8th lesson

https://www2.karlin.mff.cuni.cz/ kuncova/en/teachMat1.php
kunck6am@natur.cuni.cz

Exercises

1. Look at function $h(x)$ (the pink one)
   `https://www.geogebra.org/calculator/q6sspc9y`
   Sketch

   (a) $\frac{h(x)}{2}$ (b) $3h(x)$ (c) $-2h(x)$

2. Look at function $f(x)$ (the pink one)
   `https://www.geogebra.org/calculator/rcd6wsup`
   Sketch

   (a) $f(x+1)$ (b) $f(x-1)$ (c) $f(x)+1$ (d) $f(x)-1$

3. Look at function $g(x)$ (the pink one)
   `https://www.geogebra.org/calculator/jxfhwxca`
   Sketch

   (a) $|g(x)|$ (b) $g(|x|)$ (c) $-g(|x|)$ (d) $g(-|x|)$

4. Let $f(x) = x^2$ and $g(x) = x - 2$. Find

   (a) $f(g(3))$ (b) $g(f(3))$ (c) $f(g(x))$ (d) $g(f(x))$

5. Let $f(x) = 4 - x^2$ and $g(x) = \sqrt{x}$. Find

   (a) $f(g(x))$ (b) $g(f(x))$

6. Let $f(x) = 3x - 8$ and $g(x) = \frac{x+8}{3}$. Find

   (a) $f(g(x))$ (b) $g(f(x))$

7. Express the following functions as composition:

   (a) $(1+x^3)^{27}$ (b) e^{-x^2} (c) $-(e^x)^2$

then the area is given as a function of time by substitution:

$$A = \pi r^2 = \pi(1+t)^2.$$

We are thinking of A as a *composite function* or a "function of a function," which is written

$$A = \underbrace{f(g(t))}_{\substack{\text{Composite function;}\\ f \text{ is outside function,}\\ g \text{ is inside function}}} = \pi(g(t))^2 = \pi(1+t)^2.$$

To calculate A using the formula $\pi(1+t)^2$, the first step is to find $1+t$, and the second step is to square and multiply by π. The first step corresponds to the inside function $g(t) = 1+t$, and the second step corresponds to the outside function $f(r) = \pi r^2$.

Example 1 If $f(x) = x^2$ and $g(x) = x - 2$, find each of the following:
(a) $f(g(3))$ (b) $g(f(3))$ (c) $f(g(x))$ (d) $g(f(x))$

Solution
(a) Since $g(3) = 1$, we have $f(g(3)) = f(1) = 1$.
(b) Since $f(3) = 9$, we have $g(f(3)) = g(9) = 7$. Notice that $f(g(3)) \neq g(f(3))$.
(c) $f(g(x)) = f(x-2) = (x-2)^2$.
(d) $g(f(x)) = g(x^2) = x^2 - 2$. Again, notice that $f(g(x)) \neq g(f(x))$.
Notice that the horizontal shift in Figure 1.32 can be thought of as a composition $f(g(x)) = (x-2)^2$.

Example 2 Express each of the following functions as a composition:

(a) $h(t) = (1+t^3)^{27}$ (b) $k(y) = e^{-y^2}$ (c) $l(y) = -(e^y)^2$

Solution In each case think about how you would calculate a value of the function. The first stage of the calculation gives you the inside function, and the second stage gives you the outside function.

(a) For $(1+t^3)^{27}$, the first stage is cubing and adding 1, so an inside function is $g(t) = 1+t^3$. The second stage is taking the 27^{th} power, so an outside function is $f(y) = y^{27}$. Then

$$f(g(t)) = f(1+t^3) = (1+t^3)^{27}.$$

In fact, there are lots of different answers: $g(t) = t^3$ and $f(y) = (1+y)^{27}$ is another possibility.

(b) To calculate e^{-y^2} we square y, take its negative, and then take e to that power. So if $g(y) = -y^2$ and $f(z) = e^z$, then we have

$$f(g(y)) = e^{-y^2}.$$

(c) To calculate $-(e^y)^2$, we find e^y, square it, and take the negative. Using the same definitions of f and g as in part (b), the composition is

$$g(f(y)) = -(e^y)^2.$$

Since parts (b) and (c) give different answers, we see the order in which functions are composed is important.

Odd and Even Functions: Symmetry

There is a certain symmetry apparent in the graphs of $f(x) = x^2$ and $g(x) = x^3$ in Figure 1.33. For each point (x, x^2) on the graph of f, the point $(-x, x^2)$ is also on the graph; for each point (x, x^3) on the graph of g, the point $(-x, -x^3)$ is also on the graph. The graph of $f(x) = x^2$ is symmetric about the y-axis, whereas the graph of $g(x) = x^3$ is symmetric about the origin. The graph of any polynomial involving only even powers of x has symmetry about the y-axis, while polynomials with only odd powers of x are symmetric about the origin. Consequently, any functions with these symmetry properties are called *even* and *odd*, respectively.

Composition of Functions

Another way to combine functions is used frequently and plays an important role in both precalculus and calculus.

Definition: composition of functions

Suppose f and g are functions. The **composition function, $f \circ g$,** read "**f of g,**" is the function whose value at x is given by

$$(f \circ g)(x) = f(g(x)).$$

Thus to write a formula for $(f \circ g)(x)$, in the rule defining f,

replace each x in $f(x)$ by $g(x)$.

The **domain** of $f \circ g$ is the set of all real numbers x such that both $g(x)$ is defined, and $f(g(x))$ is defined.

The reason for calling the composition $f \circ g$ "f of g" is that the value of the composition function at a given number c is "f of $g(c)$."

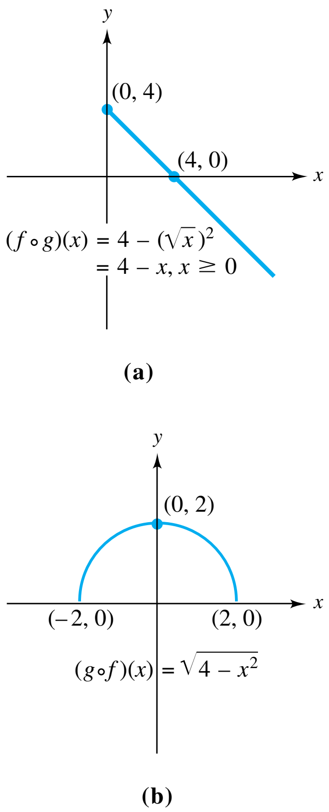


FIGURE 37

▶EXAMPLE 2 ***Two compositions*** If $f(x) = 4 - x^2$ and $g(x) = \sqrt{x}$, **(a)** write an equation and **(b)** draw a calculator graph of ***(i)*** $f \circ g$ ***(ii)*** $g \circ f$.

Solution

(a) For each composition, we follow the procedure given in the definition.

(i) $(f \circ g)(x) = f(g(x)) = 4 - (g(x))^2 = 4 - (\sqrt{x})^2$.

For $\sqrt{x}$ to be a real number we must have $x \geq 0$, and when $x \geq 0$, we can simplify the equation for $f \circ g$:

$$(f \circ g)(x) = 4 - x, \text{ where } x \geq 0.$$

(ii) $(g \circ f)(x) = g(f(x)) = \sqrt{f(x)} = \sqrt{4 - x^2}$.

Again, the domain is limited: for $4 - x^2 \geq 0$, we have $-2 \leq x \leq 2$.

(b) With a graphing calculator we can always enter the compositions in the form we wrote above, Y1 = 4 − (√X)² and Y2 = √(4 − X²).

If your calculator has a Y = menu where you can enter several functions, there are other options. For example, having entered f and g as Y1 = 4 − X² and Y2 = √X, since $f(g(x)) = 4 - (g(x))^2$, we can enter $f \circ g$ as Y3 = 4 − Y2² and $g \circ f$ as Y4 = √Y1. Observe that for $f \circ g$ we follow the defining rule for composition functions: replace each x in $f(x)$ by $g(x)$.

The calculator graphs are shown in Figure 37. Note that the limitations on the domain are obvious from the graphs and that we can also read off the ranges. The range of $f \circ g$ is $(-\infty, 4]$, and the range of $g \circ f$ is the closed interval $[0, 2]$.

Alternate Solution Sometimes it is easier to verbalize the rules that define functions. The rules for f and g state that, for any given input, f squares the input and subtracts the result from 4, while g takes the square root of its input. Thus, suppose $\sqrt{x}$ is the input. The function f squares $\sqrt{x}$ and subtracts the result from 4: $4 - (\sqrt{x})^2$. Similarly, when g is applied to $f(x)$, g takes the square root of $f(x)$. The output is: $g(f(x)) = \sqrt{f(x)} = \sqrt{4 - x^2}$. ◀

Example 2 shows that $f \circ g$ and $g \circ f$ are not the same function. In general $f \circ g$ and $g \circ f$ are different, although there are important exceptions, as the next example demonstrates.

▶EXAMPLE 3 Equal compositions If $f(x) = 3x - 8$ and $g(x) = \frac{x+8}{3}$, write an equation that gives the rule of correspondence for **(a)** $f \circ g$ **(b)** $g \circ f$.

Solution

Here the rule for f is "triple the input and then subtract 8;" for g it is "add 8 to the input and then divide the sum by 3."

(a) $$(f \circ g)(x) = f(g(x)) = f\left(\frac{x+8}{3}\right) = 3\left(\frac{x+8}{3}\right) - 8 = x.$$

(b) $$(g \circ f)(x) = g(f(x)) = g(3x - 8) = \frac{(3x-8)+8}{3} = x.$$

Thus $(f \circ g)(x) = (g \circ f)(x)$ for every number x. We say that the two functions $f \circ g$ and $g \circ f$ are equal, $f \circ g = g \circ f$. ◀

▶EXAMPLE 4 Composition equations If $f(x) = x^2 - 2x$ and $g(x) = 3 - x$, solve the equations.

(a) $(f \circ g)(x) = 0$ **(b)** $(g \circ f)(x) + x^2 + 5 = 0$

Strategy: Write each equation in a more familiar form.

Solution

(a) $(f \circ g)(x) = f(g(x)) = f(3 - x) = (3 - x)^2 - 2(3 - x) = x^2 - 4x + 3.$ Thus the given equation becomes

$$x^2 - 4x + 3 = 0 \quad \text{or} \quad (x-1)(x-3) = 0.$$

The solutions are 1 and 3.

(b) $(g \circ f)(x) = g(f(x)) = g(x^2 - 2x) = 3 - (x^2 - 2x) = -x^2 + 2x + 3.$ Replacing $(g \circ f)(x)$ by $-x^2 + 2x + 3$, the given equation becomes

$$(-x^2 + 2x + 3) + x^2 + 5 = 0 \quad \text{or} \quad 2x + 8 = 0.$$

The solution is -4. ◀

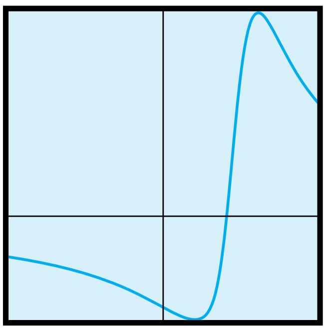
▶EXAMPLE 5 Maxima and minima from calculator graphs Let F denote the composition $g \circ f$ on the limited domain $D = [-5, 5]$, where

$$f(x) = \frac{2x - 4}{x^2 - 4x + 5} \quad \text{and} \quad g(x) = x^2 + 3x.$$

(a) Draw a calculator graph of $F(x) = g(f(x))$.
(b) From your graph, find the maximum and minimum values of F.
(c) Find the solution set for $g(f(x)) > 0$.

Solution

(a) Writing a formula for the composition $g(f(x))$ requires us to replace each x in $x^2 + 3x$ by the entire $f(x)$. The process is messy, to say the least, but some calculators are designed to make composition much easier. See the Technology Tip following this example. Not knowing the range beforehand, we may set an x-range of $[-5, 5]$ to match the domain and adjust as necessary. A calculator graph is shown in Figure 38.



[−5, 5] by [−2.1, 4.1]

FIGURE 38
$F(x) = g(f(x))$

then the area is given as a function of time by substitution:

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Solution
(a) Since $g(3) = 1$, we have $f(g(3)) = f(1) = 1$.
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Notice that the horizontal shift in Figure 1.32 can be thought of as a composition $f(g(x)) = (x-2)^2$.

Example 2 Express each of the following functions as a composition:

(a) $h(t) = (1+t^3)^{27}$ (b) $k(y) = e^{-y^2}$ (c) $l(y) = -(e^y)^2$

Solution In each case think about how you would calculate a value of the function. The first stage of the calculation gives you the inside function, and the second stage gives you the outside function.

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In fact, there are lots of different answers: $g(t) = t^3$ and $f(y) = (1+y)^{27}$ is another possibility.

(b) To calculate e^{-y^2} we square y, take its negative, and then take e to that power. So if $g(y) = -y^2$ and $f(z) = e^z$, then we have

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(c) To calculate $-(e^y)^2$, we find e^y, square it, and take the negative. Using the same definitions of f and g as in part (b), the composition is

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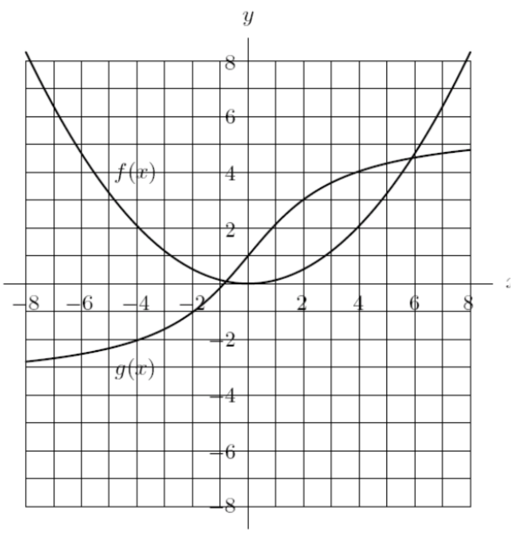
8. Find $g(f(3))$, if the f and g are at the picture:



Solution: Since $f(3) = 1$ and $g(1) = 2$, we obtain $g(f(3)) = 2$.

9. The values of functions f and g can be found in the table. Find $f(g(0))$.

x	-2	-1	0	1	2
$f(x)$	1	0	-2	2	-1
$g(x)$	-1	1	2	0	-2

Solution: Since $g(0) = 2$ and $f(2) = -1$, we obtain $f(g(0)) = -1$.

10. The values of functions f and g can be found in the table. Find x, if $f(g(x)) = 1$.

x	-2	-1	0	1	2
$f(x)$	1	0	-2	2	-1
$g(x)$	-1	1	2	0	-2

Solution: Since $f(-2) = 1$ and $g(2) = -2$, we obtain $f(g(2)) = 1$, the answer is $x = 2$.

11. Let $f(x) = x^2$. Find the image of the sets

   (a) $\{2\}$
   Solution: By the graph we obtain: $\{4\}$

   (b) $\{-3, 0, 1, 10\}$
   Solution: $\{9, 0, 1, 100\}$

(c) $(-1, 0)$
Solution: $(0, 1)$

(d) $[-2, 2]$
Solution: $[0, 4)$

(e) $(-2, 3]$
Solution: $[0, 9]$

(f) $(-2, \infty)$ **Solution:** $[0, \infty)$

12. Let $f(x) = x^2$. Find the preimage of the sets

(a) $\{4\}$
Solution: By the graph we obtain: $\{-2, 2\}$

(b) $(0, 9)$
Solution: $(-3, 0) \cup (0, 3)$

(c) $[0, 9)$
Solution: $(-3, 3)$

(d) $[1, 9]$
Solution: $[-3, 3]$

(e) $(-2, \infty)$
Solution: $\mathbb{R}$

(f) $\{-4\}$
Solution: $\emptyset$

13. Let $f(x) = \sin x$. Find the preimage of the set

(a) $\{1\}$
Solution: $\{\frac{\pi}{2} + 2k\pi, k \in \mathbb{Z}\}$

(b) $(-1, 1)$
Solution: $\mathbb{R} \setminus \{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\}$

(c) $[0, 1)$
Solution: Let $A := \bigcup_{k\in\mathbb{Z}} [0 + 2k\pi, \frac{\pi}{2} + 2k\pi)$, $B := \{\pi + 2k\pi, k \in \mathbb{Z}\}$. The solution is $A \cup B$.

(d) $(-2, -1]$
Solution: $\{-\frac{\pi}{2} + 2k\pi, k \in \mathbb{Z}\}$

(e) $(-\infty, -3]$
Solution: $\emptyset$

14. Find (or sketch) a function which maps

(a) $[0; 1]$ onto $[0; 2]$
Solution: $f(x) = 2x$

(b) $[0; \frac{\pi}{2})$ onto $[0; \infty)$
Solution: $f(x) = \tan x$

(c) $(-\frac{\pi}{2}; \frac{\pi}{2})$ onto $(-\infty; \infty)$
Solution: $f(x) = \tan x$

(d) $(0; 1)$ onto $[0; 1]$
Solution: $f(x) = \frac{1+\sin(4\pi x)}{2}$

(e) $[a; b]$ onto $[0; 1]$
Solution: $f(x) = \frac{x-a}{b-a}$