6th lesson

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Exercises

1. (a)

$$\lim_{n\to\infty} \frac{(-2)^n + 3^n}{(-2)^{n+1} + 3^{n+1}}$$

Řešení: Let us factor out the greatest term:

$$\lim \frac{(-2)^n + 3^n}{(-2)^{n+1} + 3^{n+1}} = \lim \frac{3^n}{3^{n+1}} \cdot \lim \frac{\left(\frac{-2}{3}\right)^n + 1}{\left(\frac{-2}{3}\right)^{n+1} + 1} = \frac{1}{3} \cdot \frac{0+1}{0+1} = \frac{1}{3}.$$

(b)

$$\lim_{n\to+\infty} \frac{1^n + 2^n + 3^n + 4^n + 5^n}{5,0001^n}$$

Řešení: We can factor out $5,0001^n$ or just split into five fractions and apply the Arithemics limit theorem:

$$\lim_{n\to+\infty} \frac{1^n + 2^n + 3^n + 4^n + 5^n}{5,0001^n} = \lim_{n\to+\infty} \left(\frac{1}{5,0001}\right)^n + \left(\frac{2}{5,0001}\right)^n +$$

$$+ \left(\frac{3}{5,0001}\right)^n + \left(\frac{4}{5,0001}\right)^n + \left(\frac{5}{5,0001}\right)^n = 0 + 0 + 0 + 0 + 0 = 0,$$

(c)

$$\lim_{n\to\infty} \frac{3^n + n^5 + (n+1)!}{n(n^6 + n!)}$$

Řešení: The greatest term is factorial, hence:

$$\lim_{n\to+\infty} \frac{3^n + n^5 + (n+1)!}{n(n^6 + n!)} = \lim_{n\to+\infty} \frac{(n+1)!}{n(n!)} \frac{\frac{3^n}{n!} + \frac{n^5}{n!} + 1}{\frac{n^6}{n!} + 1} = \lim_{n\to+\infty} \frac{n+1}{n} \frac{\frac{3^n}{n!} + \frac{n^5}{n!} + 1}{\frac{n^6}{n!} + 1}$$

$$= 1 \cdot \frac{0+0+1}{0+1} = 1.$$

(d)

$$\lim_{n\to\infty} \frac{\ln n + n^3 + \frac{1}{n} + e^n + 5^n}{\ln_{10} n + n^4 + 5^n + n^3 + 4^n}$$

Řešení: We factor out 5^n:

$$\lim_{n\to\infty} \frac{5^n}{5^n} \frac{\frac{\ln n}{5^n} + \frac{n^3}{5^n} + \frac{\frac{1}{n}}{5^n} + \frac{e^n}{5^n} + \frac{5^n}{5^n}}{\frac{\ln_{10} n}{5^n} + \frac{n^4}{5^n} + \frac{5^n}{5^n} + \frac{n^3}{5^n} + \frac{4^n}{5^n}} \overset{VOAL}{=} \lim_{n\to\infty} \frac{0+0+0+0+1}{0+0+1+0+0} = 1$$

(e)

$$\lim_{n\to\infty} \frac{(n+2)! + (n+1)!}{(n+2)! - (n+1)!}$$

Řešení: Let us factor out $(n+1)!$:

$$\lim_{n\to\infty} \frac{(n+2)! + (n+1)!}{(n+2)! - (n+1)!} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)!} \cdot \frac{(n+2)+1}{n+2-1} = \lim_{n\to\infty} \frac{n+3}{n+1} =$$
$$\lim_{n\to\infty} \frac{n}{n} \cdot \frac{1+3/n}{1+1/n} \overset{VOAL}{=} 1$$

2. Find infimum, minimum, maximum and supremum in $\mathbb{R}$:

(a)
$$\mathbb{N}$$

(b)
$$(0;2]$$

(c)
$$(0;1) \cap \mathbb{Q}$$

(d)
$$\{x \in \mathbb{Z}; x \geq -\sqrt{6}\}$$

(e)
$$\{(-1)^n \sqrt{n}; n \in \mathbb{N}\}$$

(f)
$$\{\arctan x; x \in \mathbb{R}\}$$

(g)
$$\{1 - \frac{1}{n}; n \in \mathbb{N}\}$$

(h)
$$\{\frac{1+(-1)^n}{2}; n \in \mathbb{N}\}$$

(i)
$$\{\cos \frac{n\pi}{2}; n \in \mathbb{N}\}$$

(j)
$$\{(-1)^n n; n \in \mathbb{N}\}$$

	inf	min	max	sup
a	1	1	$\nexists$	∞
b	0	$\nexists$	2	2
c	0	$\nexists$	$\nexists$	1
d	-2	-2	$\nexists$	∞
e	$-\infty$	$\nexists$	$\nexists$	∞
f	$-\frac{\pi}{2}$	$\nexists$	$\nexists$	$\frac{\pi}{2}$
g	0	0	$\nexists$	1
h	0	0	1	1
i	-1	-1	1	1
j	$-\infty$	$\nexists$	$\nexists$	∞

3. Find lim sup and lim inf of the following sequences. Can we find also limits?

(a)

$$x_n = 1 - \frac{1}{n}$$

Solution: Since the sequence is convergent, we have.

$$\limsup x_n = \liminf x_n = \lim x_n = 1.$$

(b)

$$x_n = (-1)^{n-1}\left(2 + \frac{3}{n}\right)$$

Solution: Let us consider even and odd subsequence: $x_{2n} = (2 + \frac{3}{n})$ a $x_{2n+1} = -(2 + \frac{3}{n})$. Hence

$$\limsup x_n = 2, \qquad \liminf x_n = -2.$$

(c)

$$x_n = \frac{(-1)^n}{n} + \frac{1 + (-1)^n}{2}$$

Solution: Let us consider even and odd subsequence: $x_{2n} = \frac{1}{n} + 1$ a $x_{2n+1} = \frac{1}{n}$. Hence

$$\limsup x_n = 1, \qquad \liminf x_n = 0.$$

(d)

$$x_n = 1 + \frac{n}{n+1} \cos \frac{n\pi}{2}$$

Solution: The sequence $\cos \frac{n\pi}{2}$ attains only values $0, -1, 0, 1$. All odd terms are zero, even terms are:

$$x_{2n} = 1 - \frac{2n}{2n+1} \to 0, \qquad x_{4n} = 1 + \frac{4n}{4n+1} \to 2.$$

Hence

$$\limsup x_n = 2, \qquad \liminf x_n = 0.$$

(e)

$$x_n = 1 + 2(-1)^{n+1} + 3 \cdot (-1)^{\frac{n(n-1)}{2}}$$

Solution: The term $n(n-1)/2$ is even for $n = 4k$ and $n = 4k + 1$, for $n = 4k + 2$ and $n = 4k + 3$ it is odd. Hence we can find four subsequences:

$$x_{4n} = 1+2+3 = 6, \quad x_{4n+1} = 1-2+3 = 2, \quad x_{4n+2} = 1+2-3 = 0, \quad x_{4n+3} = 1-2-3 = -4.$$

Hence $\limsup x_n = \sup x_n = 6$ and $\liminf x_n = \inf x_n = -4$.

(f)

$$x_n = (-1)^n n$$

Solution: Odd and even subsequences $x_{2n} = 2n$ and $x_{2n+1} = -2n - 1$ goes to $\pm\infty$. We then have

$$\limsup x_n = \sup x_n = +\infty, \qquad \liminf x_n = \inf x_n = -\infty.$$

(g)

$$x_n = -n[2 + (-1)^n]$$

Solution: The sequence is convergent, because $-n[2+(-1)^n] \le -n \to -\infty$. Therefore we obtain

$$\limsup x_n = \liminf x_n = \inf x_n = -\infty.$$