2nd lesson

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Theory

Definice 1. A *statement* (or proposition) is a sentence which can be declared to be either true or false. (But not both simultaneously.)

Exercises

1. Which sentences are statements? Which statements are true?

YES	It is raining (right now).
NO	Let the sunshine in!
YES	We have fish and chips.
YES	For every natural number there exists a bigger prime number.
YES	$\forall n \in \mathbb{N}\ \exists p : p > n$ and p is prime.
YES	Today is Friday or October.
NO	What's your favourite animal?
YES	Some mammals lay eggs.
YES	There exists a mammal, which lays eggs.
NO	This sentence is false.
YES	$\pi + e$ is irrational number.

depends on place and time;)	It is raining (right now).
depends on place and time;)	We have fish and chips.
True	For every natural number there exists a bigger prime number.
True	$\forall n \in \mathbb{N}\ \exists p : p > n$ and p is prime.
True, it was October	Today is Friday or October.
True	Some mammals lay eggs.
True	There exists a mammal, which lays eggs.
Nobody knows.	$\pi + e$ is irrational number.

2. Negate the following statements:

   (a) All classroms have at least one chair that is broken.
   There exists a classroom in which no chair is broken.

   (b) No classroom has only chairs that are not broken.
   There exists a classroom that has only chairs that are not broken

(c) Every student in this class loves dogs or cats.

There exists a student, who loves neither dogs nor cats. (For example, s/he is really afraid of dogs and avoids cats, but loves rabbits.)

(d) Every student in this class loves dogs and cats.

There exists a student, who does not love both dogs and cats. (For example, s/he loves dogs but not cats.)

(e) If a student loves cats, than s/he loves dogs.

There exists a student, who loves cats, but does not love dogs.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Function f is strictly increasing if $f(x) < f(y)$ whenever $x < y$.

(a) Express the statement using quantifiers.

$$\forall x, y \in \mathbb{R}, x < y : \ f(x) < f(y)$$

(b) Negate the statement.

$$\exists x, y \in \mathbb{R}, x < y : \ f(x) \geq f(y)$$

(c) Function f is nonincreasing if $f(x) \geq f(y)$ whenever $x < y$. Explain the difference between function, which is not increasing and function, which is nonincreasing. Give examples of such functions.

Nonincreasing: $y = -x$, $y = -\mathrm{sgn} x$. (Function still nonstrictly decreases.) Not increasing: $y = \sin x$. (There are two points, function does not increase between them.)

4. Complete the truth table:

A	B	$\neg A$	$\neg B$	$A \vee B$	$A \wedge B$	$A \implies B$	$A \iff B$
1	1	0	0	1	1	1	1
1	0	0	1	1	0	0	0
0	1	1	0	1	0	1	0
0	0	1	1	0	0	1	1

5. Let A, B, C be statements. Prove by truth table that following are tautologies:

   (a) $\neg(A \implies B) \Leftrightarrow (A \wedge \neg B)$ (b) $(A \implies B) \Leftrightarrow (\neg A \vee B)$

   (c) $((A \implies C) \wedge (C \implies B)) \implies (A \implies B)$

6. Let A and B be sets. Use the Venn diagram to show that: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

7. Let U be the set of all students of the Charles University. Further, let B be all tudents visiting a Business course, E students visiting an English course and M students visiting a Math course.

   Express by formula and by Venn diagram a set of students taking

   (a) at least one of these courses;
   (b) both Math and English, but not a Business course;
   (c) exactly one course.

8. Let A, B and X be sets. Prove de Morgan's laws:

   (a) $(A \cup B)^c = A^c \cap B^c$,
   (b) $(A \cap B)^c = A^c \cup B^c$.

Discussion

This example illustrates an alternative to using truth tables to establish the equivalence of two propositions. An alternative proof is obtained by excluding all possible ways in which the propositions may fail to be equivalent. Here is another example.

EXAMPLE 2.3.2. *Show $\neg(p \to q)$ is equivalent to $p \wedge \neg q$.*

Solution 1. Build a truth table containing each of the statements.

p	q	$\neg q$	$p \to q$	$\neg(p \to q)$	$p \wedge \neg q$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

Since the truth values for $\neg(p \to q)$ and $p \wedge \neg q$ are exactly the same for all possible combinations of truth values of p and q, the two propositions are equivalent.

Solution 2. We consider how the two propositions could *fail* to be equivalent. This can happen only if the first is true and the second is false or vice versa.

Case 1. Suppose $\neg(p \to q)$ is true and $p \wedge \neg q$ is false.
$\neg(p \to q)$ would be true if $p \to q$ is false. Now this only occurs if p is true and q is false. However, if p is true and q is false, then $p \wedge \neg q$ will be true. Hence this case is not possible.

Case 2. Suppose $\neg(p \to q)$ is false and $p \wedge \neg q$ is true.
$p \wedge \neg q$ is true only if p is true and q is false. But in this case, $\neg(p \to q)$ will be true. So this case is not possible either.

Since it is not possible for the two propositions to have different truth values, they must be equivalent.

EXERCISE 2.3.1. *Use a truth table to show that the propositions $p \leftrightarrow q$ and $\neg(p \oplus q)$ are equivalent.*

EXERCISE 2.3.2. *Use the method of Solution 2 in Example 2.3.2 to show that the propositions $p \leftrightarrow q$ and $\neg(p \oplus q)$ are equivalent.*

Definition 1.2.2 *Sentences $\mathbb{B}$ and $\mathbb{C}$ are* **logically equivalent** *if the standard truth tables for $\mathbb{B}$ and $\mathbb{C}$ have the same final column. We write $\mathbb{B} \equiv \mathbb{C}$ to denote that $\mathbb{B}$ and $\mathbb{C}$ are logically equivalent.*

The use of the word "if" in mathematical definitions (as in the preceding definition of logical equivalence) is a common practice in mathematical discourse and is always interpreted to mean "if and only if." This broader interpretation of "if" is used only in the context of definitions, while for theorems, lemmas, and other mathematical statements, we adhere to the strict, formal interpretation of the if–then logical connective. Thus, when we are reading a mathematical *definition* and encounter the word "if," we read the definition as an "if and only if" statement asserting the exact meaning of the identified word, allowing us to move freely back and forth between the defined word and the definition.

For example, if two sentences are logically equivalent, then the two sentences have the same final column in their standard truth tables. In addition, if two sentences have the same final column in their standard truth tables, then the two sentences are logically equivalent. You will want to develop a facility in this process of transitioning back and forth between defined mathematical words and the corresponding formal definitions.

We develop a good understanding of logical equivalences by considering some pairs of sentences that are logically equivalent, and some that are not.

Example 1.2.6 We prove that $(p \to q) \equiv [(\sim p) \vee q]$.

The basic truth table for the implication $p \to q$ and the standard truth table for $(\sim p) \vee q$ given in example 1.2.1 have the same final columns, as demonstrated below.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$\sim p$	$(\sim p) \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

■

Example 1.2.7 We prove that both $[(\sim p) \vee p] \not\equiv [(\sim p) \vee q]$ and $[(\sim p) \vee p] \not\equiv (p \to q)$.

Using the result of example 1.2.6, neither $(p \to q)$ nor $[(\sim p) \vee q]$ is logically equivalent to a contradiction. A contradiction has truth value F in every row of the final column of its standard truth table, while both of these sentences have T in the first row (and also in the third and fourth rows) of their respective final columns. In example 1.2.2, we found that $(\sim p) \wedge p$ is a contradiction. Alternatively, observe that the first sentence in each pair has one sentence variable, while the second sentence has two sentence variables, and so they cannot be logically equivalent.

■

A particularly important pair of logical equivalences is referred to as De Morgan's laws in honor of the nineteenth century English mathematician Augustus De Morgan, who first identified the significance of these relations for mathematical logic, set theory,

Theorem 2.5.1 *Let $\mathfrak{c}$ and $\mathfrak{c}'$ be contradictions and $\mathfrak{t}$ and $\mathfrak{t}'$ be tautologies. Then*

$$\mathfrak{c} \Leftrightarrow \mathfrak{c}', \qquad \mathfrak{t} \Leftrightarrow \mathfrak{t}', \qquad \mathfrak{c} \Leftrightarrow \neg\mathfrak{t}. \tag{2.7}$$

Proof: These follow directly from Definition 2.5.1 and Theorem 2.4.2. Because $\mathfrak{c}$ and $\mathfrak{c}'$ are both false, they have the same truth value. So they are logically equivalent. The same argument applies to $\mathfrak{t}$ and $\mathfrak{t}'$, because both of them have truth value "T." Finally $\neg\mathfrak{t}$ has truth value "F" so it is a contradiction. As a result, we may identify $\neg\mathfrak{t}$ with $\mathfrak{c}'$ in the first relation in (2.7). This yields $\mathfrak{c} \Leftrightarrow \neg\mathfrak{t}$. ■

This theorem indicates that up to logical equivalence there are a unique contradiction and a unique tautology, and that the former is the negation of the latter.

Exercise 2.5.2 Let $\mathfrak{a}, \mathfrak{c}$, and $\mathfrak{t}$ be respectively an arbitrary statement, a contradiction, and a tautology. Show that $\mathfrak{a} \Rightarrow \mathfrak{t}$ and $\mathfrak{c} \Rightarrow \mathfrak{a}$ are tautologies.
Solution: Because $\mathfrak{t}$ is true, according to Table 2.3, $\mathfrak{a} \Rightarrow \mathfrak{t}$ is true irrespective of whether $\mathfrak{a}$ is true or false. Therefore, $\mathfrak{a} \Rightarrow \mathfrak{t}$ is a tautology. Similarly, $\mathfrak{c} \Rightarrow \mathfrak{a}$ is a tautology, because $\mathfrak{c}$ is false and according to Table 2.3 this suffices to hold that $\mathfrak{c} \Rightarrow \mathfrak{a}$ is true regardless of the truth value of $\mathfrak{a}$. ■

A strange outcome of this exercise is that contradictions imply tautologies! The reader must not view all tautologies as unimportant or useless. For example, consider the statement (c) of Proposition 2.4.1, i.e.,

$$\mathfrak{d} := (((\mathfrak{a} \Leftrightarrow \mathfrak{b}) \wedge (\mathfrak{b} \Leftrightarrow \mathfrak{c})) \Rightarrow (\mathfrak{a} \Leftrightarrow \mathfrak{c})).$$

Since we have proven that $\mathfrak{d}$ is true regardless of the nature of its constituent statements, $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$, by Definition 2.5.1, $\mathfrak{d}$ is a tautology! Indeed, a large number of theorems in mathematics concern establishing that certain compound statements are tautologies. The following theorem is an example. It provides the basis for one of the most important methods of establishing the validity of an implication, namely the method of *proof by deduction* (Section 3.4).

Theorem 2.5.2 (Two-step deduction) *Let $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ be statements. Then the statement $\mathfrak{d} := ((\mathfrak{a} \Rightarrow \mathfrak{c}) \wedge (\mathfrak{c} \Rightarrow \mathfrak{b}))$ implies $\mathfrak{a} \Rightarrow \mathfrak{b}$, i.e., $\mathfrak{e} := (\mathfrak{d} \Rightarrow (\mathfrak{a} \Rightarrow \mathfrak{b}))$ is a tautology.*
Proof: We determine the truth value of $\mathfrak{e}$ by considering all possible truth values of $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$. Constructing the relevant truth table (Table 2.9) we find that indeed $\mathfrak{e}$ is always true; it is a tautology. ■

Exercise 2.5.3 *Let $\mathfrak{e}, \mathfrak{f}$ and $\mathfrak{g}$ be statements. Show that*

$$\mathfrak{h} := (\,((\mathfrak{e} \wedge \mathfrak{f}) \wedge (\mathfrak{e} \Rightarrow \mathfrak{g})) \;\Rightarrow\; (\mathfrak{f} \wedge \mathfrak{g})\,)$$

$\mathfrak{a}$	$\mathfrak{b}$	$\mathfrak{c}$	$\mathfrak{a} \Rightarrow \mathfrak{c}$	$\mathfrak{c} \Rightarrow \mathfrak{b}$	$\mathfrak{a} \Rightarrow \mathfrak{b}$	$\mathfrak{d}$	$\mathfrak{e}$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	F	T
T	F	T	T	F	F	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	F	T	F	T
F	F	F	T	T	T	T	T

Table 2.9: Truth table establishing that $\mathfrak{d} := ((\mathfrak{a} \Rightarrow \mathfrak{c}) \wedge (\mathfrak{c} \Rightarrow \mathfrak{b}))$ implies $\mathfrak{a} \Rightarrow \mathfrak{b}$. Here $\mathfrak{e} := (\mathfrak{d} \Rightarrow (\mathfrak{a} \Rightarrow \mathfrak{b}))$.

is a tautology.
Solution: Again we can establish $\mathfrak{h}$ by constructing its truth table (Problem 2.3). Here we give an alternative proof that is based on our knowledge of implications and conjunctions. Our aim is to show that $\mathfrak{h}$ cannot be false. First, we recall that an implication is false only if its hypothesis is true and its conclusion is false, and a conjunction is true only if its constituent statements are both true. We start our argument by expressing $\mathfrak{h}$ as the implication: $\mathfrak{h} = (\mathfrak{a} \Rightarrow \mathfrak{b})$ where

$$\begin{aligned} \mathfrak{a} &:= ((\mathfrak{e} \wedge \mathfrak{f}) \wedge (\mathfrak{e} \Rightarrow \mathfrak{g})), \\ \mathfrak{b} &:= (\mathfrak{f} \wedge \mathfrak{g}). \end{aligned}$$

$\mathfrak{h}$ can be false only if $\mathfrak{a}$ is true and $\mathfrak{b}$ is false. To ensure that $\mathfrak{a}$ is true,

(1) $\mathfrak{e} \wedge \mathfrak{f}$ must be true, which implies $\mathfrak{e}$ and $\mathfrak{f}$ are both true, and

(2) $\mathfrak{e} \Rightarrow \mathfrak{g}$ must be true.

Combining (1) and (2), we see that because both $\mathfrak{e}$ and $\mathfrak{e} \Rightarrow \mathfrak{g}$ are true, $\mathfrak{g}$ must be true. But according to (1), $\mathfrak{f}$ is also true. This shows that there is no way we can ensure that $\mathfrak{b}$ is false. Therefore, it is impossible for $\mathfrak{h}$ to be false; it is true regardless of the truth values of its constituents, i.e., it is a tautology. ■

Our solution of Exercise 2.5.3 involves two parts. First we actually consider the possibility that the statement we wish to prove is false. We then show that this never happens. This approach is called the method of *proof by contradiction* that we will examine more thoroughly in Section 3.5. We use a similar approach to solve the following exercise problem.

Disjoint Sets

Two sets A and B are **disjoint** if they have no elements in common, that is, if $A \cap B = \emptyset$.

An examination of Figure 1.2 or referring to the definition of A^c indicates that for any set A, A and A^c are disjoint. That is,

$$A \cap A^c = \emptyset$$

✧ Additional Laws for Sets

There are a number of laws for sets. They are referred to as commutative, associative, distributive, and De Morgan laws. We will consider two of these laws in the following examples.

HISTORICAL NOTE

Augustus De Morgan, 1806–1871

It was De Morgan who got George Boole interested in set theory and formal logic and then made significant advances upon Boole's epochal work. He discovered the De Morgan laws referred to in the last section. Boole and De Morgan are together considered the founders of the algebra of sets and of mathematical logic. De Morgan was a champion of religious and intellectual toleration and on several occasions resigned his professorships in protest of the abridgments of academic freedom of others.

EXAMPLE 5 **Establishing a De Morgan Law** Use a Venn diagram to show that

$$(A \cup B)^c = A^c \cap B^c$$

Solution We first consider the right side of this equation. Figure 1.6 shows a Venn diagram of A^c and B^c and $A^c \cap B^c$. We then notice from Figure 1.3 that this is $(A \cup B)^c$.

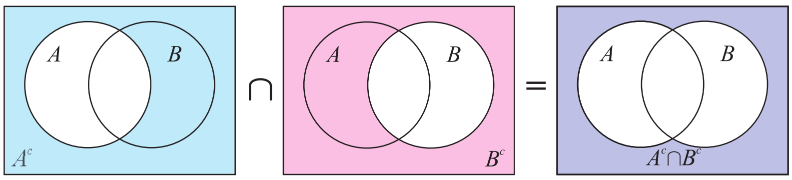

Figure 1.6 ✦

EXAMPLE 6 **Establishing the Distributive Law for Union** Use a Venn diagram to show that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution Consider first the left side of this equation. In Figure 1.7a the sets A, $B \cap C$, and the union of these two are shown. Now for the right side of the equation refer to Figure 1.7b, where the sets $A \cup B$, $A \cup C$, and the intersection of these two sets are shown. We have the same set in both cases. ✦

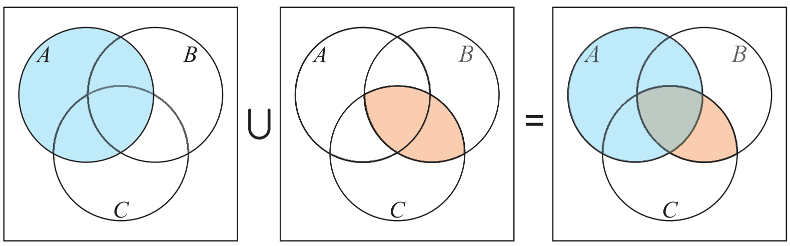

Figure 1.7a

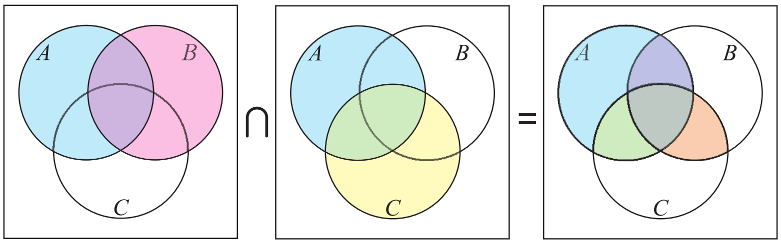

Figure 1.7b

We can summarize the laws we have found in the following list.

Laws for Set Operations

$A \cup B = B \cup A$	Commutative law for union
$A \cap B = B \cap A$	Commutative law for intersection
$A \cup (B \cup C) = (A \cup B) \cup C$	Associative law for union
$A \cap (B \cap C) = (A \cap B) \cap C$	Associative law for intersection
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive law for union
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive law for intersection
$(A \cup B)^c = A^c \cap B^c$	De Morgan law
$(A \cap B)^c = A^c \cup B^c$	De Morgan law

✧ Applications

EXAMPLE 7 Using Set Operations to Write Expressions Let U be the universal set consisting of the set of all students taking classes at the University of Hawaii and

$B = \{x | x \text{ is currently taking a business course}\}$
$E = \{x | x \text{ is currently taking an English course}\}$
$M = \{x | x \text{ is currently taking a math course}\}$

Write an expression using set operations and show the region on a Venn diagram for each of the following:

a. The set of students at the University of Hawaii taking a course in at least one of the above three fields.

b. The set of all students at the University of Hawaii taking both an English course and a math course but not a business course.

c. The set of all students at the University of Hawaii taking a course in exactly one of the three fields above.

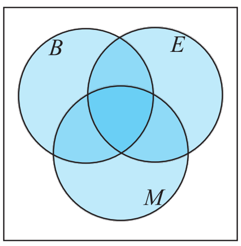

Figure 1.8a

Solution

a. This is $B \cup E \cup M$. See Figure 1.8a.

b. This can be described as the set of students taking an English course (E) and also (intersection) a math course (M) and also (intersection) not a business course (B^c) or

$$E \cap M \cap B^c$$

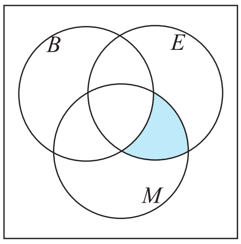

Figure 1.8b

This is the set of points in the universal set that are in both E and M but not in B and is shown in Figure 1.8b.

c. We describe this set as the set of students taking business but not taking English or math ($B \cap E^c \cap M^c$) together with (union) the set of students taking English but not business or math ($E \cap B^c \cap M^c$) together with (union) the set of students taking math but not business or English ($M \cap B^c \cap E^c$) or

$$(B \cap E^c \cap M^c) \cup (B^c \cap E \cap M^c) \cup (B^c \cap E^c \cap M)$$

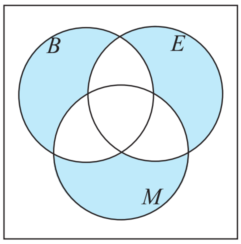

Figure 1.8c

This is the union of the three sets shown in Figure 1.8c. The first, $B \cap E^c \cap M^c$, consists of those points in B that are outside E and also outside M. The second set $E \cap B^c \cap M^c$ consists of those points in E that are outside B and M. The third set $M \cap B^c \cap E^c$ is the set of points in M that are outside B and E. The union of these three sets is then shown on the right in Figure 1.8c. ✦

REMARK: The word **only** means the same as exactly one. So a student taking only a business course would be written as $B \cap E^c \cap M^c$.

Self-Help Exercises 1.1

1. Let $U = \{1,2,3,4,5,6,7\}$, $A = \{l,2,3,4\}$, $B = \{3,4,5\}$, $C = \{2,3,4,5,6\}$. Find the following:
   a. $A \cup B$ **b.** $A \cap B$
   c. A^c **d.** $(A \cup B) \cap C$
   e. $(A \cap B) \cup C$ **f.** $A^c \cup B \cup C$

2. Let U denote the set of all corporations in this country and P those that made profits during the last year, D those that paid a dividend during the last year, and L those that increased their labor force during the last year. Describe the following using the three sets P, D, L, and set operations. Show the regions in a Venn diagram.
   a. Corporations in this country that had profits and also paid a dividend last year
   b. Corporations in this country that either had profits or paid a dividend last year
   c. Corporations in this country that did not have profits last year
   d. Corporations in this country that had profits, paid a dividend, and did not increase their labor force last year
   e. Corporations in this country that had profits or paid a dividend, and did not increase their labor force last year

1.1 Exercises

In Exercises 1 through 4, determine whether the statements are true or false.

1. **a.** $\emptyset \in A$ **b.** $A \in A$

2. **a.** $0 = \emptyset$ **b.** $\{x,y\} \in \{x,y,z\}$

3. **a.** $\{x|0 < x < -1\} = \emptyset$
   b. $\{x|0 < x < -1\} = 0$

4. **a.** $\{x|x(x-1) = 0\} = \{0,1\}$
   b. $\{x|x^2+1 < 0\} = \emptyset$

5. If $A = \{u,v,y,z\}$, determine whether the following statements are true or false.
   a. $w \in A$ **b.** $x \notin A$
   c. $\{u,x\} \cup A$ **d.** $\{y,z,v,u\} = A$

- (*Disjoint*). Given A_n and A_m with $n \neq m$, we can assume, without loss of generality, that $n < m$. Suppose that there existed some $x \in A_n \cap A_m$. Then by definition of these sets, there exists some odd numbers k and ℓ such that $x = 2^{n-1}k = 2^{m-1}\ell$. However since $n < m$, we have that $n \leq m - 1$, and therefore we can write $2^{m-1} = (2^n)(2^i)$ with $i \geq 0$. Hence we have $2^{n-1}k = 2^n 2^i \ell$. Dividing both sides by 2^{n-1} yields $k = (2)(2^i)\ell$, which contradicts the assumption that k is odd. Therefore $A_n \cap A_m = \emptyset$.
- (*Union is* $\mathbb{N}$). We want to show that $\bigcup_{n=1}^{\infty} A_n = \mathbb{N}$.
  - ($\subseteq$). Since each A_n is a subset of $\mathbb{N}$, the union of these sets is a subset of $\mathbb{N}$ as well.
  - ($\supseteq$). Given any $x \in \mathbb{N}$, we can write $x = 2^{n-1}k$ for some $n \in \mathbb{N}$ where k is odd. Then $x \in A_n$, as desired.

■

Exercise 4: Finite De Morgan's Laws (Abbott Exercise 1.2.5)

Let A and B be subsets of a set X. Show that the following set equalities hold.

(a) $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$.
(b) $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$.

These properties are sometimes called *De Morgan's Laws*.

Proof. (a)

- ($\subseteq$). If $x \in X \setminus (A \cap B)$ this means that x is not in $A \cap B$. Therefore either $x \notin A$ or $x \notin B$. Hence either $x \in X \setminus A$ or $x \in X \setminus B$. It follows that $(X \setminus A) \cup (X \setminus B)$.
- ($\supseteq$). If $x \in (X \setminus A) \cup (X \setminus B)$ then either $x \in X \setminus A$ or $x \in X \setminus B$. Hence either $x \notin A$ or $x \notin B$. Therefore $x \notin A \cap B$, so $x \in X \setminus (A \cap B)$.

(b)

- ($\subseteq$). If $x \in X \setminus (A \cup B)$, then $x \notin A \cup B$. Hence $x \notin A$ and $x \notin B$, so $x \in X \setminus A$ and $x \in X \setminus B$. Therefore $x \in (X \setminus A) \cap (X \setminus B)$.
- ($\supseteq$). If $x \in (X \setminus A) \cap (X \setminus B)$, then $x \in X \setminus A$ and $x \in X \setminus B$. Hence $x \notin A$ and $x \notin B$, so we have that $x \notin A \cup B$, and therefore $x \in X \setminus (A \cup B)$.

■

Exercise 5: Infinite De Morgan's Laws

Let A_n for each $n \in \mathbb{N}$ be subsets of a set X. Show that the following set equalities hold.

(a) $X \setminus (\bigcap_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty}(X \setminus A_n)$.
(b) $X \setminus (\bigcup_{n=1}^{\infty} A_n) = \bigcap_{n=1}^{\infty}(X \setminus A_n)$.

These properties are also referred to as *De Morgan's Laws*.

Proof. (a)

- ($\subseteq$). If $x \in X \setminus (\bigcap_{n=1}^{\infty} A_n)$ then $x \notin \bigcap_{n=1}^{\infty} A_n$, therefore there exists some n such that $x \notin A_n$. Hence $x \in X \setminus A_n$ for some n, and therefore $x \in \bigcup_{n=1}^{\infty}(X \setminus A_n)$.
- ($\supseteq$). If $x \in \bigcup_{n=1}^{\infty}(X \setminus A_n)$, then $x \in X \setminus A_n$ for some n, and therefore $x \notin A_n$ for some n, so $x \notin \bigcap_{n=1}^{\infty} A_n$, and therefore $x \in X \setminus (\bigcap_{n=1}^{\infty} A_n)$.

(b)

- ($\subseteq$). If $x \in X \setminus (\bigcup_{n=1}^{\infty} A_n)$, then $x \notin \bigcup_{n=1}^{\infty} A_n$, so $x \notin A_n$ for all n. Hence $x \in X \setminus A_n$ for all n, and therefore $x \in \bigcap_{n=1}^{\infty}(X \setminus A_n)$.
- ($\supseteq$). If $x \in \bigcap_{n=1}^{\infty}(X \setminus A_n)$, then $x \in X \setminus A_n$ for all n, so $x \notin A_n$ for all n. Therefore $x \notin \bigcup_{n=1}^{\infty} A_n$, and therefore $x \in X \setminus (\bigcup_{n=1}^{\infty} A_n)$.

■