5th lesson

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Theory

Theorem 1 (Arithemics of limits)**.** Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be sequences (of real numbers). Further let $\lim_{n\to\infty} a_n = A \in \mathbb{R}^*$ and $\lim_{n\to\infty} b_n = B \in \mathbb{R}^*$. Then

(a) $\lim_{n\to\infty} (a_n + b_n) = A + B$,

(b) $\lim_{n\to\infty} (a_n \cdot b_n) = A \cdot B$,

(c) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$,

if the right sides are well defined.

Exercises

1. Find limits:

(a) $\lim\limits_{n\to\infty} \sqrt{n+2} + \sqrt{n}$ **Solution:**
We just substitute for n - Arithmetic of limits theorem.

$$\lim_{n\to\infty} \sqrt{n+2} + \sqrt{n} = \infty + \infty = \infty$$

(b) $\lim\limits_{n\to\infty} \sqrt{n+2} - \sqrt{n}$
Solution: Let us use the formulae $A^2 - B^2 = (A-B)(A+B)$.

$$\lim_{n\to\infty} \sqrt{n+2} - \sqrt{n} = \lim_{n\to\infty} (\sqrt{n+2} - \sqrt{n}) \frac{\sqrt{n+2} + \sqrt{n}}{\sqrt{n+2} + \sqrt{n}} = \lim_{n\to\infty} \frac{n+2-n}{\sqrt{n+2} + \sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{2}{\sqrt{n+2} + \sqrt{n}} == \lim_{n\to\infty} \frac{2}{\sqrt{n}} \frac{1}{\sqrt{1+2/n} + 1} = 0 \frac{1}{1+1} = 0$$

(c) $\lim\limits_{n\to\infty} \dfrac{\sqrt{n^2+1}}{n}$
Solution: Let us factor out n:

$$\lim_{n\to\infty} \frac{\sqrt{n^2+1}}{n} = \lim_{n\to\infty} \frac{n\sqrt{1+1/n^2}}{n} = \sqrt{1+0} = 1$$

(d) $\lim\limits_{n\to\infty} \dfrac{\sqrt{n-1} - \sqrt{n}}{\sqrt{n^2-3} - \sqrt{(n+2)^2}}$
Solution: Let us expand the fraction (twice):

$$\lim_{n\to\infty} \frac{\sqrt{n-1} - \sqrt{n}}{\sqrt{n^2-3} - \sqrt{(n+2)^2}} \cdot \frac{\sqrt{n-1} + \sqrt{n}}{\sqrt{n-1} + \sqrt{n}} \cdot \frac{\sqrt{n^2-3} + \sqrt{(n+2)^2}}{\sqrt{n^2-3} + \sqrt{(n+2)^2}} =$$

$$= \lim_{n\to\infty} \frac{n-1-n}{n^2-3-(n+2)^2} \cdot \frac{\sqrt{n^2-3}+\sqrt{(n+2)^2}}{\sqrt{n-1}+\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{-1}{-4n-7} \cdot \frac{\sqrt{n^2-3}+\sqrt{(n+2)^2}}{\sqrt{n-1}+\sqrt{n}}.$$

Now, let us factor out n from the numerator and $n\sqrt{n}$ from the denominator.

$$= \lim_{n\to\infty} \frac{-n}{n\sqrt{n}} \cdot \frac{1}{-4-7/n} \frac{\sqrt{1-3/n^2}+(1+2/n)}{\sqrt{1-1/n}+1} \cdot = 0 \cdot \frac{1}{-4-0} \cdot \frac{1+1+0}{1-0+1} = 0$$

2. (a)

$$\lim_{n\to\infty} \frac{(-2)^n+3^n}{(-2)^{n+1}+3^{n+1}}$$

Řešení: Let us factor out the greatest term:

$$\lim \frac{(-2)^n+3^n}{(-2)^{n+1}+3^{n+1}} = \lim \frac{3^n}{3^{n+1}} \cdot \lim \frac{\left(\frac{-2}{3}\right)^n+1}{\left(\frac{-2}{3}\right)^{n+1}+1} = \frac{1}{3} \cdot \frac{0+1}{0+1} = \frac{1}{3}.$$

(b)

$$\lim_{n\to+\infty} \frac{1^n+2^n+3^n+4^n+5^n}{5,0001^n}$$

Řešení: We can factor out $5,0001^n$ or just split into five fractions and apply the Arithemics limit theorem:

$$\lim_{n\to+\infty} \frac{1^n+2^n+3^n+4^n+5^n}{5,0001^n} = \lim_{n\to+\infty} \left(\frac{1}{5,0001}\right)^n + \left(\frac{2}{5,0001}\right)^n +$$

$$+ \left(\frac{3}{5,0001}\right)^n + \left(\frac{4}{5,0001}\right)^n + \left(\frac{5}{5,0001}\right)^n = 0+0+0+0+0 = 0,$$

(c)

$$\lim_{n\to\infty} \frac{3^n+n^5+(n+1)!}{n(n^6+n!)}$$

Řešení: The greatest term is factorial, hence:

$$\lim_{n\to+\infty} \frac{3^n+n^5+(n+1)!}{n(n^6+n!)} = \lim_{n\to+\infty} \frac{(n+1)!}{n(n!)} \frac{\frac{3^n}{n!}+\frac{n^5}{n!}+1}{\frac{n^6}{n!}+1} = \lim_{n\to+\infty} \frac{n+1}{n} \frac{\frac{3^n}{n!}+\frac{n^5}{n!}+1}{\frac{n^6}{n!}+1}$$

$$= 1 \cdot \frac{0+0+1}{0+1} = 1.$$

(d)

$$\lim_{n\to\infty} \frac{\ln n + n^3 + \frac{1}{n} + e^n + 5^n}{\ln_{10} n + n^4 + 5^n + n^3 + 4^n}$$

Řešení: We factor out 5^n:

$$\lim_{n\to\infty} \frac{5^n}{5^n} \frac{\frac{\ln n}{5^n}+\frac{n^3}{5^n}+\frac{\frac{1}{n}}{5^n}+\frac{e^n}{5^n}+\frac{5^n}{5^n}}{\frac{\ln_{10} n}{5^n}+\frac{n^4}{5^n}+\frac{5^n}{5^n}+\frac{n^3}{5^n}+\frac{4^n}{5^n}} \overset{VOAL}{=} \lim_{n\to\infty} \frac{0+0+0+0+1}{0+0+1+0+0} = 1$$

(e)

$$\lim_{n\to\infty} \frac{(n+2)! + (n+1)!}{(n+2)! - (n+1)!}$$

Řešení: Let us factor out $(n+1)!$:

$$\lim_{n\to\infty} \frac{(n+2)! + (n+1)!}{(n+2)! - (n+1)!} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)!} \cdot \frac{(n+2)+1}{n+2-1} = \lim_{n\to\infty} \frac{n+3}{n+1} =$$

$$\lim_{n\to\infty} \frac{n}{n} \cdot \frac{1+3/n}{1+1/n} \overset{VOAL}{=} 1$$

3. True or false? (Divergent can mean both tending to infinity or oscillating.)

   (a) Let x_n be convergent sequence and y_n be divergent sequence. Decide if sequences a) $x_n + y_n$, b) $x_n y_n$ are divergent.

   Solution: a) True. b) False, $x_n = \frac{1}{n}$, $y_n = n$, then $x_n y_n = 1$, which is convergent sequence.

   (b) Let x_n a y_n be divergent sequences. Decide if sequences a) $x_n + y_n$, b) $x_n y_n$ are divergent.

   Solution: Both are false. Let $x_n = (-1)^n$, $y_n = (-1)^{n+1}$, then $x_n + y_n = 0$ a $x_n y_n = -1$, which both are convergent.

   (c) Let $\lim x_n = 0$ a y_n be an arbitrary sequence. Decide if $\lim(x_n y_n) = 0$?

   Solution: False. Let $x_n = \frac{1}{n}$ and $y_n = n$. Then $\lim_{n\to\infty} x_n y_n = 1$.

   (d) Let $\lim(x_n y_n) = 0$. Decide if either $\lim x_n = 0$ or $\lim y_n = 0$?

   Solution: False. Let us consider x_n such that $x_n = 1$ for even n and $y_n = 0$ for odd n. Further let y_n be similar, with $y_n = 0$ for even n and $y_n = 1$ for odd n. Then $x_n y_n = 0$ for all n (and of course $\lim_{n\to\infty} x_n y_n = 0$, but $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$ does not exist.