3rd lesson

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Theory

Theorem 1 (Arithemics of limits)**.** Let $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be sequences (of real numbers). Further let $\lim_{n\to\infty} a_n = A \in \mathbb{R}^*$ and $\lim_{n\to\infty} b_n = B \in \mathbb{R}^*$. Then

(a) $\lim_{n\to\infty} (a_n + b_n) = A + B$,

(b) $\lim_{n\to\infty} (a_n \cdot b_n) = A \cdot B$,

(c) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$,

if the right sides are well defined.

Exercises

1. Write down a few first terms of the sequences, sketch the graph, find the limit.

   (a) $\lim\limits_{n\to\infty} n = \infty$

   (b) $\lim\limits_{n\to\infty} n^2 = \infty$

   (c) $\lim\limits_{n\to\infty} \sqrt{n} = \infty$

   (d) $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

   (e) $\lim\limits_{n\to\infty} \frac{1}{n^2} = 0$

   (f) $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

   (g) $\lim\limits_{n\to\infty} e^n = \infty$

   (h) $\lim\limits_{n\to\infty} e^{-n} = 0$

   (i) $\lim\limits_{n\to\infty} \ln n = \infty$

   (j) $\lim\limits_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$

   (k) $\lim\limits_{n\to\infty} 2^n = \infty$

   (l) $\lim\limits_{n\to\infty} n! = \infty$

2. Find limits:

   (a) $(-1)^n \ \nexists$

   (b) $(-1)^n n \ \nexists$

   (c) $(-1)^n \frac{1}{n} = 0$

   (d) $\cos(\pi n)\sqrt{n} = \lim\limits_{n\to\infty} (-1)^n \sqrt{n} \ \nexists$

3. Find limits:

   (a)
$$\lim_{n\to\infty} -n^8 + 2n^3 - 4$$

   Solution: Factor out the "largest" term of the expression. The use the Arithmetic of limits theorem (several times).

$$\lim_{n\to\infty} -n^8+2n^3-4 = \lim_{n\to\infty} n^8 \left(-1 + \frac{2}{n^5} - \frac{4}{n^8}\right) \overset{AL}{=} \lim_{n\to\infty} n^8 \cdot \lim_{n\to\infty} \left(-1 + \frac{2}{n^5} - \frac{4}{n^8}\right) \overset{AL}{=}$$

$$\lim_{n\to\infty} n^8 \cdot \left(\lim_{n\to\infty} -1 + \lim_{n\to\infty} \frac{2}{n^5} - \lim_{n\to\infty} \frac{4}{n^8}\right) = \infty(-1+0-0) = -\infty$$

(b)

$$\lim_{n\to\infty} \frac{2n^5+2n-7}{n^5-6n^2+4}$$

Solution:

$$\lim_{n\to\infty} \frac{2n^5+2n-7}{n^5-6n^2+4} = \lim_{n\to\infty} \frac{n^5(2+\frac{2}{n^4}-\frac{7}{n^5})}{n^5(1-\frac{6}{n^3}+\frac{4}{n^5})} = \frac{\lim_{n\to\infty}(2+\frac{2}{n^4}-\frac{7}{n^5})}{\lim_{n\to\infty}(1-\frac{6}{n^3}+\frac{4}{n^5})} =$$

$$= \frac{\lim_{n\to\infty} 2 + \lim_{n\to\infty} \frac{2}{n^4} - \lim_{n\to\infty} \frac{7}{n^5}}{\lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{6}{n^3} + \lim_{n\to\infty} \frac{4}{n^5}} = \frac{2+0-0}{1-0+0} = 2$$

(c)

$$\lim_{n\to\infty} \frac{5n^2+n-5}{n^3+8}$$

Solution:

$$\lim_{n\to\infty} \frac{5n^2+n-5}{n^3+8} = \lim_{n\to\infty} \frac{n^3\left(\frac{5}{n}+\frac{1}{n}-\frac{5}{n^3}\right)}{n^3\left(1+\frac{8}{n^3}\right)} = \frac{0+0-0}{1+0} = 0$$

(d) $\lim\limits_{n\to\infty} \frac{\sqrt[3]{n^2}}{n+1}$

Solution:

$$\lim_{n\to\infty} \frac{\sqrt[3]{n^2}}{n+1} = \lim_{n\to\infty} \frac{\sqrt[3]{n^2}}{n} \cdot \frac{1}{1+\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} \cdot \frac{1}{1+\frac{1}{n}}$$

$$\stackrel{AL}{=} \lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} \cdot \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{1}{n}} = 0 \cdot \frac{1}{1+0} = 0.$$