4th lesson

https://www2.karlin.mff.cuni.cz/∼kuncova/en/teachMat1.php
kunck6am@natur.cuni.cz

Theory

Theorem 1 (Arithemics of limits)**.** Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be sequences (of real numbers). Further let $\lim_{n\to\infty} a_n = A \in \mathbb{R}^*$ and $\lim_{n\to\infty} b_n = B \in \mathbb{R}^*$. Then

(a) $\lim_{n\to\infty}(a_n + b_n) = A + B$,

(b) $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$,

(c) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$,

if the right sides are well defined.

Exercises

1. Find limits:

   (a) $(-1)^n \nexists$

   (b) $(-1)^n n \nexists$

   (c) $(-1)^n \dfrac{1}{n} = 0$

   (d) $\cos(\pi n)\sqrt{n} = \lim\limits_{n\to\infty} (-1)^n \sqrt{n} \nexists$

2. Find limits:

   (a)
$$\lim_{n\to\infty} -n^8 + 2n^3 - 4$$

   Solution: Factor out the ”largest” term of the expression. The use the Arithmetic of limits theorem (several times).

$$\lim_{n\to\infty} -n^8+2n^3-4 = \lim_{n\to\infty} n^8 \left(-1 + \frac{2}{n^5} - \frac{4}{n^8}\right) \overset{AL}{=} \lim_{n\to\infty} n^8 \cdot \lim_{n\to\infty} \left(-1 + \frac{2}{n^5} - \frac{4}{n^8}\right) \overset{AL}{=}$$

$$\lim_{n\to\infty} n^8 \cdot \left(\lim_{n\to\infty} -1 + \lim_{n\to\infty} \frac{2}{n^5} - \lim_{n\to\infty} \frac{4}{n^8}\right) = \infty(-1+0-0) = -\infty$$

   (b)
$$\lim_{n\to\infty} \frac{2n^5 + 2n - 7}{n^5 - 6n^2 + 4}$$

Solution:

$$\lim_{n\to\infty}\frac{2n^5+2n-7}{n^5-6n^2+4}=\lim_{n\to\infty}\frac{n^5(2+\frac{2}{n^4}-\frac{7}{n^5})}{n^5(1-\frac{6}{n^3}+\frac{4}{n^5})}=\frac{\lim_{n\to\infty}(2+\frac{2}{n^4}-\frac{7}{n^5})}{\lim_{n\to\infty}(1-\frac{6}{n^3}+\frac{4}{n^5})}=$$

$$=\frac{\lim_{n\to\infty}2+\lim_{n\to\infty}\frac{2}{n^4}-\lim_{n\to\infty}\frac{7}{n^5}}{\lim_{n\to\infty}1-\lim_{n\to\infty}\frac{6}{n^3}+\lim_{n\to\infty}\frac{4}{n^5}}=\frac{2+0-0}{1-0+0}=2$$

(c)

$$\lim_{n\to\infty}\frac{5n^2+n-5}{n^3+8}$$

Solution:

$$\lim_{n\to\infty}\frac{5n^2+n-5}{n^3+8}=\lim_{n\to\infty}\frac{n^3\left(\frac{5}{n}+\frac{1}{n}-\frac{5}{n^3}\right)}{n^3\left(1+\frac{8}{n^3}\right)}=\frac{0+0-0}{1+0}=0$$

(d) $\lim\limits_{n\to\infty}\frac{\sqrt[3]{n^2}}{n+1}$

Solution:

$$\lim_{n\to\infty}\frac{\sqrt[3]{n^2}}{n+1}=\lim_{n\to\infty}\frac{\sqrt[3]{n^2}}{n}\cdot\frac{1}{1+\frac{1}{n}}=\lim_{n\to\infty}\frac{1}{\sqrt[3]{n}}\cdot\frac{1}{1+\frac{1}{n}}$$

$$\stackrel{AL}{=}\lim_{n\to\infty}\frac{1}{\sqrt[3]{n}}\cdot\frac{\lim_{n\to\infty}1}{\lim_{n\to\infty}1+\lim_{n\to\infty}\frac{1}{n}}=0\cdot\frac{1}{1+0}=0.$$

3. Find limits:

(a) $\lim\limits_{n\to\infty}\sqrt{n+2}+\sqrt{n}$ **Solution:**
We just substitute for n - Arithmetic of limits theorem.

$$\lim_{n\to\infty}\sqrt{n+2}+\sqrt{n}=\infty+\infty=\infty$$

(b) $\lim\limits_{n\to\infty}\sqrt{n+2}-\sqrt{n}$

Solution: Let us use the formulae $A^2-B^2=(A-B)(A+B)$.

$$\lim_{n\to\infty}\sqrt{n+2}-\sqrt{n}=\lim_{n\to\infty}(\sqrt{n+2}-\sqrt{n})\frac{\sqrt{n+2}+\sqrt{n}}{\sqrt{n+2}+\sqrt{n}}=\lim_{n\to\infty}\frac{n+2-n}{\sqrt{n+2}+\sqrt{n}}$$

$$=\lim_{n\to\infty}\frac{2}{\sqrt{n+2}+\sqrt{n}}==\lim_{n\to\infty}\frac{2}{\sqrt{n}}\frac{1}{\sqrt{1+2/n}+1}=0\frac{1}{1+1}=0$$

(c) $\lim\limits_{n\to\infty}\frac{\sqrt{n^2+1}}{n}$

Solution: Let us factor out n:

$$\lim_{n\to\infty}\frac{\sqrt{n^2+1}}{n}=\lim_{n\to\infty}\frac{n\sqrt{1+1/n^2}}{n}=\sqrt{1+0}=1$$

(d) $\lim_{n\to\infty} \dfrac{\sqrt{n-1}-\sqrt{n}}{\sqrt{n^2-3}-\sqrt{(n+2)^2}}$

Solution: Let us expand the fraction (twice):

$$\lim_{n\to\infty} \frac{\sqrt{n-1}-\sqrt{n}}{\sqrt{n^2-3}-\sqrt{(n+2)^2}} \cdot \frac{\sqrt{n-1}+\sqrt{n}}{\sqrt{n-1}+\sqrt{n}} \cdot \frac{\sqrt{n^2-3}+\sqrt{(n+2)^2}}{\sqrt{n^2-3}+\sqrt{(n+2)^2}} =$$

$$= \lim_{n\to\infty} \frac{n-1-n}{n^2-3-(n+2)^2} \cdot \frac{\sqrt{n^2-3}+\sqrt{(n+2)^2}}{\sqrt{n-1}+\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{-1}{-4n-7} \cdot \frac{\sqrt{n^2-3}+\sqrt{(n+2)^2}}{\sqrt{n-1}+\sqrt{n}}.$$

Now, let us factor out n from the numerator and $n\sqrt{n}$ from the denominator.

$$= \lim_{n\to\infty} \frac{-n}{n\sqrt{n}} \cdot \frac{1}{-4-7/n} \frac{\sqrt{1-3/n^2}+(1+2/n)}{\sqrt{1-1/n}+1} \cdot = 0 \cdot \frac{1}{-4-0} \cdot \frac{1+1+0}{1-0+1} = 0$$

Bonus

4. Find limits:

(a) $\lim_{n\to\infty} \dfrac{(n+4)^{100}-(n+3)^{100}}{(n+2)^{100}-n^{100}}$

Solution: Let us use the binomial expansion

$$\lim_{n\to\infty} \frac{(n^{100}+100\cdot 4n^{99}+\cdots)-(n^{100}+100\cdot 3n^{99}+\cdots)}{(n^{100}+100\cdot 2n^{99}+\cdots)-n^{100}} =$$

$$= \lim_{n\to\infty} \frac{100n^{99}+34650\cdot n^{98}+\cdots}{200n^{99}+19800n^{98}\cdots} = \lim_{n\to\infty} \frac{n^{99}(100+\frac{34650}{n}\cdots)}{n^{99}(200+\frac{19800}{n}\cdots)} = \frac{1}{2}$$

(b) $\lim_{n\to\infty} \sqrt[3]{n+1}-\sqrt[3]{n}$

Solution: We use the formulae $A^3-B^3=(A-B)(A^2+AB+B^2)$ and then we expand the fraction. Here $A=\sqrt[3]{n+1}$ and $B=\sqrt[3]{n}$. Finally we factor out the leading term - $\sqrt[3]{n^2}$.

$$\lim_{n\to\infty} \sqrt[3]{n+1}-\sqrt[3]{n} = \lim_{n\to\infty} (\sqrt[3]{n+1}-\sqrt[3]{n}) \cdot \frac{(n+1)^{2/3}+\sqrt[3]{n+1}\sqrt[3]{n}+n^{2/3}}{(n+1)^{2/3}+\sqrt[3]{n+1}\sqrt[3]{n}+n^{2/3}} =$$

$$= \lim_{n\to\infty} \frac{n+1-n}{(n+1)^{2/3}+\sqrt[3]{n+1}\sqrt[3]{n}+n^{2/3}} = \lim_{n\to\infty} \frac{n^{2/3}}{n^{2/3}} \frac{\frac{1}{n^{2/3}}}{\sqrt[3]{1+\frac{2}{n}+\frac{1}{n^2}}+\sqrt[3]{1+\frac{1}{n}}+1} =$$

$$= \frac{0}{\sqrt[3]{1+0+0}+\sqrt[3]{1+0}+1} = 0$$

(c) $\lim_{n\to\infty} \sqrt{n}(\sqrt{n+1} - \sqrt{n})$

Solution: Let us use the formulae $A^2 - B^2 = (A-B)(A+B)$, here $A = \sqrt{n+1}$, $B = \sqrt{n}$. Then we factor out the leading term, which is $\sqrt{n}$.

$$\lim_{n\to\infty} \sqrt{n}(\sqrt{n+1} - \sqrt{n}) \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \sqrt{n}\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} =$$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}} \frac{1}{\sqrt{1+\frac{1}{n}}+\sqrt{1}} \overset{AL}{=} \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}$$

(d) $\lim_{n\to\infty} \sqrt[3]{n^3+1} - \sqrt{n^2+1}$

Solution: We need to use the formulae for $A^6 - B^6 = (A-B)(A^5 + A^4B + A^3B^2 + A^2B^3 + AB^4 + B^5)$. Hence we obtain:

$$\lim_{n\to\infty} \sqrt[3]{n^3+1} - \sqrt{n^2+1}$$

$$= \lim_{n\to\infty} (\sqrt[3]{n^3+1} - \sqrt{n^2+1}) \cdot$$

$$\cdot \frac{(\sqrt[3]{n^3+1})^5 + (\sqrt[3]{n^3+1})^4(\sqrt{n^2+1}) + (\sqrt[3]{n^3+1})^3(\sqrt{n^2+1})^2 + \cdots + (\sqrt{n^2+1})^5}{(\sqrt[3]{n^3+1})^5 + (\sqrt[3]{n^3+1})^4(\sqrt{n^2+1}) + (\sqrt[3]{n^3+1})^3(\sqrt{n^2+1})^2 + \cdots + (\sqrt{n^2+1})^5}$$

$$= \lim_{n\to\infty} \frac{(n^3+1)^2 - (n^2+1)^3}{(\sqrt[3]{n^3+1})^5 + (\sqrt[3]{n^3+1})^4(\sqrt{n^2+1}) + (\sqrt[3]{n^3+1})^3(\sqrt{n^2+1})^2 + \cdots + (\sqrt{n^2+1})^5}$$

$$= \lim_{n\to\infty} \frac{-3n^4 + 2n^3 - 3n^2}{(\sqrt[3]{n^3+1})^5 + (\sqrt[3]{n^3+1})^4(\sqrt{n^2+1}) + (\sqrt[3]{n^3+1})^3(\sqrt{n^2+1})^2 + \cdots + (\sqrt{n^2+1})^5}$$

The leading exponent in the numerator is n^4, whereas in the denominator it is n^5. We factor out the leading exponents:

$$= \lim_{n\to\infty} \frac{n^4}{n^5} \frac{-3 + 2\frac{1}{n} - 3\frac{1}{n^2}}{(\sqrt[3]{1+1/n^3})^5 + (\sqrt[3]{1+1/n^3})^4(\sqrt{1+1/n^2}) + \cdots + (\sqrt{1+1/n^2})^5}$$

$$= 0\frac{-3+0-0}{1+1+1+1+1+1} = 0.$$

(e) $\lim_{n\to\infty} \frac{\sqrt[3]{n^2+7} - \sqrt[3]{n^2+1}}{\sqrt[3]{n^2+6} - \sqrt[3]{n^2}}$

Solution: We expand the fraction using the formulae $(A-B)(A^2 + AB + B^2) = A^3 - B^3$ both to the numerator and the denominator.

$$\lim_{n\to\infty} \frac{\sqrt[3]{n^2+7} - \sqrt[3]{n^2+1}}{\sqrt[3]{n^2+6} - \sqrt[3]{n^2}} \cdot \frac{\sqrt[3]{(n^2+7)^2} + \sqrt[3]{n^2+1}\sqrt[3]{n^2+7} + \sqrt[3]{(n^2+1)^2}}{\sqrt[3]{(n^2+7)^2} + \sqrt[3]{n^2+1}\sqrt[3]{n^2+7} + \sqrt[3]{(n^2+1)^2}} \cdot$$

$$\cdot \frac{\sqrt[3]{(n^2+6)^2} + \sqrt[3]{n^2}\sqrt[3]{n^2+6} + \sqrt[3]{(n^2)^2}}{\sqrt[3]{(n^2+6)^2} + \sqrt[3]{n^2}\sqrt[3]{n^2+6} + \sqrt[3]{(n^2)^2}}$$

$$= \lim_{n\to\infty} \frac{6}{6} \cdot \frac{\sqrt[3]{(n^2+6)^2} + \sqrt[3]{n^2}\sqrt[3]{n^2+6} + \sqrt[3]{(n^2)^2}}{\sqrt[3]{(n^2+7)^2} + \sqrt[3]{n^2+1}\sqrt[3]{n^2+7} + \sqrt[3]{(n^2+1)^2}}$$

Now we factor out the leading term - $n^{4/3}$.

$$= \lim_{n\to\infty} \frac{n^{4/3}}{n^{4/3}} \cdot \frac{\sqrt[3]{(1+6/n^2)^2} + \sqrt[3]{1+6/n^2} + 1}{\sqrt[3]{(n^2+7)^2} + \sqrt[3]{n^2+1}\sqrt[3]{n^2+7} + \sqrt[3]{(n^2+1)^2}}$$

$$= 1 \cdot \frac{1+1+1}{1+1+1} = 1.$$