Section 5.5

Trigonometric Equations

Objectives

1. Find all solutions of a trigonometric equation.
2. Solve equations with multiple angles.
3. Solve trigonometric equations quadratic in form.
4. Use factoring to separate different functions in trigonometric equations.
5. Use identities to solve trigonometric equations.
6. Use a calculator to solve trigonometric equations.

Exponential functions display the manic energies of uncontrolled growth. By contrast, trigonometric functions repeat their behavior. Do they embody in their regularity some basic rhythm of the universe? The cycles of periodic phenomena provide events that we can comfortably count on. When will the moon look just as it does at this moment? When can I count on 13.5 hours of daylight? When will my breathing be exactly as it is right now? Models with trigonometric functions embrace the periodic rhythms of our world. Equations containing trigonometric functions are used to answer questions about these models.

Trigonometric Equations and Their Solutions

1 Find all solutions of a trigonometric equation.

1a)

A **trigonometric equation** is an equation that contains a trigonometric expression with a variable, such as $\sin x$. We have seen that some trigonometric equations are identities, such as $\sin^2 x + \cos^2 x = 1$. These equations are true for every value of the variable for which the expressions are defined. In this section, we consider trigonometric equations that are true for only some values of the variable. The values that satisfy such an equation are its **solutions**. (There are trigonometric equations that have no solution.)

An example of a trigonometric equation is

$$\sin x = \tfrac{1}{2}.$$

A solution of this equation is $\frac{\pi}{6}$ because $\sin\frac{\pi}{6} = \frac{1}{2}$. By contrast, π is not a solution because $\sin\pi = 0 \neq \frac{1}{2}$.

Is $\frac{\pi}{6}$ the only solution of $\sin x = \frac{1}{2}$? The answer is no. Because of the periodic nature of the sine function, there are infinitely many values of x for which $\sin x = \frac{1}{2}$. **Figure 5.7** shows five of the solutions, including $\frac{\pi}{6}$, for $-\frac{3\pi}{2} \le x \le \frac{7\pi}{2}$. Notice that the x-coordinates of the points where the graph of $y = \sin x$ intersects the line $y = \frac{1}{2}$ are the solutions of the equation $\sin x = \frac{1}{2}$.

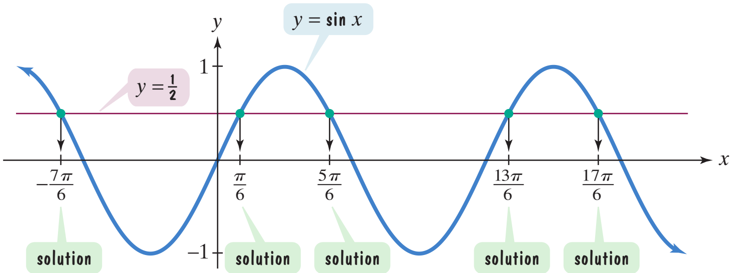


Figure 5.7 The equation $\sin x = \frac{1}{2}$ has five solutions when x is restricted to the interval $\left[-\frac{3\pi}{2}, \frac{7\pi}{2}\right]$.

How do we represent all solutions of $\sin x = \frac{1}{2}$? Because the period of the sine function is 2π, first find all solutions in $[0, 2\pi)$. The solutions are

$$x = \frac{\pi}{6} \quad \text{and} \quad x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}.$$

The sine is positive in quadrants I and II.

2a)

Any multiple of 2π can be added to these values and the sine is still $\frac{1}{2}$. Thus, all solutions of $\sin x = \frac{1}{2}$ are given by

$$x = \frac{\pi}{6} + 2n\pi \quad \text{or} \quad x = \frac{5\pi}{6} + 2n\pi,$$

where n is any integer. By choosing any two integers, such as $n = 0$ and $n = 1$, we can find some solutions of $\sin x = \frac{1}{2}$. Thus, four of the solutions are determined as follows:

Let $n = 0$.

$$x = \frac{\pi}{6} + 2 \cdot 0\pi = \frac{\pi}{6} \qquad x = \frac{5\pi}{6} + 2 \cdot 0\pi = \frac{5\pi}{6}$$

Let $n = 1$.

$$x = \frac{\pi}{6} + 2 \cdot 1\pi = \frac{\pi}{6} + 2\pi = \frac{\pi}{6} + \frac{12\pi}{6} = \frac{13\pi}{6} \qquad x = \frac{5\pi}{6} + 2 \cdot 1\pi = \frac{5\pi}{6} + 2\pi = \frac{5\pi}{6} + \frac{12\pi}{6} = \frac{17\pi}{6}.$$

These four solutions are shown among the five solutions in **Figure 5.7**.

Equations Involving a Single Trigonometric Function

To solve an equation containing a single trigonometric function:

- Isolate the function on one side of the equation.
- Solve for the variable.

EXAMPLE 1 Finding All Solutions of a Trigonometric Equation

1d)

Solve the equation: $3 \sin x - 2 = 5 \sin x - 1$.

Solution The equation contains a single trigonometric function, $\sin x$.

Step 1 Isolate the function on one side of the equation. We can solve for $\sin x$ by collecting terms with $\sin x$ on the left side and constant terms on the right side.

$$3 \sin x - 2 = 5 \sin x - 1 \quad \text{This is the given equation.}$$

$$3 \sin x - 5 \sin x - 2 = 5 \sin x - 5 \sin x - 1 \quad \text{Subtract } 5 \sin x \text{ from both sides.}$$

$$-2 \sin x - 2 = -1 \quad \text{Simplify.}$$

$$-2 \sin x = 1 \quad \text{Add 2 to both sides.}$$

$$\sin x = -\tfrac{1}{2} \quad \text{Divide both sides by } -2 \text{ and solve for } \sin x.$$

Step 2 Solve for the variable. We must solve for x in $\sin x = -\frac{1}{2}$. Because $\sin \frac{\pi}{6} = \frac{1}{2}$, the solutions of $\sin x = -\frac{1}{2}$ in $[0, 2\pi)$ are

$$x = \pi + \frac{\pi}{6} = \frac{6\pi}{6} + \frac{\pi}{6} = \frac{7\pi}{6} \qquad x = 2\pi - \frac{\pi}{6} = \frac{12\pi}{6} - \frac{\pi}{6} = \frac{11\pi}{6}.$$

The sine is negative in quadrant III. The sine is negative in quadrant IV.

Because the period of the sine function is 2π, the solutions of the equation are given by

$$x = \frac{7\pi}{6} + 2n\pi \quad \text{and} \quad x = \frac{11\pi}{6} + 2n\pi,$$

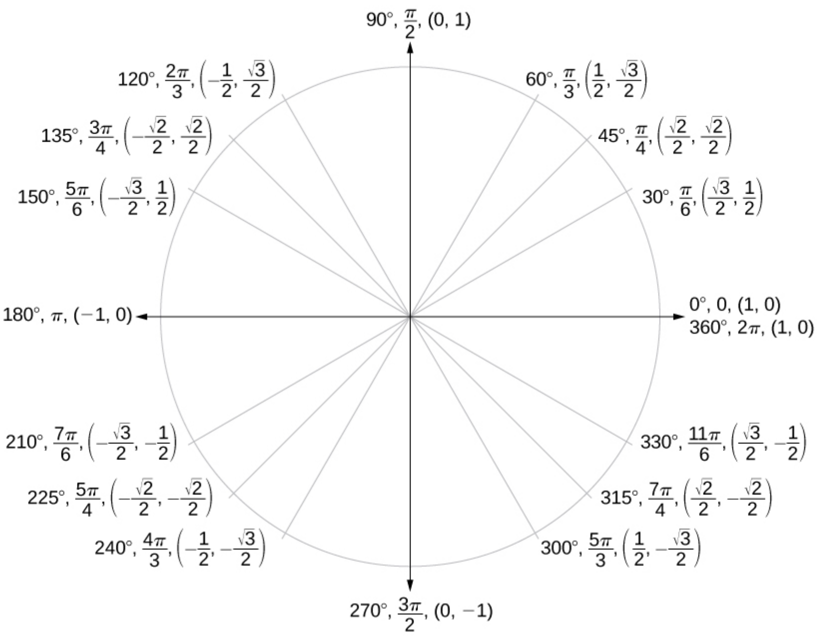
where n is any integer.

advantage of using the identities we developed in the previous sections.

General Strategy for solving trig equations

$+2k\pi$ $[0, 2\pi)$

Unit Circle



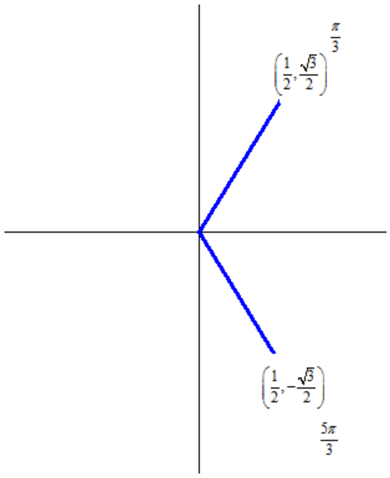
Example 7.45

Solving a Linear Trigonometric Equation Involving the Cosine Function

Find all possible exact solutions for the equation $\cos\theta = \frac{1}{2}$.

Solution

From the unit circle, we know that cosine is positive in QI and QIV. Cosine is an x value on the unit circle, so we want the angles on the unit circle where x is 1/2. Let's find the the angles on the unit circle.



$$\cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

These are the solutions in the interval $[0, 2\pi]$. All possible solutions are given by

$$\frac{\pi}{3} \pm 2k\pi \text{ and } \frac{5\pi}{3} \pm 2k\pi$$

where k is an integer.

Example 7.46

Solving a Linear Equation Involving the Sine Function

Find all possible exact solutions for the equation $\sin t = \frac{1}{2}$.

Solution

First we want to solve in one full cycle. We know sine is positive in QI and QII. Since sine is a y value we want the angles in QI and QII whose y values are $\frac{1}{2}$.

Example 1 (Continued):

1f

Add $2n\pi$ to the values of x

$$x = \frac{\pi}{4} + 2n\pi \qquad \text{and} \qquad x = \frac{7\pi}{4} + 2n\pi$$

Example 2: Find all of the solutions for the equation $\tan x = \sqrt{3}$.

1e)

Solution:

Identify the quadrants for the solutions on the interval $[0, \pi)$

Note: On this problem we are using the interval $[0, \pi)$ instead of $[0, 2\pi)$ because tangent has a period of π.

Tangent is positive in quadrants I

Solve for the variable

$$x = \frac{\pi}{3}$$

Add $n\pi$ to the value of x

$$x = \frac{\pi}{3} + n\pi$$

Solving trigonometric equations with a multiple angle

The trigonometric equations to be solved will not always have just "x" as the angle. There will be times where you will have angles such as 3x or $\frac{x}{2}$. For equations like this, you will begin by solving the equation for all of the possible solutions by adding $2n\pi$ or $n\pi$ (depending on the trigonometric function involved) to values. You would then substitute values in for n starting at 0 and continuing until all of the values within the specified interval have been found.

2a)

Any multiple of 2π can be added to these values and the sine is still $\frac{1}{2}$. Thus, all solutions of $\sin x = \frac{1}{2}$ are given by

$$x = \frac{\pi}{6} + 2n\pi \quad \text{or} \quad x = \frac{5\pi}{6} + 2n\pi,$$

where n is any integer. By choosing any two integers, such as $n = 0$ and $n = 1$, we can find some solutions of $\sin x = \frac{1}{2}$. Thus, four of the solutions are determined as follows:

Let $n = 0$.

$$x = \frac{\pi}{6} + 2 \cdot 0\pi = \frac{\pi}{6} \qquad x = \frac{5\pi}{6} + 2 \cdot 0\pi = \frac{5\pi}{6}$$

Let $n = 1$.

$$x = \frac{\pi}{6} + 2 \cdot 1\pi = \frac{\pi}{6} + 2\pi = \frac{\pi}{6} + \frac{12\pi}{6} = \frac{13\pi}{6}$$

$$x = \frac{5\pi}{6} + 2 \cdot 1\pi \qquad x = \frac{5\pi}{6} + 2\pi = \frac{5\pi}{6} + \frac{12\pi}{6} = \frac{17\pi}{6}.$$

These four solutions are shown among the five solutions in **Figure 5.7**.

Equations Involving a Single Trigonometric Function

To solve an equation containing a single trigonometric function:

- Isolate the function on one side of the equation.
- Solve for the variable.

EXAMPLE 1 Finding All Solutions of a Trigonometric Equation

1d)

Solve the equation: $3 \sin x - 2 = 5 \sin x - 1$.

Solution The equation contains a single trigonometric function, $\sin x$.

Step 1 Isolate the function on one side of the equation. We can solve for $\sin x$ by collecting terms with $\sin x$ on the left side and constant terms on the right side.

$3 \sin x - 2 = 5 \sin x - 1$	This is the given equation.
$3 \sin x - 5 \sin x - 2 = 5 \sin x - 5 \sin x - 1$	Subtract 5 sin x from both sides.
$-2 \sin x - 2 = -1$	Simplify.
$-2 \sin x = 1$	Add 2 to both sides.
$\sin x = -\frac{1}{2}$	Divide both sides by -2 and solve for sin x.

Step 2 Solve for the variable. We must solve for x in $\sin x = -\frac{1}{2}$. Because $\sin\frac{\pi}{6} = \frac{1}{2}$, the solutions of $\sin x = -\frac{1}{2}$ in $[0, 2\pi)$ are

$$x = \pi + \frac{\pi}{6} = \frac{6\pi}{6} + \frac{\pi}{6} = \frac{7\pi}{6} \qquad x = 2\pi - \frac{\pi}{6} = \frac{12\pi}{6} - \frac{\pi}{6} = \frac{11\pi}{6}.$$

The sine is negative in quadrant III.

The sine is negative in quadrant IV.

Because the period of the sine function is 2π, the solutions of the equation are given by

$$x = \frac{7\pi}{6} + 2n\pi \quad \text{and} \quad x = \frac{11\pi}{6} + 2n\pi,$$

where n is any integer.

14.4 Solving Trigonometric Equations

What you should learn

GOAL 1 Solve a trigonometric equation.

GOAL 2 Solve **real-life** trigonometric equations, such as an equation for the number of hours of daylight in Prescott, Arizona, in **Example 6**.

Why you should learn it

▼ To solve many types of **real-life** problems, such as finding the position of the sun at sunrise in **Ex. 58**.

REAL LIFE

GOAL 1 Solving a Trigonometric Equation

In Lesson 14.3 you verified trigonometric identities. In this lesson you will solve trigonometric equations. To see the difference, consider the following equations:

$$\sin^2 x + \cos^2 x = 1 \qquad \textbf{Equation 1}$$

$$\sin x = 1 \qquad \textbf{Equation 2}$$

Equation 1 is an identity because it is true for all real values of x. Equation 2, however, is true only for some values of x. When you find these values, you are solving the equation.

EXAMPLE 1 *Solving a Trigonometric Equation*

Solve $2 \sin x - 1 = 0$.

Solution

1e)

First isolate $\sin x$ on one side of the equation.

$2 \sin x - 1 = 0$ **Write original equation.**

$2 \sin x = 1$ **Add 1 to each side.**

$\sin x = \frac{1}{2}$ **Divide each side by 2.**

One solution of $\sin x = \frac{1}{2}$ in the interval $0 \le x < 2\pi$ is $x = \sin^{-1} \frac{1}{2} = \frac{\pi}{6}$. Another such solution is $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

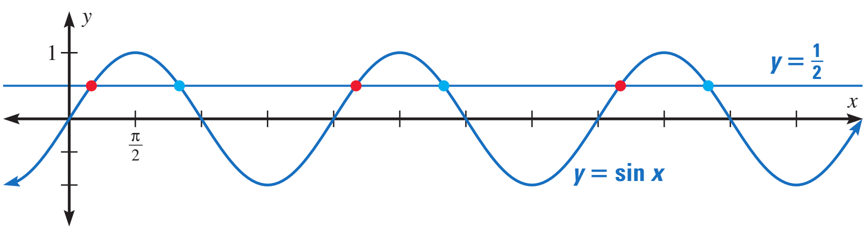
Moreover, because $y = \sin x$ is a periodic function, there are infinitely many other solutions. You can write the general solution as

$$x = \frac{\pi}{6} + 2n\pi \quad \text{or} \quad x = \frac{5\pi}{6} + 2n\pi$$

where n is any integer.

✓**CHECK** You can check your answer graphically. Graph $y = \sin x$ and $y = \frac{1}{2}$ in the same coordinate plane and find the points where the graphs intersect.

You can see that there are infinitely many such points.



Student Help

Look Back
For help with inverse trigonometric functions, see p. 792.

Trigonometric Equations

Just as we can have polynomial, rational, exponential, or logarithmic equation, for example, we can also have trigonometric equations that must be solved. A trigonometric equation is one that contains a trigonometric function with a variable. For example, sin x + 2 = 1 is an example of a trigonometric equation. The equations can be something as simple as this or more complex like $\sin^2 x - 2\cos x - 2 = 0$. The steps taken to solve the equation will depend on the form in which it is written and whether we are looking to find all of the solutions or just those within a specified interval such as $[0, 2\pi)$.

Solving for all solutions of a trigonometric equation

Back when we were solving for theta, θ, using the inverse trigonometric function we were limiting the interval for θ depending on the trigonometric function. For example, θ was limited to the interval of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for the inverse sine function. However, when we are solving a trigonometric equation for all of the solutions we will not limit the interval and must adjust the values to take into account the periodic nature of the trigonometric function. The functions sine, cosine, secant, and cosecant all have a period of 2π so we must add the term 2nπ to include all of the solutions. Tangent and cotangent have a period of π so for these two functions the term nπ would be added to obtain all of the solutions.

1L **Example 1:** Find all of the solutions for the equation $2\cos x = \sqrt{2}$.

Solution:

Isolate the function on one side of the equation

$$2\cos x = \sqrt{2}$$
$$\cos x = \frac{\sqrt{2}}{2}$$

Identify the quadrants for the solutions on the interval $[0, 2\pi)$

Cosine is positive in quadrants I and IV

Solve for the variable

$$x = \frac{\pi}{4} \text{ (quadrant I)} \qquad x = 2\pi - \frac{\pi}{4} = \frac{7\pi}{4} \text{ (quadrant IV)}$$

Example 1 (Continued):

1f Add $2n\pi$ to the values of x

$$x = \frac{\pi}{4} + 2n\pi \qquad \text{and} \qquad x = \frac{7\pi}{4} + 2n\pi$$

Example 2: Find all of the solutions for the equation $\tan x = \sqrt{3}$.

1e) Solution:

Identify the quadrants for the solutions on the interval $[0, \pi)$

Note: On this problem we are using the interval $[0, \pi)$ instead of $[0, 2\pi)$ because tangent has a period of π.

Tangent is positive in quadrants I

Solve for the variable

$$x = \frac{\pi}{3}$$

Add $n\pi$ to the value of x

$$x = \frac{\pi}{3} + n\pi$$

Solving trigonometric equations with a multiple angle

The trigonometric equations to be solved will not always have just "x" as the angle. There will be times where you will have angles such as 3x or $\frac{x}{2}$. For equations like this, you will begin by solving the equation for all of the possible solutions by adding $2n\pi$ or $n\pi$ (depending on the trigonometric function involved) to values. You would then substitute values in for n starting at 0 and continuing until all of the values within the specified interval have been found.

10.7 Trigonometric Equations and Inequalities

In Sections 10.2, 10.3 and most recently 10.6, we solved some basic equations involving the trigonometric functions. Below we summarize the techniques we've employed thus far. Note that we use the neutral letter 'u' as the argument[1] of each circular function for generality.

Strategies for Solving Basic Equations Involving Trigonometric Functions

- To solve $\cos(u) = c$ or $\sin(u) = c$ for $-1 \leq c \leq 1$, first solve for u in the interval $[0, 2\pi)$ and add integer multiples of the period 2π. If $c < -1$ or of $c > 1$, there are no real solutions.
- To solve $\sec(u) = c$ or $\csc(u) = c$ for $c \leq -1$ or $c \geq 1$, convert to cosine or sine, respectively, and solve as above. If $-1 < c < 1$, there are no real solutions.
- To solve $\tan(u) = c$ for any real number c, first solve for u in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and add integer multiples of the period π.
- To solve $\cot(u) = c$ for $c \neq 0$, convert to tangent and solve as above. If $c = 0$, the solution to $\cot(u) = 0$ is $u = \frac{\pi}{2} + \pi k$ for integers k.

Using the above guidelines, we can comfortably solve $\sin(x) = \frac{1}{2}$ and find the solution $x = \frac{\pi}{6} + 2\pi k$ or $x = \frac{5\pi}{6} + 2\pi k$ for integers k. How do we solve something like $\sin(3x) = \frac{1}{2}$? Since this equation has the *form* $\sin(u) = \frac{1}{2}$, we know the solutions take the form $u = \frac{\pi}{6} + 2\pi k$ or $u = \frac{5\pi}{6} + 2\pi k$ for integers k. Since the argument of sine here is $3x$, we have $3x = \frac{\pi}{6} + 2\pi k$ or $3x = \frac{5\pi}{6} + 2\pi k$ for integers k. To solve for x, we divide both sides[2] of these equations by 3, and obtain $x = \frac{\pi}{18} + \frac{2\pi}{3}k$ or $x = \frac{5\pi}{18} + \frac{2\pi}{3}k$ for integers k. This is the technique employed in the example below.

Example 10.7.1. Solve the following equations and check your answers analytically. List the solutions which lie in the interval $[0, 2\pi)$ and verify them using a graphing utility.

1. $\cos(2x) = -\frac{\sqrt{3}}{2}$
2. $\csc\left(\frac{1}{3}x - \pi\right) = \sqrt{2}$
3. $\cot(3x) = 0$
4. $\sec^2(x) = 4$
5. $\tan\left(\frac{x}{2}\right) = -3$
6. $\sin(2x) = 0.87$

Solution.

1. The solutions to $\cos(u) = -\frac{\sqrt{3}}{2}$ are $u = \frac{5\pi}{6} + 2\pi k$ or $u = \frac{7\pi}{6} + 2\pi k$ for integers k. Since the argument of cosine here is $2x$, this means $2x = \frac{5\pi}{6} + 2\pi k$ or $2x = \frac{7\pi}{6} + 2\pi k$ for integers k. Solving for x gives $x = \frac{5\pi}{12} + \pi k$ or $x = \frac{7\pi}{12} + \pi k$ for integers k. To check these answers analytically, we substitute them into the original equation. For any integer k we have

$$\begin{aligned} \cos\left(2\left[\frac{5\pi}{12} + \pi k\right]\right) &= \cos\left(\frac{5\pi}{6} + 2\pi k\right) \\ &= \cos\left(\frac{5\pi}{6}\right) \qquad \text{(the period of cosine is } 2\pi) \\ &= -\frac{\sqrt{3}}{2} \end{aligned}$$

[1] See the comments at the beginning of Section 10.5 for a review of this concept.
[2] Don't forget to divide the $2\pi k$ by 3 as well!

3j) Similarly, we find $\cos\left(2\left[\frac{7\pi}{12}+\pi k\right]\right) = \cos\left(\frac{7\pi}{6}+2\pi k\right) = \cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$. To determine which of our solutions lie in $[0, 2\pi)$, we substitute integer values for k. The solutions we keep come from the values of $k = 0$ and $k = 1$ and are $x = \frac{5\pi}{12}$, $\frac{7\pi}{12}$, $\frac{17\pi}{12}$ and $\frac{19\pi}{12}$. Using a calculator, we graph $y = \cos(2x)$ and $y = -\frac{\sqrt{3}}{2}$ over $[0, 2\pi)$ and examine where these two graphs intersect. We see that the x-coordinates of the intersection points correspond to the decimal representations of our exact answers.

1k)

2. Since this equation has the form $\csc(u) = \sqrt{2}$, we rewrite this as $\sin(u) = \frac{\sqrt{2}}{2}$ and find $u = \frac{\pi}{4} + 2\pi k$ or $u = \frac{3\pi}{4} + 2\pi k$ for integers k. Since the argument of cosecant here is $\left(\frac{1}{3}x - \pi\right)$,

$$\frac{1}{3}x - \pi = \frac{\pi}{4} + 2\pi k \quad \text{or} \quad \frac{1}{3}x - \pi = \frac{3\pi}{4} + 2\pi k$$

To solve $\frac{1}{3}x - \pi = \frac{\pi}{4} + 2\pi k$, we first add π to both sides

$$\frac{1}{3}x = \frac{\pi}{4} + 2\pi k + \pi$$

A common error is to treat the '$2\pi k$' and 'π' terms as 'like' terms and try to combine them when they are not.[3] We can, however, combine the 'π' and '$\frac{\pi}{4}$' terms to get

$$\frac{1}{3}x = \frac{5\pi}{4} + 2\pi k$$

We now finish by multiplying both sides by 3 to get

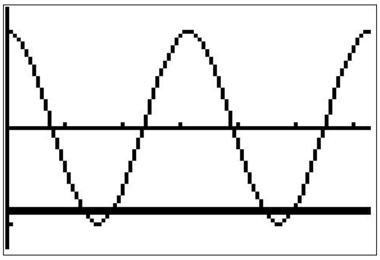
$$x = 3\left(\frac{5\pi}{4} + 2\pi k\right) = \frac{15\pi}{4} + 6\pi k$$

Solving the other equation, $\frac{1}{3}x - \pi = \frac{3\pi}{4} + 2\pi k$ produces $x = \frac{21\pi}{4} + 6\pi k$ for integers k. To check the first family of answers, we substitute, combine line terms, and simplify.

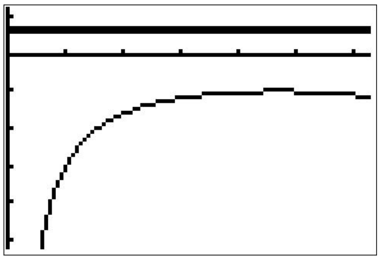
$$\begin{aligned} \csc\left(\frac{1}{3}\left[\frac{15\pi}{4} + 6\pi k\right] - \pi\right) &= \csc\left(\frac{5\pi}{4} + 2\pi k - \pi\right) \\ &= \csc\left(\frac{\pi}{4} + 2\pi k\right) \\ &= \csc\left(\frac{\pi}{4}\right) \qquad \text{(the period of cosecant is } 2\pi\text{)} \\ &= \sqrt{2} \end{aligned}$$

The family $x = \frac{21\pi}{4} + 6\pi k$ checks similarly. Despite having infinitely many solutions, we find that *none* of them lie in $[0, 2\pi)$. To verify this graphically, we use a reciprocal identity to rewrite the cosecant as a sine and we find that $y = \frac{1}{\sin\left(\frac{1}{3}x - \pi\right)}$ and $y = \sqrt{2}$ do not intersect at all over the interval $[0, 2\pi)$.

[3] Do you see why?



$y = \cos(2x)$ and $\boldsymbol{y = -\frac{\sqrt{3}}{2}}$



$y = \frac{1}{\sin\left(\frac{1}{3}x - \pi\right)}$ and $\boldsymbol{y = \sqrt{2}}$

4i)

3. Since $\cot(3x) = 0$ has the form $\cot(u) = 0$, we know $u = \frac{\pi}{2} + \pi k$, so, in this case, $3x = \frac{\pi}{2} + \pi k$ for integers k. Solving for x yields $x = \frac{\pi}{6} + \frac{\pi}{3}k$. Checking our answers, we get

$$\begin{aligned} \cot\left(3\left[\tfrac{\pi}{6} + \tfrac{\pi}{3}k\right]\right) &= \cot\left(\tfrac{\pi}{2} + \pi k\right) \\ &= \cot\left(\tfrac{\pi}{2}\right) \qquad \text{(the period of cotangent is } \pi\text{)} \\ &= 0 \end{aligned}$$

As k runs through the integers, we obtain six answers, corresponding to $k = 0$ through $k = 5$, which lie in $[0, 2\pi)$: $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}$ and $\frac{11\pi}{6}$. To confirm these graphically, we must be careful. On many calculators, there is no function button for cotangent. We choose[4] to use the quotient identity $\cot(3x) = \frac{\cos(3x)}{\sin(3x)}$. Graphing $y = \frac{\cos(3x)}{\sin(3x)}$ and $y = 0$ (the x-axis), we see that the x-coordinates of the intersection points approximately match our solutions.

1g)

4. The complication in solving an equation like $\sec^2(x) = 4$ comes not from the argument of secant, which is just x, but rather, the fact the secant is being squared. To get this equation to look like one of the forms listed on page 857, we extract square roots to get $\sec(x) = \pm 2$. Converting to cosines, we have $\cos(x) = \pm\frac{1}{2}$. For $\cos(x) = \frac{1}{2}$, we get $x = \frac{\pi}{3} + 2\pi k$ or $x = \frac{5\pi}{3} + 2\pi k$ for integers k. For $\cos(x) = -\frac{1}{2}$, we get $x = \frac{2\pi}{3} + 2\pi k$ or $x = \frac{4\pi}{3} + 2\pi k$ for integers k. If we take a step back and think of these families of solutions geometrically, we see we are finding the measures of all angles with a reference angle of $\frac{\pi}{3}$. As a result, these solutions can be combined and we may write our solutions as $x = \frac{\pi}{3} + \pi k$ and $x = \frac{2\pi}{3} + \pi k$ for integers k. To check the first family of solutions, we note that, depending on the integer k, $\sec\left(\frac{\pi}{3} + \pi k\right)$ doesn't always equal $\sec\left(\frac{\pi}{3}\right)$. However, it is true that for all integers k, $\sec\left(\frac{\pi}{3} + \pi k\right) = \pm\sec\left(\frac{\pi}{3}\right) = \pm 2$. (Can you show this?) As a result,

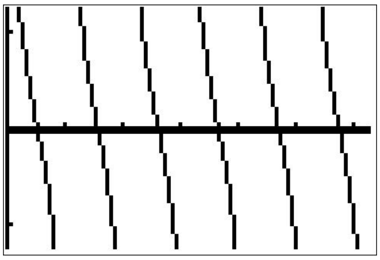
$$\begin{aligned} \sec^2\left(\tfrac{\pi}{3} + \pi k\right) &= \left(\pm\sec\left(\tfrac{\pi}{3}\right)\right)^2 \\ &= (\pm 2)^2 \\ &= 4 \end{aligned}$$

The same holds for the family $x = \frac{2\pi}{3} + \pi k$. The solutions which lie in $[0, 2\pi)$ come from the values $k = 0$ and $k = 1$, namely $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$ and $\frac{5\pi}{3}$. To confirm graphically, we use

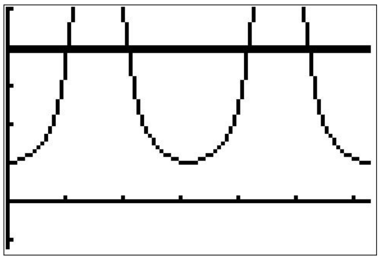
[4]The reader is encouraged to see what happens if we had chosen the reciprocal identity $\cot(3x) = \frac{1}{\tan(3x)}$ instead. The graph on the calculator *appears* identical, but what happens when you try to find the intersection points?

a reciprocal identity to rewrite the secant as cosine. The x-coordinates of the intersection points of $y = \frac{1}{(\cos(x))^2}$ and $y = 4$ verify our answers.

11)



$y = \frac{\cos(3x)}{\sin(3x)}$ and $\boldsymbol{y = 0}$



$y = \frac{1}{\cos^2(x)}$ and $\boldsymbol{y = 4}$

5. The equation $\tan\left(\frac{x}{2}\right) = -3$ has the form $\tan(u) = -3$, whose solution is $u = \arctan(-3) + \pi k$. Hence, $\frac{x}{2} = \arctan(-3) + \pi k$, so $x = 2\arctan(-3) + 2\pi k$ for integers k. To check, we note

12)

$$\begin{aligned} \tan\left(\frac{2\arctan(-3)+2\pi k}{2}\right) &= \tan\left(\arctan(-3) + \pi k\right) \\ &= \tan\left(\arctan(-3)\right) && \text{(the period of tangent is } \pi\text{)} \\ &= -3 && \text{(See Theorem 10.27)} \end{aligned}$$

To determine which of our answers lie in the interval $[0, 2\pi)$, we first need to get an idea of the value of $2\arctan(-3)$. While we could easily find an approximation using a calculator,[5] we proceed analytically. Since $-3 < 0$, it follows that $-\frac{\pi}{2} < \arctan(-3) < 0$. Multiplying through by 2 gives $-\pi < 2\arctan(-3) < 0$. We are now in a position to argue which of the solutions $x = 2\arctan(-3) + 2\pi k$ lie in $[0, 2\pi)$. For $k = 0$, we get $x = 2\arctan(-3) < 0$, so we discard this answer and all answers $x = 2\arctan(-3) + 2\pi k$ where $k < 0$. Next, we turn our attention to $k = 1$ and get $x = 2\arctan(-3) + 2\pi$. Starting with the inequality $-\pi < 2\arctan(-3) < 0$, we add 2π and get $\pi < 2\arctan(-3) + 2\pi < 2\pi$. This means $x = 2\arctan(-3) + 2\pi$ lies in $[0, 2\pi)$. Advancing k to 2 produces $x = 2\arctan(-3) + 4\pi$. Once again, we get from $-\pi < 2\arctan(-3) < 0$ that $3\pi < 2\arctan(-3) + 4\pi < 4\pi$. Since this is outside the interval $[0, 2\pi)$, we discard $x = 2\arctan(-3) + 4\pi$ and all solutions of the form $x = 2\arctan(-3) + 2\pi k$ for $k > 2$. Graphically, we see $y = \tan\left(\frac{x}{2}\right)$ and $y = -3$ intersect only once on $[0, 2\pi)$ at $x = 2\arctan(-3) + 2\pi \approx 3.7851$.

6. To solve $\sin(2x) = 0.87$, we first note that it has the form $\sin(u) = 0.87$, which has the family of solutions $u = \arcsin(0.87) + 2\pi k$ or $u = \pi - \arcsin(0.87) + 2\pi k$ for integers k. Since the argument of sine here is $2x$, we get $2x = \arcsin(0.87) + 2\pi k$ or $2x = \pi - \arcsin(0.87) + 2\pi k$ which gives $x = \frac{1}{2}\arcsin(0.87) + \pi k$ or $x = \frac{\pi}{2} - \frac{1}{2}\arcsin(0.87) + \pi k$ for integers k. To check,

[5] Your instructor will let you know if you should abandon the analytic route at this point and use your calculator. But seriously, what fun would that be?

Try It

7.29 Find all solutions for $\tan x = \sqrt{3}$.

Example 7.51

Identify all Solutions to the Equation Involving Tangent

Identify all exact solutions to the equation $2(\tan x + 3) = 5 + \tan x,\ 0 \le x < 2\pi$.

Solution

We can solve this equation using only algebra. Isolate the expression $\tan x$ on the left side of the equals sign.

$$\begin{aligned} 2(\tan x) + 2(3) &= 5 + \tan x \\ 2\tan x + 6 &= 5 + \tan x \\ 2\tan x - \tan x &= 5 - 6 \\ \tan x &= -1 \end{aligned}$$

There are two angles on the unit circle that have a tangent value of $-1: x = \frac{3\pi}{4}$ and $x = \frac{7\pi}{4}$.

Example 7.52

1e)

Solve $2\cos(4\theta) + 1 = 0$ EXACTLY

a) give all solutions

b) Give all solutions in $[0,\quad 2\pi)$

Isolate the trig function

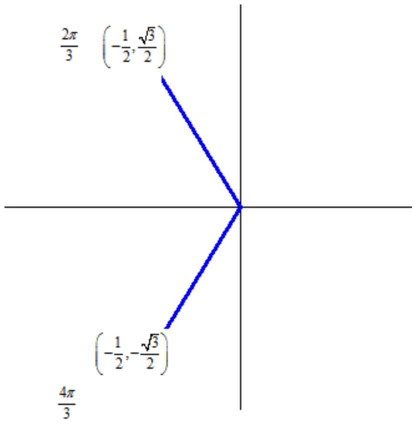
Get $\cos(4\theta)$ by itself

$$\begin{aligned} 2\cos(4\theta) &= -1 \\ \cos(4\theta) &= -\frac{1}{2} \end{aligned}$$

Ask "What quadrant is the angle in"?

Cosine is negative in QII and QIII

Now find the angles on the unit circle where x is -1/2:



Find angles in one cycle where your trig function has the given value

Cosine is $-\frac{1}{2}$ at the angle $\frac{2\pi}{3}$ and at $\frac{4\pi}{3}$

Set our angle to each of these + k*period

$$4\theta = \frac{2\pi}{3} + 2k\pi \qquad or \qquad 4\theta = \frac{4\pi}{3} + 2k\pi$$

Solve for θ

$$\frac{4\theta}{4} = \frac{\frac{2\pi}{3}}{4} + \frac{2k\pi}{4} \qquad or \qquad \frac{4\theta}{4} = \frac{\frac{4\pi}{3}}{4} + \frac{2k\pi}{4}$$

$$\theta = \frac{\pi}{6} + \frac{k\pi}{2} \qquad or \qquad \theta = \frac{\pi}{3} + \frac{k\pi}{2}$$

a) General Solution

$$\theta = \frac{\pi}{6} + \frac{k\pi}{2} \qquad or \qquad \theta = \frac{\pi}{3} + \frac{k\pi}{2}$$

or in set notation

$$\left\{\frac{\pi}{6} + \frac{k\pi}{2}, \quad \frac{\pi}{3} + \frac{k\pi}{2}\right\}$$

b) Solution in $[0, \quad 2\pi)$

To find the solution in a specific interval, we make a table and keep only the values in that interval.

EXAMPLE 2 *Solving a Trigonometric Equation in an Interval*

Solve $4\tan^2 x - 1 = 0$ in the interval $0 \le x < 2\pi$.

SOLUTION

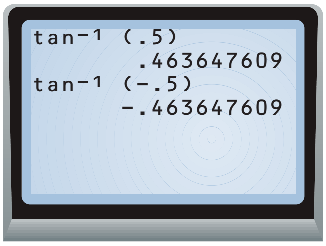
$4\tan^2 x - 1 = 0$ — Write original equation.

$4\tan^2 x = 1$ — Add 1 to each side.

$\tan^2 x = \frac{1}{4}$ — Divide each side by 4.

$\tan x = \pm\frac{1}{2}$ — Take square roots of each side.

Use a calculator to find values of x for which $\tan x = \pm\frac{1}{2}$, as shown at the right.



> **STUDENT HELP**
> **Study Tip**
> Note that to find the general solution of a trigonometric equation, you must add multiples of the period to the solutions in one cycle.

The general solution of the equation is

$$x \approx 0.464 + n\pi$$

or

$$x \approx -0.464 + n\pi$$

where n is any integer. The solutions that are in the interval $0 \le x < 2\pi$ are:

$x \approx 0.464$ $\qquad$ $x \approx 0.464 + \pi \approx 3.61$

$x \approx -0.464 + \pi \approx 2.628$ $\qquad$ $x \approx -0.464 + 2\pi \approx 5.82$

✓**CHECK** Check these solutions by substituting them back into the original equation.

EXAMPLE 3 *Factoring to Solve a Trigonometric Equation*

2a)

Solve $\sin^2 x \cos x = 4\cos x$.

SOLUTION

$\sin^2 x \cos x = 4\cos x$ — Write original equation.

$\sin^2 x \cos x - 4\cos x = 0$ — Subtract $4\cos x$ from each side.

$\cos x\,(\sin^2 x - 4) = 0$ — Factor out $\cos x$.

$\cos x\,(\sin x + 2)(\sin x - 2) = 0$ — Factor difference of squares.

Set each factor equal to 0 and solve for x, if possible.

$\cos x = 0$ $\qquad$ $\sin x + 2 = 0$ $\qquad$ $\sin x - 2 = 0$

$x = \frac{\pi}{2}$ or $x = \frac{3\pi}{2}$ $\qquad$ $\sin x = -2$ $\qquad$ $\sin x = 2$

> **STUDENT HELP**
> **Study Tip**
> Remember not to divide both sides of an equation by a variable expression, such as $\cos x$.

Because neither $\sin x = -2$ nor $\sin x = 2$ has a solution, the only solutions in the interval $0 \le x < 2\pi$ are $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.

▶ The general solution is $x = \frac{\pi}{2} + 2n\pi$ or $x = \frac{3\pi}{2} + 2n\pi$ where n is any integer.

Each of the problems in Example 10.7.1 featured one trigonometric function. If an equation involves two different trigonometric functions or if the equation contains the same trigonometric function but with different arguments, we will need to use identities and Algebra to reduce the equation to the same form as those given on page 857.

Example 10.7.2. Solve the following equations and list the solutions which lie in the interval $[0, 2\pi)$. Verify your solutions on $[0, 2\pi)$ graphically.

1. $3\sin^3(x) = \sin^2(x)$
2. $\sec^2(x) = \tan(x) + 3$
3. $\cos(2x) = 3\cos(x) - 2$
4. $\cos(3x) = 2 - \cos(x)$
5. $\cos(3x) = \cos(5x)$
6. $\sin(2x) = \sqrt{3}\cos(x)$
7. $\sin(x)\cos\left(\frac{x}{2}\right) + \cos(x)\sin\left(\frac{x}{2}\right) = 1$
8. $\cos(x) - \sqrt{3}\sin(x) = 2$

Solution.

23)

1. We resist the temptation to divide both sides of $3\sin^3(x) = \sin^2(x)$ by $\sin^2(x)$ (What goes wrong if you do?) and instead gather all of the terms to one side of the equation and factor.

$$\begin{aligned} 3\sin^3(x) &= \sin^2(x) \\ 3\sin^3(x) - \sin^2(x) &= 0 \\ \sin^2(x)(3\sin(x) - 1) &= 0 \qquad \text{Factor out } \sin^2(x) \text{ from both terms.} \end{aligned}$$

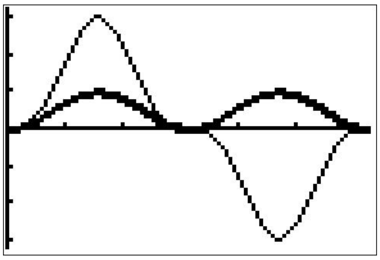
We get $\sin^2(x) = 0$ or $3\sin(x) - 1 = 0$. Solving for $\sin(x)$, we find $\sin(x) = 0$ or $\sin(x) = \frac{1}{3}$. The solution to the first equation is $x = \pi k$, with $x = 0$ and $x = \pi$ being the two solutions which lie in $[0, 2\pi)$. To solve $\sin(x) = \frac{1}{3}$, we use the arcsine function to get $x = \arcsin\left(\frac{1}{3}\right) + 2\pi k$ or $x = \pi - \arcsin\left(\frac{1}{3}\right) + 2\pi k$ for integers k. We find the two solutions here which lie in $[0, 2\pi)$ to be $x = \arcsin\left(\frac{1}{3}\right)$ and $x = \pi - \arcsin\left(\frac{1}{3}\right)$. To check graphically, we plot $y = 3(\sin(x))^3$ and $y = (\sin(x))^2$ and find the x-coordinates of the intersection points of these two curves. Some extra zooming is required near $x = 0$ and $x = \pi$ to verify that these two curves do in fact intersect four times.[6]

2. Analysis of $\sec^2(x) = \tan(x) + 3$ reveals two different trigonometric functions, so an identity is in order. Since $\sec^2(x) = 1 + \tan^2(x)$, we get

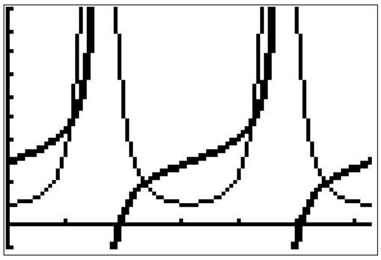
$$\begin{aligned} \sec^2(x) &= \tan(x) + 3 \\ 1 + \tan^2(x) &= \tan(x) + 3 \quad (\text{Since } \sec^2(x) = 1 + \tan^2(x).) \\ \tan^2(x) - \tan(x) - 2 &= 0 \\ u^2 - u - 2 &= 0 \qquad \text{Let } u = \tan(x). \\ (u + 1)(u - 2) &= 0 \end{aligned}$$

[6] Note that we are *not* counting the point $(2\pi, 0)$ in our solution set since $x = 2\pi$ is not in the interval $[0, 2\pi)$. In the forthcoming solutions, remember that while $x = 2\pi$ may be a solution to the equation, it isn't counted among the solutions in $[0, 2\pi)$.

This gives $u = -1$ or $u = 2$. Since $u = \tan(x)$, we have $\tan(x) = -1$ or $\tan(x) = 2$. From $\tan(x) = -1$, we get $x = -\frac{\pi}{4} + \pi k$ for integers k. To solve $\tan(x) = 2$, we employ the arctangent function and get $x = \arctan(2) + \pi k$ for integers k. From the first set of solutions, we get $x = \frac{3\pi}{4}$ and $x = \frac{5\pi}{4}$ as our answers which lie in $[0, 2\pi)$. Using the same sort of argument we saw in Example 10.7.1, we get $x = \arctan(2)$ and $x = \pi + \arctan(2)$ as answers from our second set of solutions which lie in $[0, 2\pi)$. Using a reciprocal identity, we rewrite the secant as a cosine and graph $y = \frac{1}{(\cos(x))^2}$ and $y = \tan(x) + 3$ to find the x-values of the points where they intersect.



$y = 3(\sin(x))^3$ and $\boldsymbol{y = (\sin(x))^2}$



$y = \frac{1}{(\cos(x))^2}$ and $\boldsymbol{y = \tan(x) + 3}$

3. In the equation $\cos(2x) = 3\cos(x) - 2$, we have the same circular function, namely cosine, on both sides but the arguments differ. Using the identity $\cos(2x) = 2\cos^2(x) - 1$, we obtain a 'quadratic in disguise' and proceed as we have done in the past.

2c)

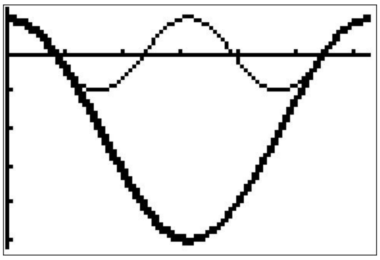
$$\begin{aligned} \cos(2x) &= 3\cos(x) - 2 \\ 2\cos^2(x) - 1 &= 3\cos(x) - 2 \quad \text{(Since } \cos(2x) = 2\cos^2(x) - 1.) \\ 2\cos^2(x) - 3\cos(x) + 1 &= 0 \\ 2u^2 - 3u + 1 &= 0 \qquad \text{Let } u = \cos(x). \\ (2u - 1)(u - 1) &= 0 \end{aligned}$$

This gives $u = \frac{1}{2}$ or $u = 1$. Since $u = \cos(x)$, we get $\cos(x) = \frac{1}{2}$ or $\cos(x) = 1$. Solving $\cos(x) = \frac{1}{2}$, we get $x = \frac{\pi}{3} + 2\pi k$ or $x = \frac{5\pi}{3} + 2\pi k$ for integers k. From $\cos(x) = 1$, we get $x = 2\pi k$ for integers k. The answers which lie in $[0, 2\pi)$ are $x = 0$, $\frac{\pi}{3}$, and $\frac{5\pi}{3}$. Graphing $y = \cos(2x)$ and $y = 3\cos(x) - 2$, we find, after a little extra effort, that the curves intersect in three places on $[0, 2\pi)$, and the x-coordinates of these points confirm our results.

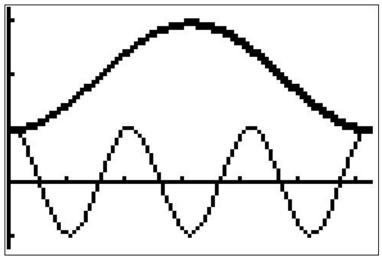
4. To solve $\cos(3x) = 2 - \cos(x)$, we use the same technique as in the previous problem. From Example 10.4.3, number 4, we know that $\cos(3x) = 4\cos^3(x) - 3\cos(x)$. This transforms the equation into a polynomial in terms of $\cos(x)$.

$$\begin{aligned} \cos(3x) &= 2 - \cos(x) \\ 4\cos^3(x) - 3\cos(x) &= 2 - \cos(x) \\ 2\cos^3(x) - 2\cos(x) - 2 &= 0 \\ 4u^3 - 2u - 2 &= 0 \qquad \text{Let } u = \cos(x). \end{aligned}$$

To solve $4u^3 - 2u - 2 = 0$, we need the techniques in Chapter 3 to factor $4u^3 - 2u - 2$ into $(u-1)\left(4u^2 + 4u + 2\right)$. We get either $u - 1 = 0$ or $4u^2 + 2u + 2 = 0$, and since the discriminant of the latter is negative, the only real solution to $4u^3 - 2u - 2 = 0$ is $u = 1$. Since $u = \cos(x)$, we get $\cos(x) = 1$, so $x = 2\pi k$ for integers k. The only solution which lies in $[0, 2\pi)$ is $x = 0$. Graphing $y = \cos(3x)$ and $y = 2 - \cos(x)$ on the same set of axes over $[0, 2\pi)$ shows that the graphs intersect at what appears to be $(0, 1)$, as required.



$y = \cos(2x)$ and $\boldsymbol{y = 3\cos(x) - 2}$



$y = \cos(3x)$ and $\boldsymbol{y = 2 - \cos(x)}$

5. While we could approach $\cos(3x) = \cos(5x)$ in the same manner as we did the previous two problems, we choose instead to showcase the utility of the Sum to Product Identities. From $\cos(3x) = \cos(5x)$, we get $\cos(5x) - \cos(3x) = 0$, and it is the presence of 0 on the right hand side that indicates a switch to a product would be a good move.[7] Using Theorem 10.21, we have that $\cos(5x) - \cos(3x) = -2\sin\left(\frac{5x+3x}{2}\right)\sin\left(\frac{5x-3x}{2}\right) = -2\sin(4x)\sin(x)$. Hence, the equation $\cos(5x) = \cos(3x)$ is equivalent to $-2\sin(4x)\sin(x) = 0$. From this, we get $\sin(4x) = 0$ or $\sin(x) = 0$. Solving $\sin(4x) = 0$ gives $x = \frac{\pi}{4}k$ for integers k, and the solution to $\sin(x) = 0$ is $x = \pi k$ for integers k. The second set of solutions is contained in the first set of solutions,[8] so our final solution to $\cos(5x) = \cos(3x)$ is $x = \frac{\pi}{4}k$ for integers k. There are eight of these answers which lie in $[0, 2\pi)$: $x = 0$, $\frac{\pi}{4}$, $\frac{\pi}{2}$, $\frac{3\pi}{4}$, π, $\frac{5\pi}{4}$, $\frac{3\pi}{2}$ and $\frac{7\pi}{4}$. Our plot of the graphs of $y = \cos(3x)$ and $y = \cos(5x)$ below (after some careful zooming) bears this out.

6. In examining the equation $\sin(2x) = \sqrt{3}\cos(x)$, not only do we have different circular functions involved, namely sine and cosine, we also have different arguments to contend with, namely $2x$ and x. Using the identity $\sin(2x) = 2\sin(x)\cos(x)$ makes all of the arguments the same and we proceed as we would solving any nonlinear equation – gather all of the nonzero terms on one side of the equation and factor.

$$\begin{array}{rcll}
\sin(2x) & = & \sqrt{3}\cos(x) & \\
2\sin(x)\cos(x) & = & \sqrt{3}\cos(x) & (\text{Since } \sin(2x) = 2\sin(x)\cos(x).) \\
2\sin(x)\cos(x) - \sqrt{3}\cos(x) & = & 0 & \\
\cos(x)(2\sin(x) - \sqrt{3}) & = & 0 &
\end{array}$$

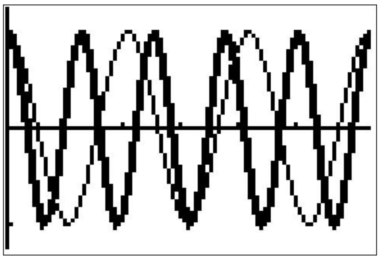
from which we get $\cos(x) = 0$ or $\sin(x) = \frac{\sqrt{3}}{2}$. From $\cos(x) = 0$, we obtain $x = \frac{\pi}{2} + \pi k$ for integers k. From $\sin(x) = \frac{\sqrt{3}}{2}$, we get $x = \frac{\pi}{3} + 2\pi k$ or $x = \frac{2\pi}{3} + 2\pi k$ for integers k. The answers

[7] As always, experience is the greatest teacher here!

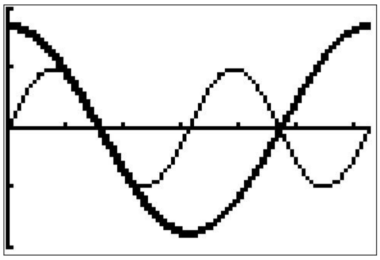
[8] As always, when in doubt, write it out!

2d)

which lie in $[0, 2\pi)$ are $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{3}$ and $\frac{2\pi}{3}$. We graph $y = \sin(2x)$ and $y = \sqrt{3}\cos(x)$ and, after some careful zooming, verify our answers.



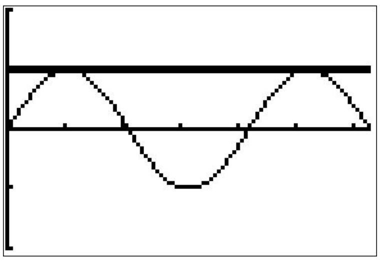
$y = \cos(3x)$ and $\boldsymbol{y = \cos(5x)}$



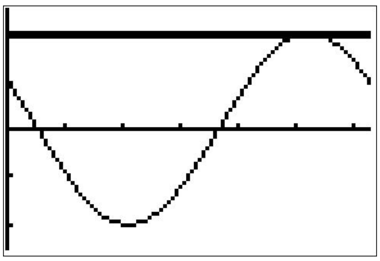
$y = \sin(2x)$ and $\boldsymbol{y = \sqrt{3}\cos(x)}$

7. Unlike the previous problem, there seems to be no quick way to get the circular functions or their arguments to match in the equation $\sin(x)\cos\left(\frac{x}{2}\right) + \cos(x)\sin\left(\frac{x}{2}\right) = 1$. If we stare at it long enough, however, we realize that the left hand side is the expanded form of the sum formula for $\sin\left(x + \frac{x}{2}\right)$. Hence, our original equation is equivalent to $\sin\left(\frac{3}{2}x\right) = 1$. Solving, we find $x = \frac{\pi}{3} + \frac{4\pi}{3}k$ for integers k. Two of these solutions lie in $[0, 2\pi)$: $x = \frac{\pi}{3}$ and $x = \frac{5\pi}{3}$. Graphing $y = \sin(x)\cos\left(\frac{x}{2}\right) + \cos(x)\sin\left(\frac{x}{2}\right)$ and $y = 1$ validates our solutions.

8. With the absence of double angles or squares, there doesn't seem to be much we can do. However, since the arguments of the cosine and sine are the same, we can rewrite the left hand side of this equation as a sinusoid.[9] To fit $f(x) = \cos(x) - \sqrt{3}\sin(x)$ to the form $A\sin(\omega t + \phi) + B$, we use what we learned in Example 10.5.3 and find $A = 2$, $B = 0$, $\omega = 1$ and $\phi = \frac{5\pi}{6}$. Hence, we can rewrite the equation $\cos(x) - \sqrt{3}\sin(x) = 2$ as $2\sin\left(x + \frac{5\pi}{6}\right) = 2$, or $\sin\left(x + \frac{5\pi}{6}\right) = 1$. Solving the latter, we get $x = -\frac{\pi}{3} + 2\pi k$ for integers k. Only one of these solutions, $x = \frac{5\pi}{3}$, which corresponds to $k = 1$, lies in $[0, 2\pi)$. Geometrically, we see that $y = \cos(x) - \sqrt{3}\sin(x)$ and $y = 2$ intersect just once, supporting our answer.



$y = \sin(x)\cos\left(\frac{x}{2}\right) + \cos(x)\sin\left(\frac{x}{2}\right)$ and $\boldsymbol{y = 1}$



$y = \cos(x) - \sqrt{3}\sin(x)$ and $\boldsymbol{y = 2}$

□

We repeat here the advice given when solving systems of nonlinear equations in section 8.7 – when it comes to solving equations involving the trigonometric functions, it helps to just try something.

[9] We are essentially 'undoing' the sum / difference formula for cosine or sine, depending on which form we use, so this problem is actually closely related to the previous one!

Trigonometric equations in quadratic form

The trigonometric equations can also be written in the quadratic form of $au^2 + bu + c = 0$ where u is a trigonometric function. The methods that can be used to solve these equations are the same as those used when solving quadratic equations – factoring, square root property, completing the square, and the quadratic formula. The method that you use will depend on the values of a, b, and c. If the equation can be factored then this would be your first option.

2e) **Example 5:** Solve the following trigonometric equation in quadratic form on the interval $[0, 2\pi)$.

$$\cos^2 x - 2\cos x = 3$$

Solution:

Group all terms on the left side so that it is equal to 0

$$\cos^2 x - 2\cos x - 3 = 0$$

Let u represent the trigonometric function cos x

$$u = \cos x$$

$$(\cos x)^2 - 2\cos x - 3 = 0$$
$$u^2 - 2u - 3 = 0$$

Factor the quadratic equation

$$(u + 1)(u - 3) = 0$$

Solve for u

$$u + 1 = 0 \quad \text{or} \quad u - 3 = 0$$
$$u = -1 \quad \text{or} \quad u = 3$$

Substitute the cosine function back in for u

$$\cos x = -1 \quad \text{or} \quad \cos x = 3$$

cos x cannot be greater than 1 so cos x = 3 has no solutions

Solve for x

$$x = \pi$$

Example 6: Solve the following trigonometric equation in quadratic form on the interval $[0, 2\pi)$.

2f)

$$\tan^2 x - 2 = 3 \tan x$$

Solution:

Group all terms on the left side so that it is equal to 0

$$\tan^2 x - 3 \tan x - 2 = 0$$

Let u represent the trigonometric function tan x

$$u = \tan x$$

$$(\tan x)^2 - 3 \tan x - 2 = 0$$
$$u^2 - 3u - 2 = 0$$

Factor the quadratic equation

The equation cannot be factored so the quadratic formula must be used

Solve for u

$a = 1$, $b = -3$, and $c = -2$

$$u = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-2)}}{2(1)}$$

$$u = \frac{3 \pm \sqrt{9+8}}{2}$$

$$u = \frac{3 \pm \sqrt{17}}{2}$$

$$u = \frac{3 - \sqrt{17}}{2} \quad \text{or} \quad u = \frac{3 + \sqrt{17}}{2}$$

Substitute the tangent function back in for u

$$\tan x = \frac{3 - \sqrt{17}}{2} \quad \text{or} \quad \tan x = \frac{3 + \sqrt{17}}{2}$$

$$\tan x \approx -0.5616 \quad \text{or} \quad \tan x \approx 3.5616$$

Example 6 (Continued):

Solve for the reference angle θ

$\theta \approx \tan^{-1}(0.5616)$ or $\theta \approx \tan^{-1}(3.5616)$
$\theta \approx 0.5117$ or $\theta \approx 1.2971$

Solve for the values of x within the interval $[0, 2\pi)$

$\tan x \approx -0.5616$

tan x is negative in quadrants II and IV

$x \approx \pi - 0.5117$ or $x \approx 2\pi - 0.5117$
$x \approx 2.6299$ or $x \approx 5.7715$

$\tan x \approx 3.5616$

tan x is positive in quadrants I and III

$x \approx 1.2971$ or $x \approx \pi + 1.2971$
$x \approx 4.4387$

The solutions to the equation (rounded to four decimal places) are 1.2971, 2.6299, 4.4387, and 5.7715.

Using identities to solve trigonometric equations

There could also be equations where two or more trigonometric functions are contained within the equation. If the functions can be separated by factoring the equation then you can solve the equation using the factoring method. However, if it is not possible to factor the equation then you must use the different trigonometric identities to rewrite the function in a single trigonometric function or in a form that can be solved by factoring.

Example 7: Use trigonometric identities to solve the following equation on the interval $[0, 2\pi)$.

2j)

$$2\sin^2 x + \cos x = 1$$

Solution:

Use the Pythagorean identity $\sin^2 x = 1 - \cos^2 x$ to replace $\sin^2 x$ in the equation

$2\sin^2 x + \cos x = 1$
$2(1 - \cos^2 x) + \cos x = 1$
$2 - 2\cos^2 x + \cos x = 1$

Example 7 (Continued):

Group all terms on the left side so that it is equal to 0

$2 - 2\cos^2 x + \cos x - 1 = 0$
$-2\cos^2 x + \cos x + 1 = 0$

Multiply the equation by -1 to make the leading coefficient positive

$-1(-2\cos^2 x + \cos x + 1 = 0)$
$2\cos^2 x - \cos x - 1 = 0$

Let u represent the trigonometric function cos x

$u = \cos x$

$2(\cos x)^2 - \cos x - 1 = 0$
$2u^2 - u - 1 = 0$

Factor the quadratic equation

$(2u + 1)(u - 1) = 0$

Solve for u

$2u + 1 = 0$	or	$u - 1 = 0$
$2u = -1$	or	$u = 1$
$u = -\frac{1}{2}$		

Substitute the cosine function back in for u

$\cos x = -\frac{1}{2}$ or $\cos x = 1$

Identify the quadrants for the solutions on the interval $[0, 2\pi)$

Cosine is negative in quadrants II and III and is 1 at 0

Example 7 (Continued):

Solve for x

2g)

$\cos x = -\frac{1}{2}$

Cosine is equal to $\frac{1}{2}$ at $\frac{\pi}{3}$ so the angles in quadrants II and III are

$\pi - \frac{\pi}{3} = \frac{2\pi}{3}$ (quadrant II) and $\pi + \frac{\pi}{3} = \frac{4\pi}{3}$ (quadrant III)

$x = \frac{2\pi}{3}$ and $x = \frac{4\pi}{3}$

$\cos x = 1$

$x = 0$

Add $2n\pi$ to the angle and solve for x

$x = \frac{2\pi}{3} + 2n\pi$ $\quad x = \frac{4\pi}{3} + 2n\pi$ $\quad x = 0 + 2n\pi$

Now substitute values in for n starting with 0 until the angle is outside of the interval $[0, 2\pi)$

$n = 0$

$x = \frac{2\pi}{3} + 2(0)\pi$ $\quad x = \frac{4\pi}{3} + 2(0)\pi$ $\quad x = 0 + 2(0)\pi$

$x = \frac{2\pi}{3}$ $\quad x = \frac{4\pi}{3}$ $\quad x = 0$

$n = 1$

$x = \frac{2\pi}{3} + 2(1)\pi$ $\quad x = \frac{4\pi}{3} + 2(1)\pi$ $\quad x = 0 + 2(1)\pi$

When $n = 1$ we will be adding 2π to the angles which will put them outside of the interval $[0, 2\pi)$.

So the solutions for the equation are 0, $\frac{2\pi}{3}$, and $\frac{4\pi}{3}$.

Try It **7.26** Solve exactly the following linear equation on the interval $[0,\ 2\pi)$: $2\sin x + 1 = 0.$

Solving Equations Involving a Single Trigonometric Function

When we are given equations that involve only one of the six trigonometric functions, their solutions involve using algebraic techniques and the unit circle (see **m49395 (https://legacy.cnx.org/content/m49395/latest/#Figure_07_02_008)**). We need to make several considerations when the equation involves trigonometric functions other than sine and cosine. Problems involving the reciprocals of the primary trigonometric functions need to be viewed from an algebraic perspective. In other words, we will write the reciprocal function, and solve for the angles using the function. Also, an equation involving the tangent function is slightly different from one containing a sine or cosine function. First, as we know, the period of tangent is π, not 2π. Further, the domain of tangent is all real numbers with the exception of odd integer multiples of $\frac{\pi}{2}$, unless, of course, a problem places its own restrictions on the domain.

Example 7.48 Solving a Problem Involving a Single Trigonometric Function

2h)

Solve the problem exactly: $2\sin^2\theta - 1 = 0,\ 0 \le \theta < 2\pi.$

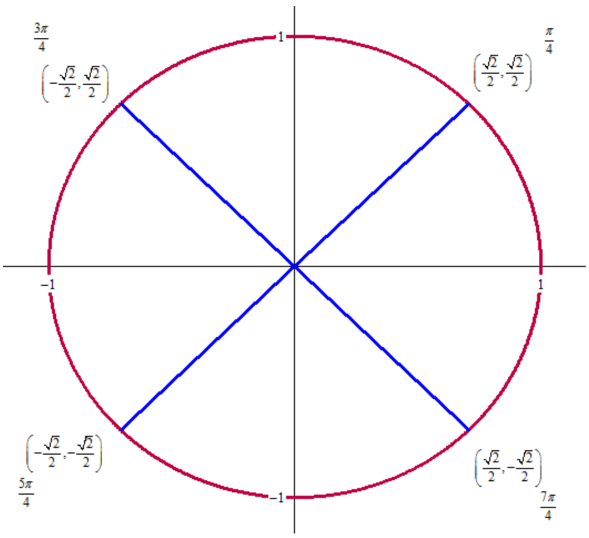
Solution

As this problem is not easily factored, we will solve using the square root property. First, we use algebra to isolate $\sin\theta$. Then we will find the angles.

Isolate $\sin\theta$

$$\begin{aligned} 2\sin^2\theta - 1 &= 0 \\ 2\sin^2\theta &= 1 \\ \sin^2\theta &= \frac{1}{2} \\ \sqrt{\sin^2\theta} &= \pm\sqrt{\frac{1}{2}} \\ \sin\theta &= \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2} \end{aligned} \tag{7.41}$$

We know sine is a y value, so let us find the angles on the unit circle where the y value is $\pm\frac{\sqrt{2}}{2}$



Thus our solution in $[0,\ 2\pi)$ is $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Watch Out!

Remember to include the $\pm$ when taking square roots!

Try It **7.27** Solve $4\cos^2\theta - 3 = 0$ algebraically. in $[0,\ 2\pi)$. Give exact answers.

Example 7.49

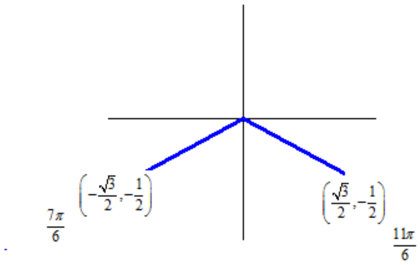
Solve $\csc\theta = -2$ exactly in $[0,\ 4\pi)$

Solution

We want all values of θ for which $\csc\theta = -2$ over the interval $[0,\ 4\pi)$

$$\csc\theta = -2$$
$$\frac{1}{\sin\theta} = -2 \quad \text{(convert to sines)}$$
$$\sin\theta = -\frac{1}{2} \quad (solve\ for\ \sin\theta)$$

We need to find the angles on the unit circle where y is $-\frac{1}{2}$. We know y is negative in QIII and QIV. From the unit circle.



Our angles in one cycle are

$$\theta = \frac{7\pi}{6} \ \ or \ \ \theta = \frac{11\pi}{6}$$

We want all angles in a larger interval, so ake a table to find angles in $[0,\ 4\pi)$

k	$\frac{7\pi}{6}+2k\pi$	$\frac{11\pi}{6}+2k\pi$
0	$\frac{7\pi}{6}$	$\frac{11\pi}{6}$
1	$\frac{7\pi}{6}+2\pi=\frac{19\pi}{6}$	$\frac{11\pi}{6}+2\pi=\frac{23\pi}{6}$
2	$\frac{7\pi}{6}+2\cdot 2\pi>4\pi$	$\frac{11\pi}{6}+2\cdot 2\pi>4\pi$

Table 7.8 Table of values between 0 and 4π

Final answer in $[0, 4\pi)$

$$\left\{\frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}\right\}$$

Analysis

As $\sin\theta=-\frac{1}{2}$, notice that all four solutions are in the third and fourth quadrants.

Try It

7.28 Solve $\sec\theta=-\sqrt{2}$ exactly in $[0, 4\pi)$

Example 7.50

2i)

Solving an Equation Involving Tangent

Solve the equation exactly: $\tan\left(\theta-\frac{\pi}{2}\right)=1,\ 0\le\theta<2\pi.$

Solution

We want to find the angles where tangent is 1. Recall that the tangent function has a period of π. On the interval $[0, \pi)$, and at the angle of $\frac{\pi}{4}$, the tangent has a value of 1. However, the angle we want is $\left(\theta-\frac{\pi}{2}\right)$. Thus, if $\tan\left(\frac{\pi}{4}\right)=1$, then

$$\begin{aligned}\theta-\frac{\pi}{2}&=\frac{\pi}{4}\pm k\pi\\ \theta&=\frac{3\pi}{4}\pm k\pi\end{aligned}$$

Over the interval $[0, 2\pi)$, we have two solutions:

$$\frac{3\pi}{4}\text{ and }\frac{3\pi}{4}+\pi=\frac{7\pi}{4}$$

Solution The period of the sine function is 2π. In the interval $[0, 2\pi)$, there are two values at which the sine function is $\frac{\sqrt{3}}{2}$. One of these values is $\frac{\pi}{3}$. The sine is positive in quadrant II; thus, the other value is $\pi - \frac{\pi}{3}$, or $\frac{2\pi}{3}$. This means that $\frac{x}{2} = \frac{\pi}{3}$ or $\frac{x}{2} = \frac{2\pi}{3}$. Because the period is 2π, all the solutions of $\sin\frac{x}{2} = \frac{\sqrt{3}}{2}$ are given by

$$\frac{x}{2} = \frac{\pi}{3} + 2n\pi \quad \text{or} \quad \frac{x}{2} = \frac{2\pi}{3} + 2n\pi \qquad n \text{ is any integer.}$$

$$x = \frac{2\pi}{3} + 4n\pi \qquad\qquad x = \frac{4\pi}{3} + 4n\pi. \qquad \text{Multiply both sides by 2 and solve for } x.$$

We see that $x = \frac{2\pi}{3} + 4n\pi$ or $x = \frac{4\pi}{3} + 4n\pi$. If $n = 0$, we obtain $x = \frac{2\pi}{3}$ from the first equation and $x = \frac{4\pi}{3}$ from the second equation. If we let $n = 1$, we are adding $4 \cdot 1 \cdot \pi$, or 4π, to $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$. These values of x exceed 2π. Thus, in the interval $[0, 2\pi)$, the only solutions of $\sin\frac{x}{2} = \frac{\sqrt{3}}{2}$ are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$.

Check Point 3 Solve the equation: $\sin\frac{x}{3} = \frac{1}{2}, 0 \le x < 2\pi$.

3 Solve trigonometric equations quadratic in form.

Trigonometric Equations Quadratic in Form

Some trigonometric equations are in the form of a quadratic equation $au^2 + bu + c = 0$, where u is a trigonometric function and $a \neq 0$. Here are two examples of trigonometric equations that are quadratic in form:

$$2\cos^2 x + \cos x - 1 = 0 \qquad 2\sin^2 x - 3\sin x + 1 = 0.$$

The form of this equation is $2u^2 + u - 1 = 0$ with $u = \cos x$.

The form of this equation is $2u^2 - 3u + 1 = 0$ with $u = \sin x$.

To solve this kind of equation, try using factoring. If the trigonometric expression does not factor, use another method, such as the quadratic formula or the square root property.

Technology

Graphic Connections

The graph of

$$y = 2\cos^2 x + \cos x - 1$$

is shown in a

$$\left[0, 2\pi, \frac{\pi}{2}\right] \text{ by } [-3, 3, 1]$$

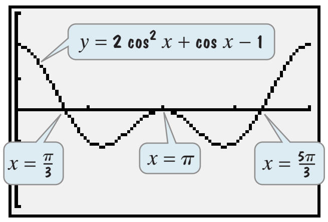
viewing rectangle. The x-intercepts,

$$\frac{\pi}{3}, \pi, \text{ and } \frac{5\pi}{3},$$

verify the three solutions of

$$2\cos^2 x + \cos x - 1 = 0$$

in $[0, 2\pi)$.



EXAMPLE 4 Solving a Trigonometric Equation Quadratic in Form

Solve the equation: $2\cos^2 x + \cos x - 1 = 0, \quad 0 \le x < 2\pi$.

Solution The given equation is in quadratic form $2u^2 + u - 1 = 0$ with $u = \cos x$. Let us attempt to solve the equation by factoring.

$$2\cos^2 x + \cos x - 1 = 0 \qquad \text{This is the given equation.}$$

$$(2\cos x - 1)(\cos x + 1) = 0 \qquad \text{Factor: Notice that } 2u^2 + u - 1 \text{ factors as } (2u - 1)(u + 1).$$

$$2\cos x - 1 = 0 \quad \text{or} \quad \cos x + 1 = 0 \qquad \text{Set each factor equal to 0.}$$

$$2\cos x = 1 \qquad \cos x = -1 \qquad \text{Solve for } \cos x.$$

$$\cos x = \tfrac{1}{2}$$

$$x = \frac{\pi}{3} \quad x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} \qquad x = \pi \qquad \text{Solve each equation for } x, 0 \le x < 2\pi.$$

The cosine is positive in quadrants I and IV.

The solutions in the interval $[0, 2\pi,)$ are $\frac{\pi}{3}$, π, and $\frac{5\pi}{3}$.

Study Tip

In solving

$$\tan x \sin^2 x = 3 \tan x,$$

do not begin by dividing both sides by $\tan x$. Division by zero is undefined. If you divide by $\tan x$, you lose the two solutions for which $\tan x = 0$, namely 0 and π.

2d)

EXAMPLE 6 Using Factoring to Separate Different Functions

Solve the equation: $\tan x \sin^2 x = 3 \tan x, \quad 0 \le x < 2\pi.$

Solution Move all terms to one side and obtain zero on the other side.

$$\tan x \sin^2 x = 3 \tan x \quad \text{This is the given equation.}$$

$$\tan x \sin^2 x - 3 \tan x = 0 \quad \text{Subtract } 3 \tan x \text{ from both sides.}$$

We now have $\tan x \sin^2 x - 3 \tan x = 0$, which contains both tangent and sine functions. Use factoring to separate the two functions.

$$\tan x(\sin^2 x - 3) = 0 \quad \text{Factor out } \tan x \text{ from the two terms on the left side.}$$

$$\tan x = 0 \quad \text{or} \quad \sin^2 x - 3 = 0 \quad \text{Set each factor equal to 0.}$$

$$x = 0 \quad x = \pi \qquad \sin^2 x = 3 \quad \text{Solve for } x.$$

$$\sin x = \pm\sqrt{3}$$

This equation has no solution because $\sin x$ cannot be greater than 1 or less than -1.

The solutions in the interval $[0, 2\pi)$ are 0 and π.

Check Point 6 Solve the equation: $\sin x \tan x = \sin x, \quad 0 \le x < 2\pi.$

5 Use identities to solve trigonometric equations.

Using Identities to Solve Trigonometric Equations

Some trigonometric equations contain more than one function on the same side and these functions cannot be separated by factoring. For example, consider the equation

$$2\cos^2 x + 3 \sin x = 0.$$

How can we obtain an equivalent equation that has only one trigonometric function? We use the identity $\sin^2 x + \cos^2 x = 1$ and substitute $1 - \sin^2 x$ for $\cos^2 x$. This forms the basis of our next example.

2e)

EXAMPLE 7 Using an Identity to Solve a Trigonometric Equation

Solve the equation: $2\cos^2 x + 3 \sin x = 0, \quad 0 \le x < 2\pi.$

Solution

$$2\cos^2 x + 3\sin x = 0 \quad \text{This is the given equation.}$$

$$2(1 - \sin^2 x) + 3 \sin x = 0 \quad \cos^2 x = 1 - \sin^2 x$$

$$2 - 2\sin^2 x + 3\sin x = 0 \quad \text{Use the distributive property.}$$

$$-2\sin^2 x + 3\sin x + 2 = 0 \quad \text{Write the equation in descending powers of } \sin x.$$

$$2\sin^2 x - 3\sin x - 2 = 0 \quad \text{Multiply both sides by } -1\text{. The equation is in quadratic form } 2u^2 - 3u - 2 = 0 \text{ with } u = \sin x.$$

It's easier to factor with a positive leading coefficient.

$$(2\sin x + 1)(\sin x - 2) = 0 \quad \text{Factor. Notice that } 2u^2 - 3u - 2 \text{ factors as } (2u + 1)(u - 2).$$

$$2\sin x + 1 = 0 \quad \text{or} \quad \sin x - 2 = 0 \quad \text{Set each factor equal to 0.}$$

2e)

$$2 \sin x = -1 \qquad \sin x = 2$$

Solve $2 \sin x + 1 = 0$ and $\sin x - 2 = 0$ for $\sin x$.

This equation has no solution because $\sin x$ cannot be greater than 1.

$$\sin x = -\frac{1}{2}$$

$$x = \pi + \frac{\pi}{6} = \frac{7\pi}{6} \qquad x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$

Solve for x.

$\sin \frac{\pi}{6} = \frac{1}{2}$. The sine is negative in quadrants III and IV.

The solutions of $2 \cos^2 x + 3 \sin x = 0$ in the interval $[0, 2\pi)$ are $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

Check Point 7 Solve the equation: $2 \sin^2 x - 3 \cos x = 0, \quad 0 \le x < 2\pi$.

2m)

EXAMPLE 8 Using an Identity to Solve a Trigonometric Equation

Solve the equation: $\cos 2x + 3 \sin x - 2 = 0, \quad 0 \le x < 2\pi$.

Solution The given equation contains a cosine function and a sine function. The cosine is a function of $2x$ and the sine is a function of x. We want one trigonometric function of the same angle. This can be accomplished by using the double-angle identity $\cos 2x = 1 - 2 \sin^2 x$ to obtain an equivalent equation involving $\sin x$ only.

$$\cos 2x + 3 \sin x - 2 = 0$$
This is the given equation.

$$1 - 2 \sin^2 x + 3 \sin x - 2 = 0$$
$\cos 2x = 1 - 2 \sin^2 x$

$$-2 \sin^2 x + 3 \sin x - 1 = 0$$
Combine like terms.

$$2 \sin^2 x - 3 \sin x + 1 = 0$$
Multiply both sides by -1. The equation is in quadratic form $2u^2 - 3u + 1 = 0$ with $u = \sin x$.

$$(2 \sin x - 1)(\sin x - 1) = 0$$
Factor. Notice that $2u^2 - 3u + 1$ factors as $(2u - 1)(u - 1)$.

$$2 \sin x - 1 = 0 \quad \text{or} \quad \sin x - 1 = 0$$
Set each factor equal to 0.

$$\sin x = \frac{1}{2} \qquad \sin x = 1$$
Solve for $\sin x$.

$$x = \frac{\pi}{6} \qquad x = \pi - \frac{\pi}{6} = \frac{5\pi}{6} \qquad x = \frac{\pi}{2}$$
Solve each equation for x, $0 \le x < 2\pi$.

The sine is positive in quadrants I and II.

The solutions in the interval $[0, 2\pi)$ are $\frac{\pi}{6}, \frac{\pi}{2}$, and $\frac{5\pi}{6}$.

Check Point 8 Solve the equation: $\cos 2x + \sin x = 0, \quad 0 \le x < 2\pi$.

Sometimes it is necessary to do something to both sides of a trigonometric equation before using an identity. For example, consider the equation

$$\sin x \cos x = \tfrac{1}{2}.$$

This equation contains both a sine and a cosine function. How can we obtain a single function? Multiply both sides by 2. In this way, we can use the double-angle identity $\sin 2x = 2 \sin x \cos x$ and obtain $\sin 2x$, a single function, on the left side.

Next, we focus on solving inequalities involving the trigonometric functions. Since these functions are continuous on their domains, we may use the sign diagram technique we've used in the past to solve the inequalities.[10]

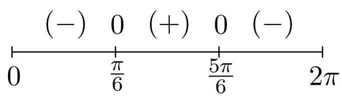
Example 10.7.3. Solve the following inequalities on $[0, 2\pi)$. Express your answers using interval notation and verify your answers graphically.

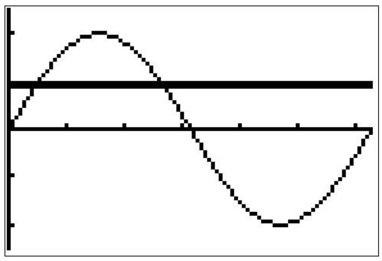
1. $2\sin(x) \leq 1$
2. $\sin(2x) > \cos(x)$
3. $\tan(x) \geq 3$

Solution.

3a)

1. We begin solving $2\sin(x) \leq 1$ by collecting all of the terms on one side of the equation and zero on the other to get $2\sin(x) - 1 \leq 0$. Next, we let $f(x) = 2\sin(x) - 1$ and note that our original inequality is equivalent to solving $f(x) \leq 0$. We now look to see where, if ever, f is undefined and where $f(x) = 0$. Since the domain of f is all real numbers, we can immediately set about finding the zeros of f. Solving $f(x) = 0$, we have $2\sin(x) - 1 = 0$ or $\sin(x) = \frac{1}{2}$. The solutions here are $x = \frac{\pi}{6} + 2\pi k$ and $x = \frac{5\pi}{6} + 2\pi k$ for integers k. Since we are restricting our attention to $[0, 2\pi)$, only $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$ are of concern to us. Next, we choose test values in $[0, 2\pi)$ other than the zeros and determine if f is positive or negative there. For $x = 0$ we have $f(0) = -1$, for $x = \frac{\pi}{2}$ we get $f\left(\frac{\pi}{2}\right) = 1$ and for $x = \pi$ we get $f(\pi) = -1$. Since our original inequality is equivalent to $f(x) \leq 0$, we are looking for where the function is negative $(-)$ or 0, and we get the intervals $\left[0, \frac{\pi}{6}\right] \cup \left[\frac{5\pi}{6}, 2\pi\right)$. We can confirm our answer graphically by seeing where the graph of $y = 2\sin(x)$ crosses or is below the graph of $y = 1$.





$y = 2\sin(x)$ and $\boldsymbol{y = 1}$

3f)

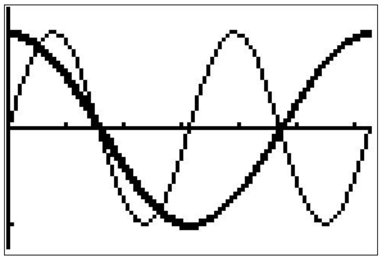
2. We first rewrite $\sin(2x) > \cos(x)$ as $\sin(2x) - \cos(x) > 0$ and let $f(x) = \sin(2x) - \cos(x)$. Our original inequality is thus equivalent to $f(x) > 0$. The domain of f is all real numbers, so we can advance to finding the zeros of f. Setting $f(x) = 0$ yields $\sin(2x) - \cos(x) = 0$, which, by way of the double angle identity for sine, becomes $2\sin(x)\cos(x) - \cos(x) = 0$ or $\cos(x)(2\sin(x) - 1) = 0$. From $\cos(x) = 0$, we get $x = \frac{\pi}{2} + \pi k$ for integers k of which only $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$ lie in $[0, 2\pi)$. For $2\sin(x) - 1 = 0$, we get $\sin(x) = \frac{1}{2}$ which gives $x = \frac{\pi}{6} + 2\pi k$ or $x = \frac{5\pi}{6} + 2\pi k$ for integers k. Of those, only $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$ lie in $[0, 2\pi)$. Next, we choose

[10] See page 214, Example 3.1.5, page 321, page 399, Example 6.3.2 and Example 6.4.2 for discussion of this technique.

our test values. For $x = 0$ we find $f(0) = -1$; when $x = \frac{\pi}{4}$ we get $f\left(\frac{\pi}{4}\right) = 1 - \frac{\sqrt{2}}{2} = \frac{2-\sqrt{2}}{2}$; for $x = \frac{3\pi}{4}$ we get $f\left(\frac{3\pi}{4}\right) = -1 + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}-2}{2}$; when $x = \pi$ we have $f(\pi) = 1$, and lastly, for $x = \frac{7\pi}{4}$ we get $f\left(\frac{7\pi}{4}\right) = -1 - \frac{\sqrt{2}}{2} = \frac{-2-\sqrt{2}}{2}$. We see $f(x) > 0$ on $\left(\frac{\pi}{6}, \frac{\pi}{2}\right) \cup \left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$, so this is our answer. We can use the calculator to check that the graph of $y = \sin(2x)$ is indeed above the graph of $y = \cos(x)$ on those intervals.

38)

$$\begin{array}{ccccccccccc} & (-) & 0 & (+) & 0 & (-) & 0 & (+) & 0 & (-) & \\ 0 & & \frac{\pi}{6} & & \frac{\pi}{2} & & \frac{5\pi}{6} & & \frac{3\pi}{2} & & 2\pi \end{array}$$



$y = \sin(2x)$ and $\boldsymbol{y = \cos(x)}$

35)

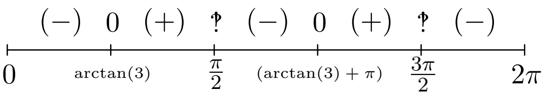
3. Proceeding as in the last two problems, we rewrite $\tan(x) \geq 3$ as $\tan(x) - 3 \geq 0$ and let $f(x) = \tan(x) - 3$. We note that on $[0, 2\pi)$, f is undefined at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, so those values will need the usual disclaimer on the sign diagram.[11] Moving along to zeros, solving $f(x) = \tan(x) - 3 = 0$ requires the arctangent function. We find $x = \arctan(3) + \pi k$ for integers k and of these, only $x = \arctan(3)$ and $x = \arctan(3) + \pi$ lie in $[0, 2\pi)$. Since $3 > 0$, we know $0 < \arctan(3) < \frac{\pi}{2}$ which allows us to position these zeros correctly on the sign diagram. To choose test values, we begin with $x = 0$ and find $f(0) = -3$. Finding a convenient test value in the interval $\left(\arctan(3), \frac{\pi}{2}\right)$ is a bit more challenging. Keep in mind that the arctangent function is increasing and is bounded above by $\frac{\pi}{2}$. This means that the number $x = \arctan(117)$ is guaranteed[12] to lie between $\arctan(3)$ and $\frac{\pi}{2}$. We see that $f(\arctan(117)) = \tan(\arctan(117)) - 3 = 114$. For our next test value, we take $x = \pi$ and find $f(\pi) = -3$. To find our next test value, we note that since $\arctan(3) < \arctan(117) < \frac{\pi}{2}$, it follows[13] that $\arctan(3) + \pi < \arctan(117) + \pi < \frac{3\pi}{2}$. Evaluating f at $x = \arctan(117) + \pi$ yields $f(\arctan(117) + \pi) = \tan(\arctan(117) + \pi) - 3 = \tan(\arctan(117)) - 3 = 114$. We choose our last test value to be $x = \frac{7\pi}{4}$ and find $f\left(\frac{7\pi}{4}\right) = -4$. Since we want $f(x) \geq 0$, we see that our answer is $\left[\arctan(3), \frac{\pi}{2}\right) \cup \left[\arctan(3) + \pi, \frac{3\pi}{2}\right)$. Using the graphs of $y = \tan(x)$ and $y = 3$, we see when the graph of the former is above (or meets) the graph of the latter.

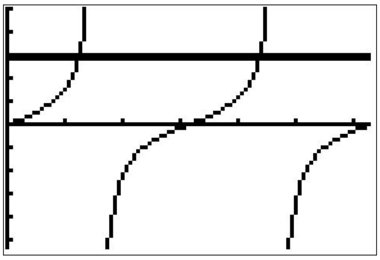
[11] See page 321 for a discussion of the non-standard character known as the interrobang.

[12] We could have chosen any value $\arctan(t)$ where $t > 3$.

[13] ... by adding π through the inequality ...

3b)





$y = \tan(x)$ and $\boldsymbol{y = 3}$

□

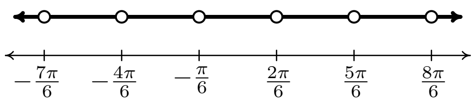
We close this section with an example that puts solving equations and inequalities to good use – finding domains of functions.

Example 10.7.4. Express the domain of the following functions using extended interval notation.[14]

1. $f(x) = \csc\left(2x + \frac{\pi}{3}\right)$
2. $f(x) = \dfrac{\sin(x)}{2\cos(x) - 1}$
3. $f(x) = \sqrt{1 - \cot(x)}$

Solution.

1. To find the domain of $f(x) = \csc\left(2x + \frac{\pi}{3}\right)$, we rewrite f in terms of sine as $f(x) = \frac{1}{\sin\left(2x + \frac{\pi}{3}\right)}$. Since the sine function is defined everywhere, our only concern comes from zeros in the denominator. Solving $\sin\left(2x + \frac{\pi}{3}\right) = 0$, we get $x = -\frac{\pi}{6} + \frac{\pi}{2}k$ for integers k. In set-builder notation, our domain is $\left\{x : x \neq -\frac{\pi}{6} + \frac{\pi}{2}k \text{ for integers } k\right\}$. To help visualize the domain, we follow the old mantra 'When in doubt, write it out!' We get $\left\{x : x \neq -\frac{\pi}{6}, \frac{2\pi}{6}, -\frac{4\pi}{6}, \frac{5\pi}{6}, -\frac{7\pi}{6}, \frac{8\pi}{6}, \ldots\right\}$, where we have kept the denominators 6 throughout to help see the pattern. Graphing the situation on a numberline, we have



Proceeding as we did on page 756 in Section 10.3.1, we let x_k denote the kth number excluded from the domain and we have $x_k = -\frac{\pi}{6} + \frac{\pi}{2}k = \frac{(3k-1)\pi}{6}$ for integers k. The intervals which comprise the domain are of the form $(x_k, x_{k+1}) = \left(\frac{(3k-1)\pi}{6}, \frac{(3k+2)\pi}{6}\right)$ as k runs through the integers. Using extended interval notation, we have that the domain is

$$\bigcup_{k=-\infty}^{\infty} \left(\frac{(3k-1)\pi}{6}, \frac{(3k+2)\pi}{6}\right)$$

We can check our answer by substituting in values of k to see that it matches our diagram.

[14]See page 756 for details about this notation.

3c)

$$\tan x \geq -\sqrt{3}$$

This example has been selected here as it involves consideration of each step as enumerated above for finding solution of inequality. Corresponding trigonometric equation, in this case, is :

$$\tan x = -\sqrt{3}$$

The acute angle is $\pi/3$. Further, tangent function is negative in second and fourth quarter (see sign diagram). Using value diagram in conjunction with sign diagram, solution of given equation in $[0, 2\pi]$ are :

Sign and value diagram

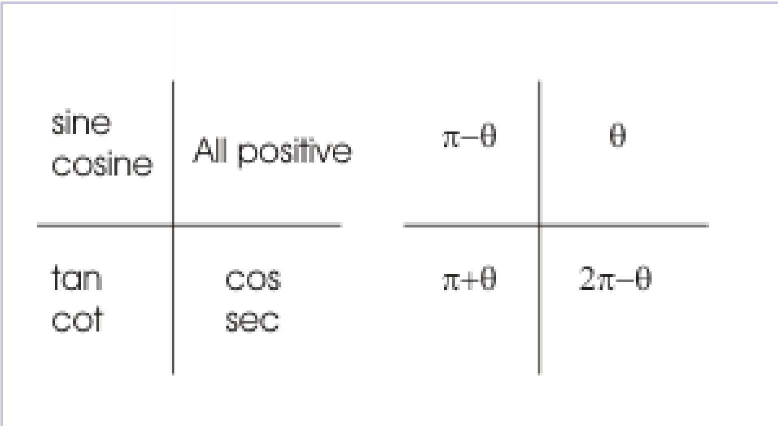


Figure 3.86: Tangent function is negative in second and fourth quarter .

$$\Rightarrow x = \pi - \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$\Rightarrow x = 2\pi - \theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$$

Here, second angle is greater than π. Hence, equivalent negative angle is :

$$\Rightarrow y = \frac{5\pi}{3} - 2\pi = -\frac{\pi}{3}$$

Tangent function, however, is not a continuous function between -$\pi/3$ and $2\pi/3$. Tangent values are greater than -$\sqrt{3}$ for angle greater than -$\pi/3$, but value asymptotes to infinity at $\pi/2$. This can be verified from the intersection graph.

Trigonometric inequality

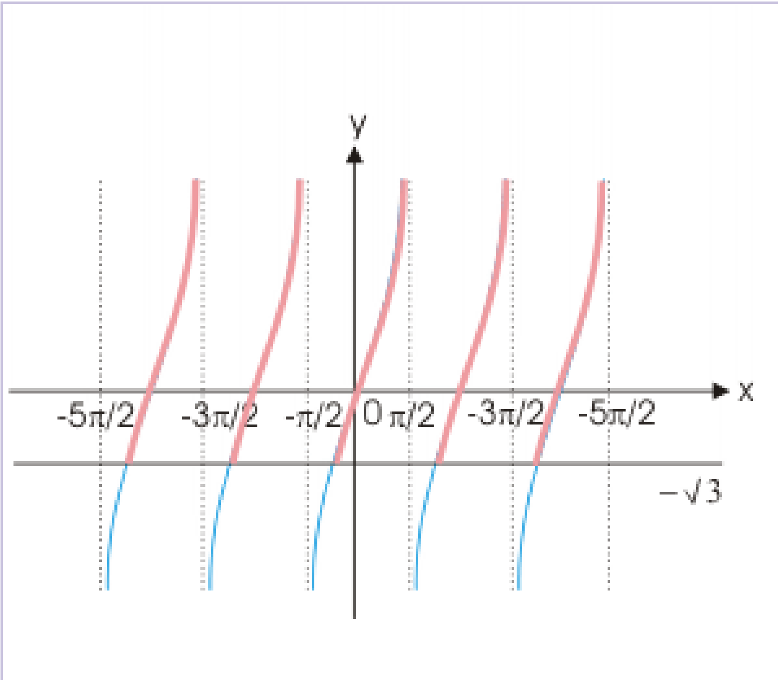


Figure 3.87: Intervals satisfying inequality, involving tangent function.

Thus, basic interval satisfying inequality is :

$$-\frac{\pi}{3} \leq x < \frac{\pi}{2}$$

It is also clear that the solution in this interval is repeated with a period of π, which is period of tangent function. Hence, solution of given inequality is :

$$n\pi - \frac{\pi}{3} \leq x < n\pi + \frac{\pi}{2}; \quad n \in Z$$

3.14.2 Examples

3d)

Example 3.36

Problem : Solve trigonometric inequality given by :

$$\sin x \geq \frac{1}{2}$$

Solution : The solution of the corresponding equal equation is obtained as :

$$\sin x = \frac{1}{2} = \sin\frac{\pi}{6}$$

$$\Rightarrow x = \frac{\pi}{6}$$

The sine function is positive in first and second quarter. Hence, second angle between "0" and "2π" is :

$$\Rightarrow x = \pi - \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

Both angles are less than "π". Thus, we do not need to convert angle into equivalent negative angle. Further, sine curve is defined for all values of "x". The base interval, therefore, is :
The valid intervals on sine plot are shown in the figure.

Trigonometric inequality

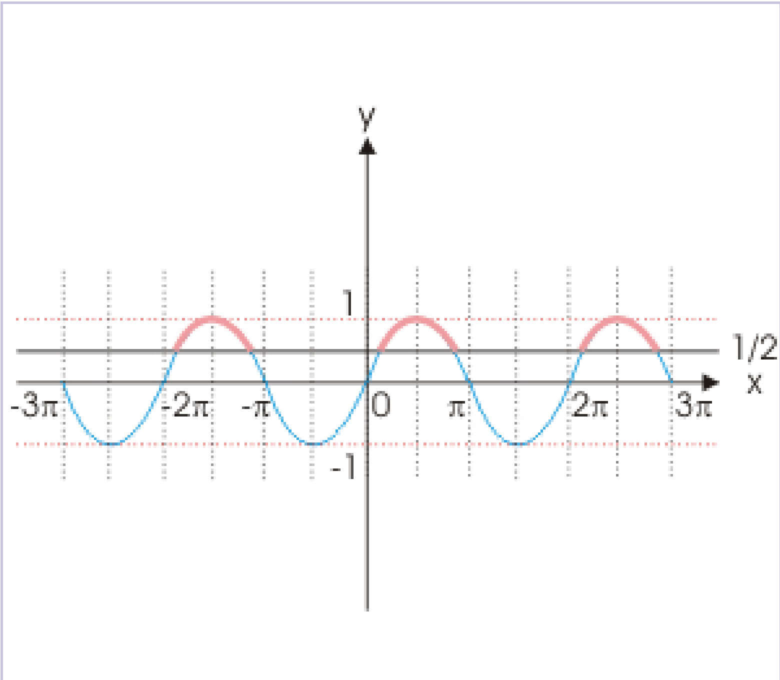


Figure 3.88: Intervals satisfying inequality

$$\frac{\pi}{6} \le x \le \frac{5\pi}{6}$$

The periodicity of sine function is "2π". Hence, we add "$2n\pi$" on either side of the base interval :

$$2n\pi + \frac{\pi}{6} \le x \le 2n\pi + \frac{5\pi}{6}, \quad n \in Z$$

Example 3.37
Problem : Solve trigonometric inequality given by :

$$\sin x > \cos x$$

Solution : In order to solve this inequality, it is required to convert it in terms of inequality of a single trigonometric function.

$$\sin x > \cos x$$

$$\Rightarrow \text{sin}x - \text{cos}x > 0$$

$$\Rightarrow \text{sin}x\text{cos}\frac{\pi}{4} - \text{cos}x\text{sin}\frac{\pi}{4} > 0$$

$$\Rightarrow \sin\left(x - \frac{\pi}{4}\right) > 0$$

Let $y = x - \pi/4$. Then,

$$\text{sin}y > 0$$

Thus, we see that problem finally reduces to solving trigonometric sine inequality. The solution of the corresponding equality is obtained as :

$$\Rightarrow \text{sin}y = 0 = \text{sin}0$$

$$\Rightarrow y = 0$$

The second angle between "0" and "2π" is "π". The base interval, therefore, is :

$$0 < y < \pi$$

The periodicity of sine function is "2π". Hence, we add "2nπ" on either side of the base interval :

$$2n\pi < y < 2n\pi + \pi, \quad n \in Z$$

Now substituting for $y = x - \pi/4$, we have :

$$2n\pi < x - \frac{\pi}{4} < 2n\pi + \pi, \quad n \in Z$$

$$\Rightarrow 2n\pi + \frac{\pi}{4} < x < 2n\pi + \frac{5\pi}{4}, \quad n \in Z$$

Example 3.38
Problem : If domain of a function, "f(x)", is [0,1], then find the domain of the function given by :

$$f\,(2\text{sin}x - 1)$$

Solution : The domain of the function is given here. We need to find the domain when argument (input) to the function is a trigonometric expression. The given domain is :

$$0 \le x \le 1$$

Changing argument of the function, the domain becomes :

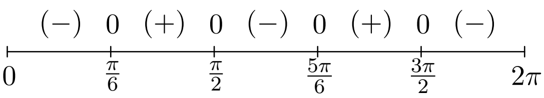
$$0 \le 2\text{sin}x - 1 \le 1 \quad \Rightarrow 1 \le 2\text{sin}x \le 2 \quad \Rightarrow 1/2 \le \text{sin}x \le 1$$

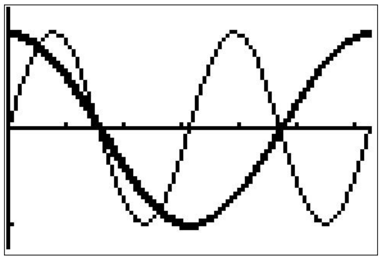
However, the range of sinx is [-1,1]. It means that the above interval is equivalent to a trigonometric inequality given by :

$$\Rightarrow \text{sin}x \ge \frac{1}{2}$$

our test values. For $x = 0$ we find $f(0) = -1$; when $x = \frac{\pi}{4}$ we get $f\left(\frac{\pi}{4}\right) = 1 - \frac{\sqrt{2}}{2} = \frac{2-\sqrt{2}}{2}$; for $x = \frac{3\pi}{4}$ we get $f\left(\frac{3\pi}{4}\right) = -1 + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}-2}{2}$; when $x = \pi$ we have $f(\pi) = 1$, and lastly, for $x = \frac{7\pi}{4}$ we get $f\left(\frac{7\pi}{4}\right) = -1 - \frac{\sqrt{2}}{2} = \frac{-2-\sqrt{2}}{2}$. We see $f(x) > 0$ on $\left(\frac{\pi}{6}, \frac{\pi}{2}\right) \cup \left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$, so this is our answer. We can use the calculator to check that the graph of $y = \sin(2x)$ is indeed above the graph of $y = \cos(x)$ on those intervals.

38)





$y = \sin(2x)$ and $\boldsymbol{y = \cos(x)}$

35)

3. Proceeding as in the last two problems, we rewrite $\tan(x) \geq 3$ as $\tan(x) - 3 \geq 0$ and let $f(x) = \tan(x) - 3$. We note that on $[0, 2\pi)$, f is undefined at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, so those values will need the usual disclaimer on the sign diagram.[11] Moving along to zeros, solving $f(x) = \tan(x) - 3 = 0$ requires the arctangent function. We find $x = \arctan(3) + \pi k$ for integers k and of these, only $x = \arctan(3)$ and $x = \arctan(3) + \pi$ lie in $[0, 2\pi)$. Since $3 > 0$, we know $0 < \arctan(3) < \frac{\pi}{2}$ which allows us to position these zeros correctly on the sign diagram. To choose test values, we begin with $x = 0$ and find $f(0) = -3$. Finding a convenient test value in the interval $\left(\arctan(3), \frac{\pi}{2}\right)$ is a bit more challenging. Keep in mind that the arctangent function is increasing and is bounded above by $\frac{\pi}{2}$. This means that the number $x = \arctan(117)$ is guaranteed[12] to lie between $\arctan(3)$ and $\frac{\pi}{2}$. We see that $f(\arctan(117)) = \tan(\arctan(117)) - 3 = 114$. For our next test value, we take $x = \pi$ and find $f(\pi) = -3$. To find our next test value, we note that since $\arctan(3) < \arctan(117) < \frac{\pi}{2}$, it follows[13] that $\arctan(3) + \pi < \arctan(117) + \pi < \frac{3\pi}{2}$. Evaluating f at $x = \arctan(117) + \pi$ yields $f(\arctan(117) + \pi) = \tan(\arctan(117) + \pi) - 3 = \tan(\arctan(117)) - 3 = 114$. We choose our last test value to be $x = \frac{7\pi}{4}$ and find $f\left(\frac{7\pi}{4}\right) = -4$. Since we want $f(x) \geq 0$, we see that our answer is $\left[\arctan(3), \frac{\pi}{2}\right) \cup \left[\arctan(3) + \pi, \frac{3\pi}{2}\right)$. Using the graphs of $y = \tan(x)$ and $y = 3$, we see when the graph of the former is above (or meets) the graph of the latter.

[11] See page 321 for a discussion of the non-standard character known as the interrobang.
[12] We could have chosen any value $\arctan(t)$ where $t > 3$.
[13] ...by adding π through the inequality ...

3g

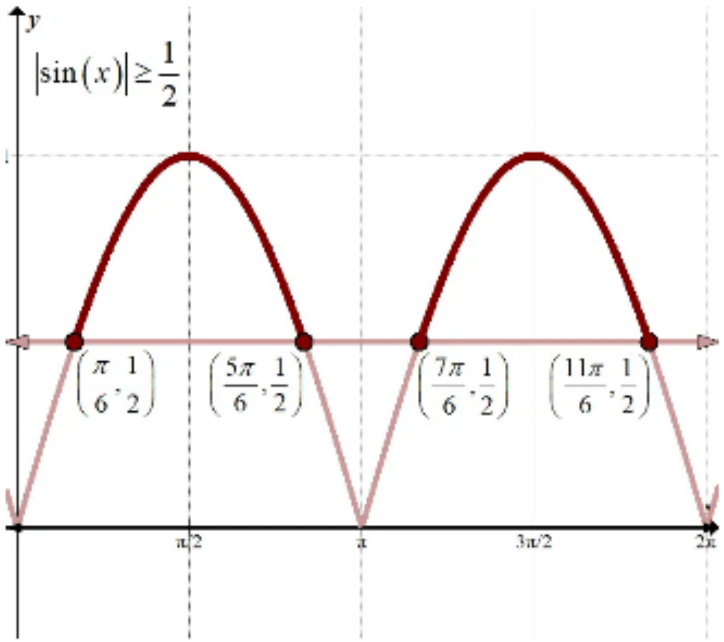
Solving Trig Inequalities Graphically

$$-6\left|\sin\theta\right| \le -3 \quad \text{over reals}$$

$$\left|\sin\theta\right| \ge \frac{1}{2} \quad \text{(switch signs!)}$$

Graph both sides of the inequality to see where $\left|\sin(x)\right|$ is greater than or equal to $\frac{1}{2}$. Just graph between **0** and 2π to see the points of intersection.

We want **above** (including) $\frac{1}{2}$, because of the $\ge$:



We see the solution is:

$$\left(\frac{\pi}{6}+2\pi k, \frac{5\pi}{6}+2\pi k\right) \cup \left(\frac{7\pi}{6}+2\pi k, \frac{11\pi}{6}+2\pi k\right)$$

3h

Solving Trig Inequalities Algebraically

Set everything to **0** and factor, so we can get our critical values:

$$3\cot^2 x + 3\cot x - \sqrt{3}\cot x < \sqrt{3}; \quad 0 \le x \le 2\pi$$

$$3\cot^2 x + 3\cot x - \sqrt{3}\cot x - \sqrt{3} < 0$$

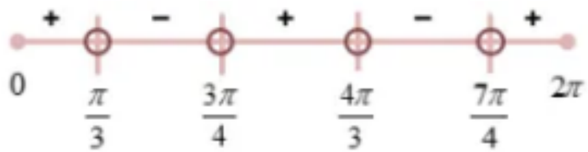
$$3\cot x\,(\cot x + 1) - \sqrt{3}\,(\cot x + 1) < 0$$

$$(\cot x + 1)\left(3\cot x - \sqrt{3}\right) < 0$$

$$\cot x = -1 \quad \left(x = \frac{3\pi}{4}, \frac{7\pi}{4}\right) \qquad \cot x = \frac{\sqrt{3}}{3} \quad \left(x = \frac{\pi}{3}, \frac{4\pi}{3}\right)$$

Draw **sign chart** with **critical values** $\frac{3\pi}{4}, \frac{7\pi}{4}, \frac{\pi}{3}, \frac{4\pi}{3}$, and endpoints **0** and 2π, since we're just going between **0** and 2π. Use **open circles** for the critical values since we have a $<$. Then check each interval with a sample value in the last inequality above and see if we get a **positive** or **negative** value.

We need to take values where the function is defined (avoid asymptotes), so let's try $\frac{\pi}{4}$ for the interval less than $\frac{\pi}{3}$ for example: $\left(\cot\frac{\pi}{4}+1\right)\left(3\cot\frac{\pi}{4}-\sqrt{3}\right) \approx 2.5$, which is **positive**:



We want the **negative** intervals, not including the critical values. We see the solution is:

$$\left(\frac{\pi}{3}, \frac{3\pi}{4}\right) \cup \left(\frac{4\pi}{3}, \frac{7\pi}{4}\right).$$

You can also use graphing calculators to directly solve the trig inequality R(x) < 0 (or > 0). This method, if allowed, is fast, accurate and convenient. To know how to proceed, read the last chapter of the book mentioned above (Amazon e-book 2010).

EXAMPLES ON SOLVING TRIG INEQUALITIES

3) <u>Example 13</u>. Solve: (2cos x – 1)/(2cos x – 1) < 0 (0 < x < 2Pi)

Solution.

Step 1: The function F(x) = f(x)/g(x) = (2cos x + 1)/(2cos x – 1) < 0 is undefined when x = 2Pi/3 and x = 4Pi/3.

Step 2. Common period 2Pi.

Step 3. Solve f(x) = 2cos x – 1 = 0 → cos x = ½ → x = Pi/3 ; x = 5Pi/3

Solve g(x) = 2cos x + 1 = 0 → cos x = -1/2 → x = 2Pi/3 ; x = 4Pi/3

Step 4. Solve F(x) < 0, algebraically, by setting up the sign chart (sign table)

x	I	0	Pi/3		2Pi/3		4Pi/3		5Pi/3		2Pi
f(x)	I	+	0	-		-		-	0	+	
g(x)	I	+		+	0	-	0	+		+	
F(x)	I	No	I	Yes	II	No	II	Yes	I	No	

Answer or solution set: (Pi/3 , 2Pi/3) ; (4Pi/3 , 5Pi/3).

<u>Example 14.</u> Solve: tan x + cot x < -4 (0 < x < Pi)

Solution.

Step 1. Transform the inequality into standard form;

sin x/cos x + cos x/sin x + 4 = 2/sin 2x + 4 < 0

F(x) = 2(2sin 2x + 1)/sin 2x < 0 undefined when x = kPi/2 and x = kPi

Step 2. The common period of F(x) is Pi since the period of sin 2x is Pi.