7th lesson

https://www2.karlin.mff.cuni.cz/ kuncova/en/teachMat1.php
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Exercises

1. Find limits:

   (a) $x_1 = \sqrt{2}$, $x_n = \sqrt{2 + x_{n-1}}$
   Řešení: For the nth term we obtain:

$$x_n = \underbrace{\sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}_{n \text{ roots}}.$$

The sequence x_n is strictly increasing. It is easily seen that $x_{n+1} > x_n$.
Let us show that $x_n \leq 2$ for every n; if we apply the second power, we have:

$$x_n \leq 2 \Leftrightarrow \underbrace{\sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}_{n \text{ roots}} \leq 2 \Leftrightarrow 2 + \underbrace{\sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}_{n-1 \text{ roots}} \leq 4 \Leftrightarrow$$

$$\Leftrightarrow \underbrace{\sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}}_{n-1 \text{ rotts}} \leq 2 \Leftrightarrow \ldots \Leftrightarrow \sqrt{2} \leq 2.$$

Since the sequence is monotonous and bounded, it has a limit L.
Since the limit of a subsequence has to be equal to the origin limit, we get:

$$\lim x_{n+1} = \lim \sqrt{2 + x_n}$$

$$L = \sqrt{2 + L}$$

$$L = 2.$$

MATH3283W LECTURE NOTES: WEEK 6

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Recursive sequences (cont.)

Examples:

(2) $a_1 = 2,\ a_{n+1} = \frac{1}{3-a_n}$.
The first few terms are
$$2, 1, \frac{1}{2}, \frac{1}{\frac{5}{2}} = \frac{2}{5}, \frac{1}{\frac{13}{5}} = \frac{5}{13}, \cdots$$
Since $\frac{5}{13} < \frac{2}{5}$, we suspect that a_n is a decreasing sequence. Let's prove it by induction:
$a_2 < a_1$ is true.
Suppose $a_{n+1} < a_n$, then we want to show that $a_{n+2} = \frac{1}{3-a_{n+1}} < a_{n+1}$. First, we need to show that $\{a_n\}$ is bounded.
Claim: $a_n \leq 2$ (by induction).
It's true for $a_1 = 2$.
If $a_n \leq 2$, then $a_{n+1} = \frac{1}{3-a_n} \leq \frac{1}{3-2} \leq 2$. So $a_n \leq 2$ is true by induction.
Now $3 - a_{n+1} > 3 - a_n \geq 1 > 0$. So $a_{n+2} = \frac{1}{3-a_{n+1}} < \frac{1}{3-a_n} = a_{n+1}$.
We also need to claim: $0 < a_n,\ \forall n$.
True for $n = 1$.
Assume it is true for n, then $3 - a_n > 3 - 2 = 1$ and $\frac{1}{3-a_n} > 0$.

By bounded convergence theorem, $a_n \to L$ for some L. Using recursive sequences,
$$L = \lim_{n\to\infty} a_{n+1} = \frac{1}{3 - \lim_{n\to\infty} a_n} = \frac{1}{3-L}$$
$$L(3-L) = 3L - L^2 = 1$$
$$L^2 - 3L + 1 = 0 \Rightarrow L = \frac{3 \pm \sqrt{9-4}}{2}$$
So $L = \frac{3-\sqrt{5}}{2} \approx .382$. (Q: why do we know $L \neq \frac{3+\sqrt{5}}{2}$?)