x-1)(x+2)} \quad \leftarrow \text{cancellation is valid provided that } x \neq 1\\ &= \lim_{x\to 1} \frac{1}{x+2}\\ &= \frac{\lim_{x\to 1} 1}{\lim_{x\to 1}(x+2)} = \frac{1}{3}.\end{aligned}$$ ■

If a limit of a rational function has the indeterminate form 0/0 as $x \to a$, then by the Factor Theorem of algebra $x - a$ must be a factor of both the numerator and the denominator. Factor those quantities and cancel the factor $x - a$.

Sometimes you can tell at a glance when *a limit does not exist*.

Theorem 2.2.5 A Limit That Does Not Exist

Let $\lim_{x\to a} f(x) = L_1 \neq 0$ and $\lim_{x\to a} g(x) = 0$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

does not exist.

PROOF We will give an indirect proof of this result based on Theorem 2.2.3. Suppose $\lim_{x\to a} f(x) = L_1 \neq 0$ and $\lim_{x\to a} g(x) = 0$ and suppose further that $\lim_{x\to a}(f(x)/g(x))$ exists and equals L_2. Then

$$\begin{aligned}L_1 = \lim_{x\to a} f(x) &= \lim_{x\to a}\left(g(x)\cdot\frac{f(x)}{g(x)}\right), \qquad g(x) \neq 0,\\ &= \left(\lim_{x\to a} g(x)\right)\left(\lim_{x\to a}\frac{f(x)}{g(x)}\right) = 0\cdot L_2 = 0.\end{aligned}$$

By contradicting the assumption that $L_1 \neq 0$, we have proved the theorem. ■

11.2 Techniques for Evaluating Limits

Dividing Out Technique

In Section 11.1, you studied several types of functions whose limits can be evaluated by direct substitution. In this section, you will study several techniques for evaluating limits of functions for which direct substitution fails.

Suppose you were asked to find the following limit.

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}$$

Direct substitution fails because -3 is a zero of the denominator. By using a table, however, it appears that the limit of the function as x approaches -3 is -5.

x	-3.01	-3.001	-3.0001	-3	-2.9999	-2.999	-2.99
$\frac{x^2 + x - 6}{x + 3}$	-5.01	-5.001	-5.0001	?	-4.9999	-4.999	-4.99

Another way to find the limit of this function is shown in Example 1.

Example 1 Dividing Out Technique

Find the limit.

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3}$$

Solution

Begin by factoring the numerator and dividing out any common factors.

$$\lim_{x \to -3} \frac{x^2 + x - 6}{x + 3} = \lim_{x \to -3} \frac{(x - 2)(x + 3)}{x + 3}$$ Factor numerator.

$$= \lim_{x \to -3} \frac{(x - 2)\cancel{(x + 3)}}{\cancel{x + 3}}$$ Divide out common factor.

$$= \lim_{x \to -3} (x - 2)$$ Simplify.

$$= -3 - 2$$ Direct substitution

$$= -5$$ Simplify.

CHECKPOINT Now try Exercise 11.

This procedure for evaluating a limit is called the **dividing out technique.** The validity of this technique stems from the fact that when two functions agree at all but a single number c, they must have identical limit behavior at $x = c$. In Example 1, the functions given by

$$f(x) = \frac{x^2 + x - 6}{x + 3} \quad \text{and} \quad g(x) = x - 2$$

agree at all values of x other than

$$x = -3.$$

So, you can use $g(x)$ to find the limit of $f(x)$.

What you should learn

- Use the dividing out technique to evaluate limits of functions.
- Use the rationalizing technique to evaluate limits of functions.
- Use technology to approximate limits of functions graphically and numerically.
- Evaluate one-sided limits of functions.
- Evaluate limits of difference quotients from calculus.

Why you should learn it

Many definitions in calculus involve the limit of a function. For instance, in Exercises 69 and 70 on page 768, the definition of the velocity of a free-falling object at any instant in time involves finding the limit of a position function.

3c

EXAMPLE 10. Evaluate infinite limit

$$\lim_{x \to 2} \frac{x^2 + 3x - 4}{x^2 - 4x + 4}$$

Factoring and sign analysis:

$$= \lim_{x \to 2} \frac{(x+4)(x-1)}{(x-2)^2} = \frac{(6) \cdot (1)}{(0^+)} = -\infty$$

EXAMPLE 11. Evaluate infinite limit

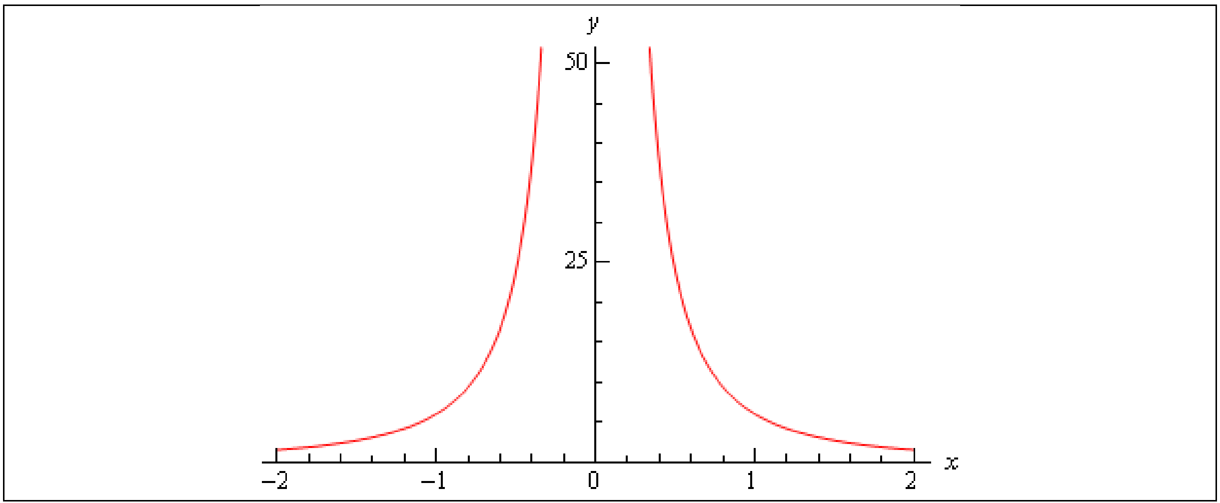
$$\lim_{x\to 0}\frac{1}{\sin x}$$

Sign analysis for one-sided limits:

$$\lim_{x\to 0^+}\frac{1}{\sin x}=\frac{1}{(0^+)}=+\infty$$

$$\lim_{x\to 0^-}\frac{1}{\sin x}=\frac{1}{(0^-)}=-\infty$$

Limit at 0 does not exist



With this next example we'll move away from just an *x* in the denominator, but as we'll see in the next couple of examples they work pretty much the same way.

3e

Example 3 Evaluate each of the following limits.

$$\lim_{x\to -2^+}\frac{-4}{x+2} \qquad \lim_{x\to -2^-}\frac{-4}{x+2} \qquad \lim_{x\to -2}\frac{-4}{x+2}$$

Solution

Let's again start with the right-hand limit. With the right-hand limit we know that we have,

$$x > -2 \qquad \Rightarrow \qquad x+2 > 0$$

Also, as *x* gets closer and closer to -2 then $x+2$ will be getting closer and closer to zero, while staying positive as noted above. So, for the right-hand limit, we'll have a negative constant divided by an increasingly small positive number. The result will be an increasingly large and negative number. So, it looks like the right-hand limit will be negative infinity.

For the left-hand limit we have,

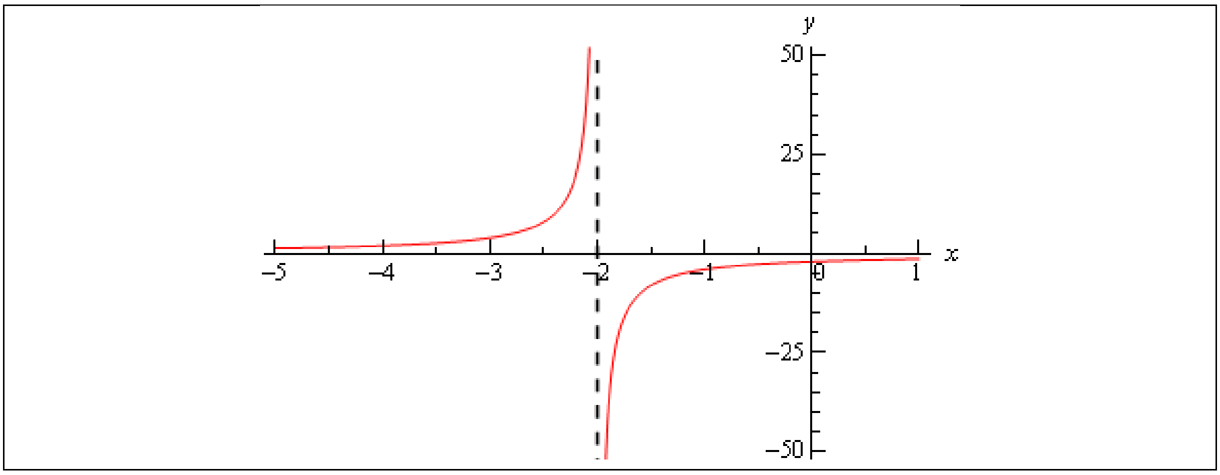
$$x < -2 \qquad \Rightarrow \qquad x+2 < 0$$

and $x+2$ will get closer and closer to zero (and be negative) as *x* gets closer and closer to -2. In this case then we'll have a negative constant divided by an increasingly small negative number. The result will then be an increasingly large positive number and so it looks like the left-hand limit will be positive infinity.

Finally, since two one sided limits are not the same the normal limit won't exist.

Here are the official answers for this example as well as a quick graph of the function for verification purposes.

$$\lim_{x\to -2^+}\frac{-4}{x+2} = -\infty \qquad \lim_{x\to -2^-}\frac{-4}{x+2} = \infty \qquad \lim_{x\to -2}\frac{-4}{x+2} \text{ doesn't exist}$$



At this point we should briefly acknowledge the idea of vertical asymptotes. Each of the three previous graphs have had one. Recall from an Algebra class that a vertical asymptote is a vertical line (the dashed line at $x = -2$ in the previous example) in which the graph will go towards infinity and/or minus infinity on one or both sides of the line.

In an Algebra class they are a little difficult to define other than to say pretty much what we just said. Now that we have infinite limits under our belt we can easily define a vertical asymptote as follows,

Definition

The function *f(x)* will have a vertical asymptote at $x = a$ if we have any of the following limits at $x = a$.

$$\lim_{x\to a^-} f(x) = \pm\infty \qquad \lim_{x\to a^+} f(x) = \pm\infty \qquad \lim_{x\to a} f(x) = \pm\infty$$

Note that it only requires one of the above limits for a function to have a vertical asymptote at $x = a$.

Using this definition we can see that the first two examples had vertical asymptotes at $x = 0$ while the third example had a vertical asymptote at $x = -2$.

We aren't really going to do a lot with vertical asymptotes here but wanted to mention them at this point since we'd reached a good point to do that.

Let's now take a look at a couple more examples of infinite limits that can cause some problems on occasion.

Example 4 Evaluate each of the following limits.

$$\lim_{x\to 4^+} \frac{3}{(4-x)^3} \qquad \lim_{x\to 4^-} \frac{3}{(4-x)^3} \qquad \lim_{x\to 4} \frac{3}{(4-x)^3}$$

Solution

Let's start with the right-hand limit. For this limit we have,

$$x > 4 \qquad \Rightarrow \qquad 4 - x < 0 \qquad \Rightarrow \qquad (4-x)^3 < 0$$

also, $4-x \to 0$ as $x \to 4$. So, we have a positive constant divided by an increasingly small negative number. The results will be an increasingly large negative number and so it looks like the right-hand limit will be negative infinity.

For the left-handed limit we have,

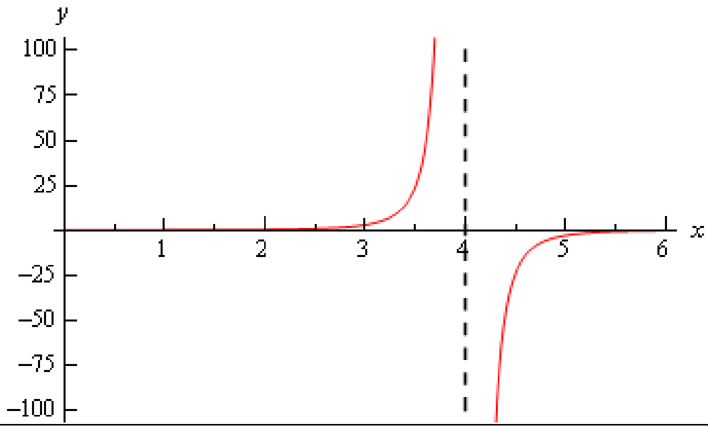
$$x<4 \qquad \Rightarrow \qquad 4-x>0 \qquad \Rightarrow \qquad (4-x)^3>0$$

and we still have, $4-x \to 0$ as $x \to 4$. In this case we have a positive constant divided by an increasingly small positive number. The results will be an increasingly large positive number and so it looks like the left-hand limit will be positive infinity.

The normal limit will not exist since the two one-sided limits are not the same. The official answers to this example are then,

$$\lim_{x\to 4^+}\frac{3}{(4-x)^3}=-\infty \qquad \lim_{x\to 4^-}\frac{3}{(4-x)^3}=\infty \qquad \lim_{x\to 4}\frac{3}{(4-x)^3} \text{ doesn't exist}$$

Here is a quick sketch to verify our limits.



All the examples to this point have had a constant in the numerator and we should probably take a quick look at an example that doesn't have a constant in the numerator.

Example 5 Evaluate each of the following limits.

$$\lim_{x\to 3^+}\frac{2x}{x-3} \qquad \lim_{x\to 3^-}\frac{2x}{x-3} \qquad \lim_{x\to 3}\frac{2x}{x-3}$$

Solution

Let's take a look at the right-handed limit first. For this limit we'll have,

$$x>3 \qquad \Rightarrow \qquad x-3>0$$

The main difference here with this example is the behavior of the numerator as we let *x* get closer and closer to 3. In this case we have the following behavior for both the numerator and denominator.

$$x-3\to 0 \quad \text{and} \quad 2x \to 6 \quad \text{as} \quad x \to 3$$

So, as we let *x* get closer and closer to 3 (always staying on the right of course) the numerator, while not a constant, is getting closer and closer to a positive constant while the denominator is getting closer and closer to zero and will be positive since we are on the right side.

This means that we'll have a numerator that is getting closer and closer to a non-zero and positive constant divided by an increasingly smaller positive number and so the result should be an increasingly larger positive number. The right-hand limit should then be positive infinity.

For the left-hand limit we'll have,

$$x < 3 \qquad \Rightarrow \qquad x - 3 < 0$$

As with the right-hand limit we'll have the following behaviors for the numerator and the denominator,

$$x - 3 \to 0 \quad \text{and} \quad 2x \to 6 \quad \text{as} \quad x \to 3$$

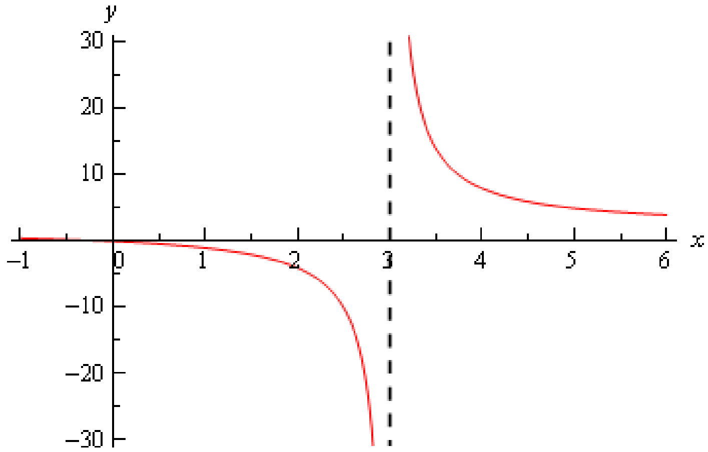
The main difference in this case is that the denominator will now be negative. So, we'll have a numerator that is approaching a positive, non-zero constant divided by an increasingly small negative number. The result will be an increasingly large and negative number.

The formal answers for this example are then,

$$\lim_{x \to 3^+} \frac{2x}{x-3} = \infty \qquad \lim_{x \to 3^-} \frac{2x}{x-3} = -\infty \qquad \lim_{x \to 3} \frac{2x}{x-3} \quad \text{doesn't exist}$$

As with most of the examples in this section the normal limit does not exist since the two one-sided limits are not the same.

Here's a quick graph to verify our limits.



So far all we've done is look at limits of rational expressions, let's do a couple of quick examples with some different functions.

Limit examples

Example 1

Evaluate

$$\lim_{x\to 4} \frac{x^2}{x^2-4}$$

If we try direct substitution, we end up with "$\frac{16}{0}$" (i.e., a non-zero constant over zero), so we'll get either $+\infty$ or $-\infty$ as we approach 4. We then need to check left- and right-hand limits to see which one it is, and to make sure the limits are equal from both sides.

- Left-hand limit:

$$\lim_{x\to 4^-} \frac{x^2}{(x-4)(x+4)}$$

As $x \to 4^-$, the function is negative since $\frac{(+)^2}{(-)(+)} = (-)$, so the left-hand limit is $-\infty$.

- Right-hand limit:

$$\lim_{x\to 4^+} \frac{x^2}{(x-4)(x+4)}$$

As $x \to 4^+$, the function is positive since $\frac{(+)^2}{(+)(+)} = (+)$, so the right-hand limit is $+\infty$.

Since the left- and right-hand limits are not equal,

$$\lim_{x\to 4} \frac{x^2}{x^2-4} \text{ DNE}$$

Example 2

Evaluate

$$\lim_{x\to -3} \frac{x^2-2x-3}{x^2+6x+9}$$

If we try direct substitution, we end up with "$\frac{12}{0}$", so we'll get either $+\infty$ or $-\infty$ as we approach -3. As in the last example, we need to check left- and right-hand limits to see which one it is, and to make sure the limits are equal from both sides.

- Left-hand limit:

$$\lim_{x\to -3^-} \frac{(x-3)(x+1)}{(x+3)^2}$$

As $x \to -3^-$, the function is positive since $\frac{(-)(-)}{(-)^2} = \frac{(+)}{(+)} = (+)$, so the left-hand limit is $+\infty$.

- Right-hand limit:

$$\lim_{x\to -3^+} \frac{(x-3)(x+1)}{(x+3)^2}$$

As $x \to -3^+$, the function is positive since $\frac{(-)(-)}{(+)^2} = \frac{(+)}{(+)} = (+)$, so the right-hand limit is also $+\infty$.

Since the left- and right-hand limits are the same,

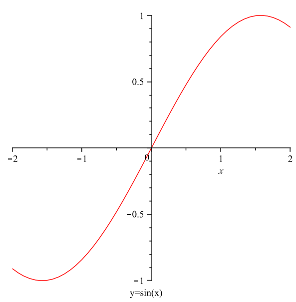
$$\lim_{x\to 4} \frac{x^2-2x-3}{x^2+6x+9} = \infty$$

Example 3

Evaluate

$$\lim_{x\to 0^+} \frac{2}{\sin(x)}$$

First of all, we note that direct substitution fails (we get "$\frac{2}{0}$"). There are a couple of different ways we can look at this problem. For either one, we observe that as $x \to 0^+$, $\sin(x)$ also goes to zero from values greater than zero (i.e., $\sin(x) \to 0^+$): So, $\lim\limits_{x\to 0^+} \dfrac{2}{\sin(x)}$ is either $+\infty$



or $-\infty$. From what we observed above, we know the function will be $\frac{(+)}{(+)} = (+)$, so the limit is $+\infty$.

The other way we can approach this is to replace $\sin(x)$ with another variable that goes to the same value as $\sin(x)$ when we take the limit. Since $\sin(x) \to 0^+$ as $x \to 0^+$, then

$$\lim_{x\to 0^+} \frac{2}{\sin(x)} = \lim_{t\to 0^+} \frac{2}{t} \quad (\text{ which still} = \infty).$$

To show this one formally, we first note that as $x \to \infty$, then

$$x^2 \to \infty$$

as well, so

$$-x^2 \to -\infty$$

and

$$3 - x^2 \to -\infty$$

also. So, we can replace the "$3 - x^2$" in the exponent with another variable (say, t) that goes to $-\infty$ without changing the limit, i.e.,

$$\lim_{x\to\infty} e^{3-x^2} = \lim_{t\to-\infty} e^t \quad (= 0 \text{ by properties mentioned in class}).$$

Example 7

Evaluate

$$\lim_{x\to-\infty} \frac{1}{e^x}$$

In this example, we first rewrite the limit as

$$\lim_{x\to-\infty} e^{-x},$$

which is $+\infty$ from properties mentioned in class.

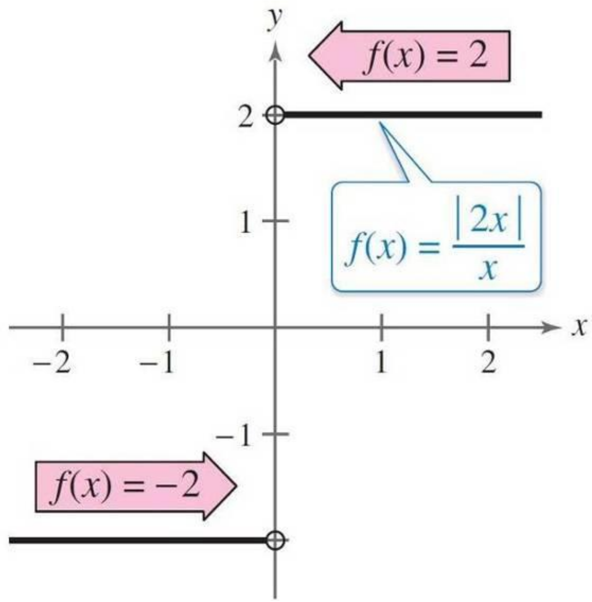
Example 6 – *Evaluating One-Sided Limits*

Find the limit as $x \to 0$ from the left and the limit as $x \to 0$ from the right for

$$f(x) = \frac{|2x|}{x}.$$

Solution:

From the graph of f, shown in the figure, you can see that $f(x) = -2$ for all $x < 0$.



Example 6 – *Solution*

cont'd

So, the limit from the left is

$$\lim_{x \to 0^-} \frac{|2x|}{x} = -2.$$

Limit from the left

Because $f(x) = 2$ for all $x > 0$, the limit from the right is

$$\lim_{x \to 0^+} \frac{|2x|}{x} = 2.$$

Limit from the right