7th lesson

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Exercises

1. Evaluate the following (use the unit circle):

   (a) $\sin \frac{2\pi}{3}$
   (b) $\sin \frac{-2\pi}{3}$
   (c) $\cos \frac{7\pi}{6}$
   (d) $\cos \frac{-7\pi}{6}$
   (e) $\tan \frac{-\pi}{4}$
   (f) $\tan \frac{7\pi}{4}$
   (g) $\cot \frac{5\pi}{4}$
   (h) $\cos \frac{5\pi}{6}$
   (i) $\sin \frac{-4\pi}{3}$
   (j) $\sin \frac{7\pi}{4}$
   (k) $\cos \frac{-2\pi}{3}$
   (l) $\tan \frac{3\pi}{4}$
   (m) $\tan \frac{-\pi}{3}$
   (n) $\tan \frac{15\pi}{4}$
   (o) $\cot \frac{-\pi}{3}$

Recall as well that one complete revolution is 2π , so the positive *x*-axis can correspond to either an angle of 0 or 2π (or 4π , or 6π, or -2π, or -4π, *etc*. depending on the direction of rotation). Likewise, the angle $\frac{\pi}{6}$ (to pick an angle completely at random) can also be any of the following angles:

$\frac{\pi}{6}+2\pi=\frac{13\pi}{6}$ (start at $\frac{\pi}{6}$ then rotate once around counter clockwise)

$\frac{\pi}{6}+4\pi=\frac{25\pi}{6}$ (start at $\frac{\pi}{6}$ then rotate around twice counter clockwise)

$\frac{\pi}{6}-2\pi=-\frac{11\pi}{6}$ (start at $\frac{\pi}{6}$ then rotate once around clockwise)

$\frac{\pi}{6}-4\pi=-\frac{23\pi}{6}$ (start at $\frac{\pi}{6}$ then rotate around twice clockwise)

etc.

In fact, $\frac{\pi}{6}$ can be any of the following angles $\frac{\pi}{6}+2\pi n,\;\; n=0,\pm 1,\pm 2,\pm 3,\ldots$ In this case *n* is the number of complete revolutions you make around the unit circle starting at $\frac{\pi}{6}$. Positive values of *n* correspond to counter clockwise rotations and negative values of *n* correspond to clockwise rotations.

So, why did we only put in the first quadrant? The answer is simple. If you know the first quadrant then you can get all the other quadrants from the first with a small application of geometry. You'll see how this is done in the following set of examples.

Example 1 Evaluate each of the following.

(a) $\sin\left(\frac{2\pi}{3}\right)$ and $\sin\left(-\frac{2\pi}{3}\right)$

(b) $\cos\left(\frac{7\pi}{6}\right)$ and $\cos\left(-\frac{7\pi}{6}\right)$

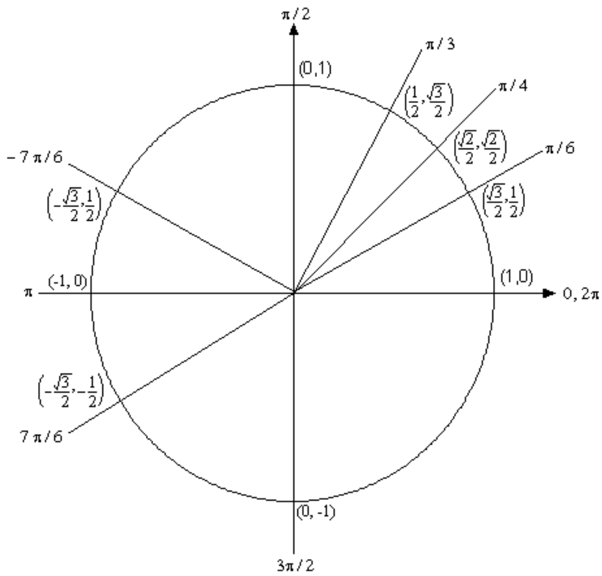
(c) $\tan\left(-\frac{\pi}{4}\right)$ and $\tan\left(\frac{7\pi}{4}\right)$

(d) $\sec\left(\frac{25\pi}{6}\right)$

Solution

(a) The first evaluation in this part uses the angle $\frac{2\pi}{3}$. That's not on our unit circle above, however notice that $\frac{2\pi}{3}=\pi-\frac{\pi}{3}$. So $\frac{2\pi}{3}$ is found by rotating up $\frac{\pi}{3}$ from the negative *x*-axis. This means the negative *x*-axis to get to this angle. So, as with the last part, both of these angles will be mirror images of $\frac{\pi}{6}$ in the third and second quadrants respectively and we can use this to determine the coordinates for both of these new angles.

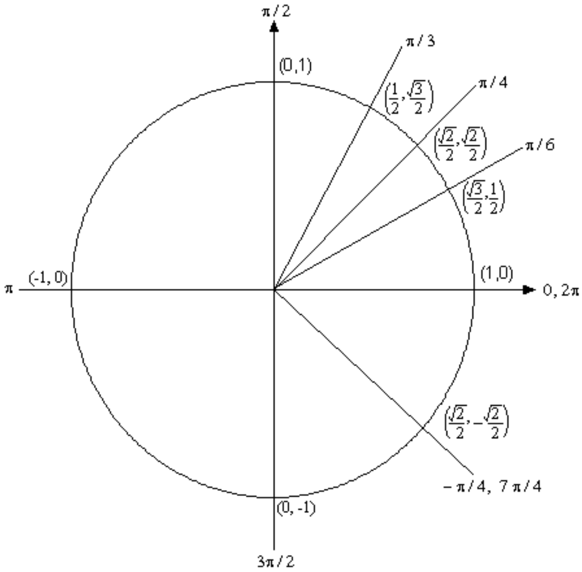
Both of these angles are shown on the following unit circle along with the coordinates for the intersection points.



From this unit circle we can see that $\cos\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$ and $\cos\left(-\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2}$. In this case the cosine function is called an **even** function and so for ANY angle we have

$$\cos(-\theta) = \cos(\theta).$$

(c) Here we should note that $\frac{7\pi}{4} = 2\pi - \frac{\pi}{4}$ so $\frac{7\pi}{4}$ and $-\frac{\pi}{4}$ are in fact the same angle! Also note that this angle will be the mirror image of $\frac{\pi}{4}$ in the fourth quadrant. The unit circle for this angle is



Now, if we remember that $\tan\left(x\right)=\dfrac{\sin\left(x\right)}{\cos\left(x\right)}$ we can use the unit circle to find the values of the tangent function. So,

$$\tan\left(\frac{7\pi}{4}\right)=\tan\left(-\frac{\pi}{4}\right)=\frac{\sin\left(-\pi/4\right)}{\cos\left(-\pi/4\right)}=\frac{-\sqrt{2}/2}{\sqrt{2}/2}=-1\,.$$

On a side note, notice that $\tan\left(\dfrac{\pi}{4}\right)=1$ and we can see that the tangent function is also called an **odd** function and so for ANY angle we will have

$$\tan\left(-\theta\right)=-\tan\left(\theta\right).$$

(d) Here we need to notice that $\dfrac{25\pi}{6}=4\pi+\dfrac{\pi}{6}$. In other words, we've started at $\dfrac{\pi}{6}$ and rotated around twice to end back up at the same point on the unit circle. This means that

$$\sec\left(\frac{25\pi}{6}\right)=\sec\left(4\pi+\frac{\pi}{6}\right)=\sec\left(\frac{\pi}{6}\right)$$

Now, let's also not get excited about the secant here. Just recall that

$$\sec\left(x\right)=\frac{1}{\cos\left(x\right)}$$

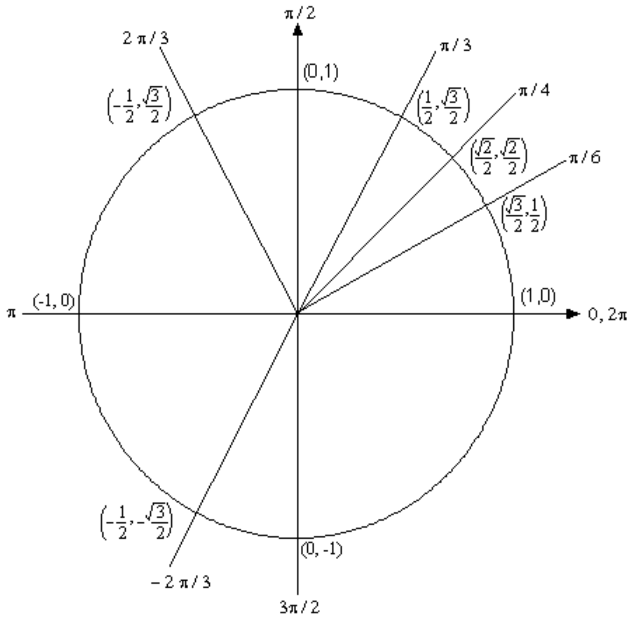
and so all we need to do here is evaluate a cosine! Therefore,

$$\sec\left(\frac{25\pi}{6}\right)=\sec\left(\frac{\pi}{6}\right)=\frac{1}{\cos\left(\frac{\pi}{6}\right)}=\frac{1}{\sqrt{3}/2}=\frac{2}{\sqrt{3}}$$

that the line for $\frac{2\pi}{3}$ will be a mirror image of the line for $\frac{\pi}{3}$ only in the second quadrant. The coordinates for $\frac{2\pi}{3}$ will be the coordinates for $\frac{\pi}{3}$ except the *x* coordinate will be negative.

Likewise, for $-\frac{2\pi}{3}$ we can notice that $-\frac{2\pi}{3} = -\pi + \frac{\pi}{3}$, so this angle can be found by rotating down $\frac{\pi}{3}$ from the negative *x*-axis. This means that the line for $-\frac{2\pi}{3}$ will be a mirror image of the line for $\frac{\pi}{3}$ only in the third quadrant and the coordinates will be the same as the coordinates for $\frac{\pi}{3}$ except both will be negative.

Both of these angles, along with the coordinates of the intersection points, are shown on the following unit circle.



From this unit circle we can see that $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\sin\left(-\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2}$.

This leads to a nice fact about the sine function. The sine function is called an **odd** function and so for ANY angle we have

$$\sin(-\theta) = -\sin(\theta)$$

(b) For this example, notice that $\frac{7\pi}{6} = \pi + \frac{\pi}{6}$ so this means we would rotate down $\frac{\pi}{6}$ from the negative *x*-axis to get to this angle. Also $-\frac{7\pi}{6} = -\pi - \frac{\pi}{6}$ so this means we would rotate up $\frac{\pi}{6}$ from

Algebra Help

You can review dividing fractions and rationalizing denominators in Appendix A.1 and Appendix A.2, respectively.

Example 1 Evaluating Trigonometric Functions

Evaluate the six trigonometric functions at each real number.

a. $t = \dfrac{\pi}{6}$ **b.** $t = \dfrac{5\pi}{4}$ **c.** $t = 0$ **d.** $t = \pi$

Solution

For each t-value, begin by finding the corresponding point (x, y) on the unit circle. Then use the definitions of trigonometric functions listed on page 293.

a. $t = \dfrac{\pi}{6}$ corresponds to the point $(x, y) = \left(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$.

$$\sin \frac{\pi}{6} = y = \frac{1}{2} \qquad \csc \frac{\pi}{6} = \frac{1}{y} = \frac{1}{1/2} = 2$$

$$\cos \frac{\pi}{6} = x = \frac{\sqrt{3}}{2} \qquad \sec \frac{\pi}{6} = \frac{1}{x} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\tan \frac{\pi}{6} = \frac{y}{x} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \cot \frac{\pi}{6} = \frac{x}{y} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

b. $t = \dfrac{5\pi}{4}$ corresponds to the point $(x, y) = \left(-\dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right)$.

$$\sin \frac{5\pi}{4} = y = -\frac{\sqrt{2}}{2} \qquad \csc \frac{5\pi}{4} = \frac{1}{y} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

$$\cos \frac{5\pi}{4} = x = -\frac{\sqrt{2}}{2} \qquad \sec \frac{5\pi}{4} = \frac{1}{x} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

$$\tan \frac{5\pi}{4} = \frac{y}{x} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1 \qquad \cot \frac{5\pi}{4} = \frac{x}{y} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1$$

c. $t = 0$ corresponds to the point $(x, y) = (1, 0)$.

$$\sin 0 = y = 0 \qquad \csc 0 = \frac{1}{y} \text{ is undefined.}$$

$$\cos 0 = x = 1 \qquad \sec 0 = \frac{1}{x} = \frac{1}{1} = 1$$

$$\tan 0 = \frac{y}{x} = \frac{0}{1} = 0 \qquad \cot 0 = \frac{x}{y} \text{ is undefined.}$$

d. $t = \pi$ corresponds to the point $(x, y) = (-1, 0)$.

$$\sin \pi = y = 0 \qquad \csc \pi = \frac{1}{y} \text{ is undefined.}$$

$$\cos \pi = x = -1 \qquad \sec \pi = \frac{1}{x} = \frac{1}{-1} = -1$$

$$\tan \pi = \frac{y}{x} = \frac{0}{-1} = 0 \qquad \cot \pi = \frac{x}{y} \text{ is undefined.}$$

CHECKPoint Now try Exercise 23.

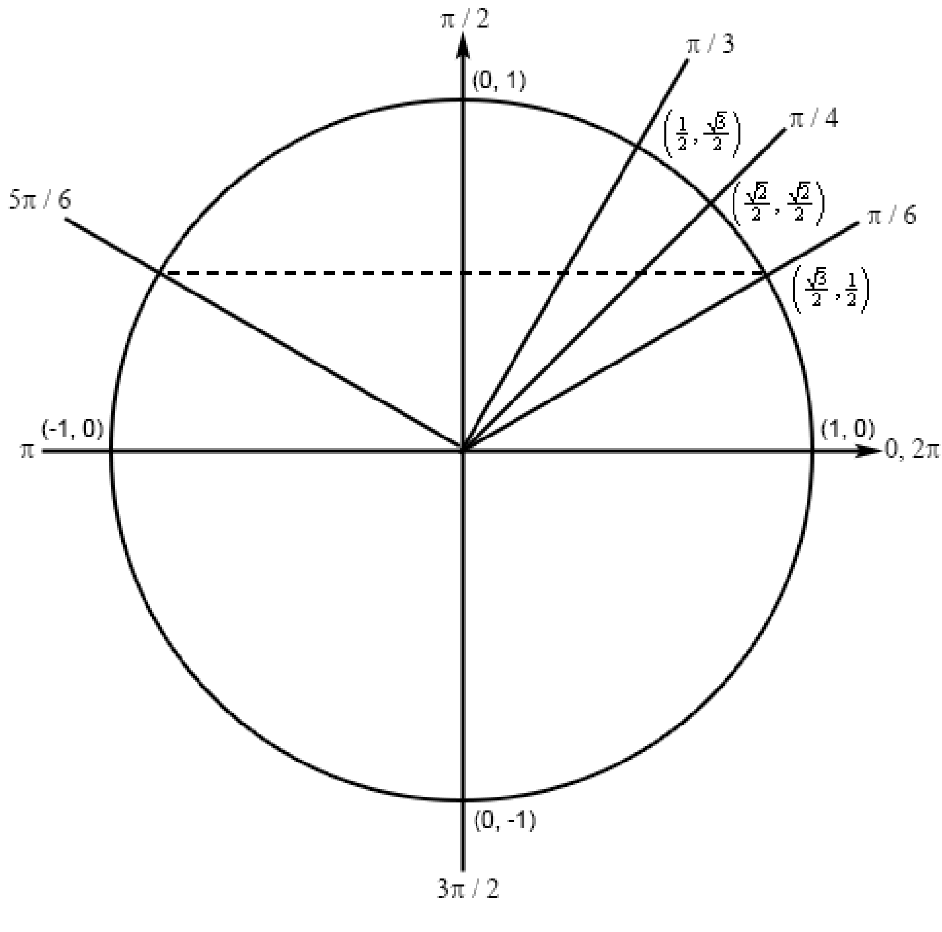
Section 1-3 : Trig Functions

1. Determine the exact value of $\cos\left(\frac{5\pi}{6}\right)$ without using a calculator.

Hint 1 : Sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1

First, we can notice that $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$ and so the terminal line for $\frac{5\pi}{6}$ will form an angle of $\frac{\pi}{6}$ with the negative *x*-axis in the second quadrant and we'll have the following unit circle for this problem.



Hint 2 : Given the obvious symmetry in the unit circle relate the coordinates of the line representing $\frac{5\pi}{6}$ to the coordinates of the line representing $\frac{\pi}{6}$ and use those to answer the question.

Step 2

The coordinates of the line representing $\frac{5\pi}{6}$ will be the same as the coordinates of the line representing $\frac{\pi}{6}$ except that the *x* coordinate will now be negative. So, our new coordinates will then be $\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)$ and so the answer is,

$$\cos\left(\frac{5\pi}{6}\right)=-\frac{\sqrt{3}}{2}$$

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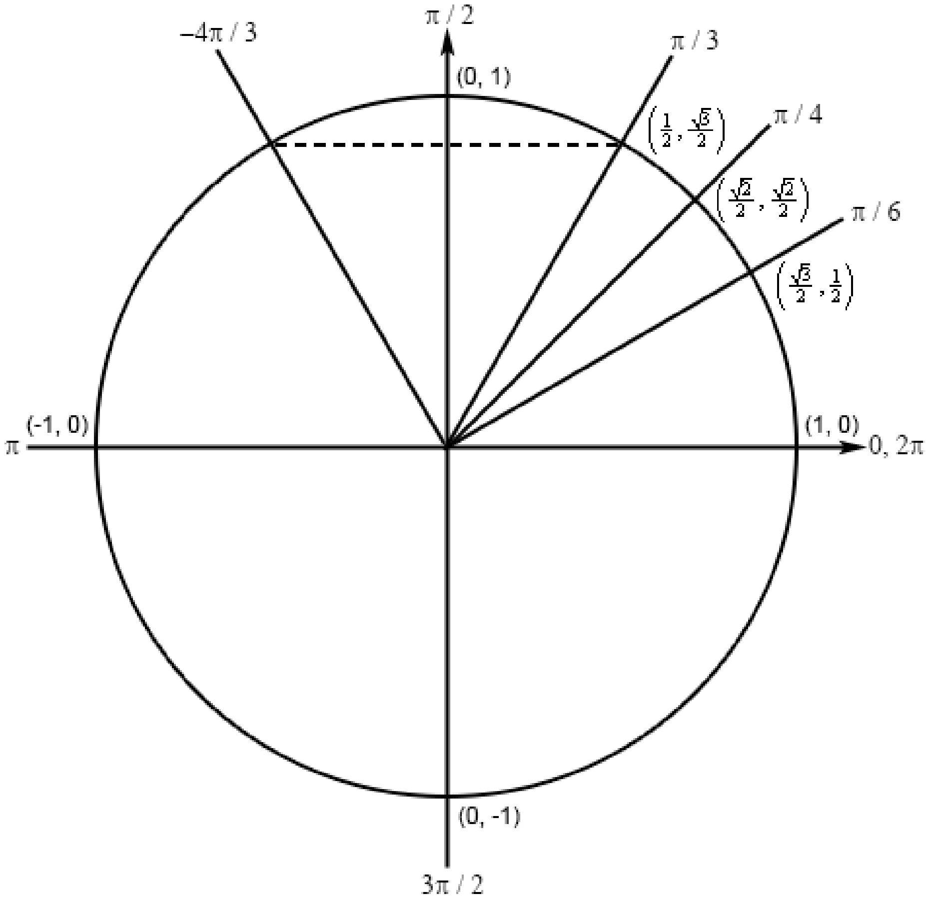
2. Determine the exact value of $\sin\left(-\frac{4\pi}{3}\right)$ without using a calculator.

Hint 1 : Sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1

First we can notice that $-\pi-\frac{\pi}{3}=-\frac{4\pi}{3}$ and so (remembering that negative angles are rotated clockwise) we can see that the terminal line for $-\frac{4\pi}{3}$ will form an angle of $\frac{\pi}{3}$ with the negative *x*-axis in the second quadrant and we'll have the following unit circle for this problem.



Hint 2 : Given the obvious symmetry in the unit circle relate the coordinates of the line representing $-\frac{4\pi}{3}$ to the coordinates of the line representing $\frac{\pi}{3}$ and use those to answer the question.

Step 2

The coordinates of the line representing $-\frac{4\pi}{3}$ will be the same as the coordinates of the line representing $\frac{\pi}{3}$ except that the *x* coordinate will now be negative. So, our new coordinates will then be $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ and so the answer is,

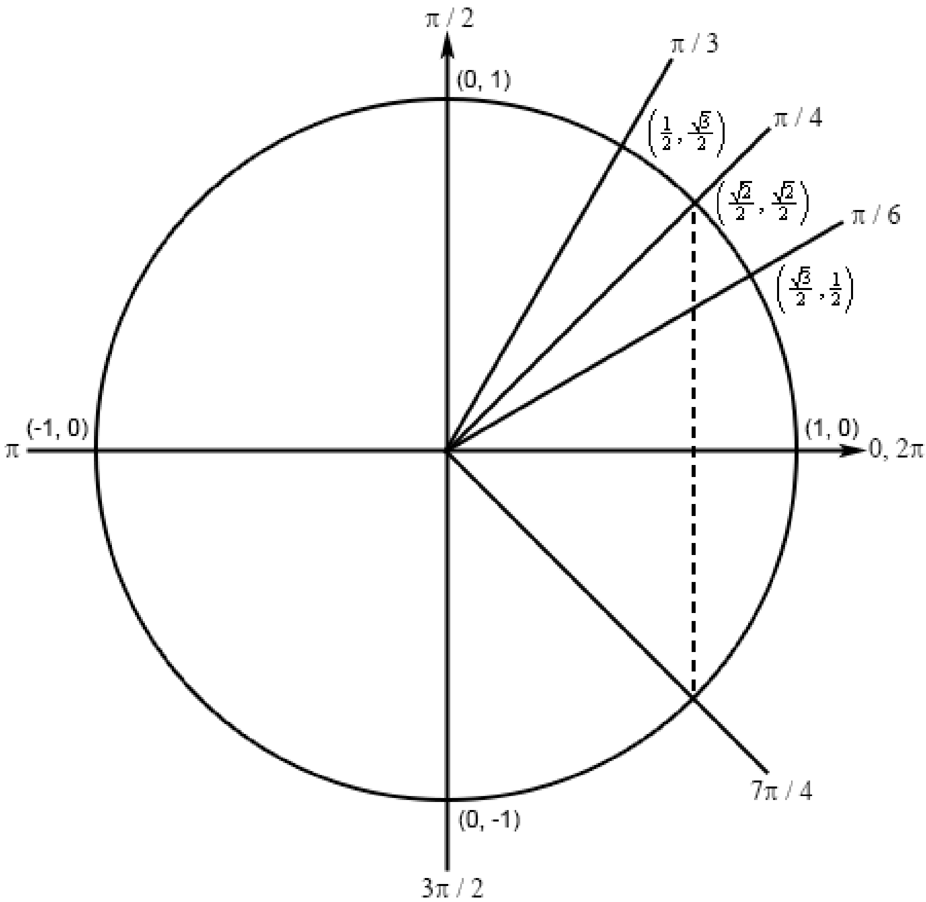
$$\sin\left(-\frac{4\pi}{3}\right)=\frac{\sqrt{3}}{2}$$

3. Determine the exact value of $\sin\left(\frac{7\pi}{4}\right)$ without using a calculator.

Hint 1 : Sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1

First we can notice that $2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$ and so the terminal line for $\frac{7\pi}{4}$ will form an angle of $\frac{\pi}{4}$ with the positive *x*-axis in the fourth quadrant and we'll have the following unit circle for this problem.



Hint 2 : Given the obvious symmetry in the unit circle relate the coordinates of the line representing $\frac{7\pi}{4}$ to the coordinates of the line representing $\frac{\pi}{4}$ and use those to answer the question.

Step 2

The coordinates of the line representing $\frac{7\pi}{4}$ will be the same as the coordinates of the line representing $\frac{\pi}{4}$ except that the *y* coordinate will now be negative. So, our new coordinates will then be $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ and so the answer is,

$$\sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

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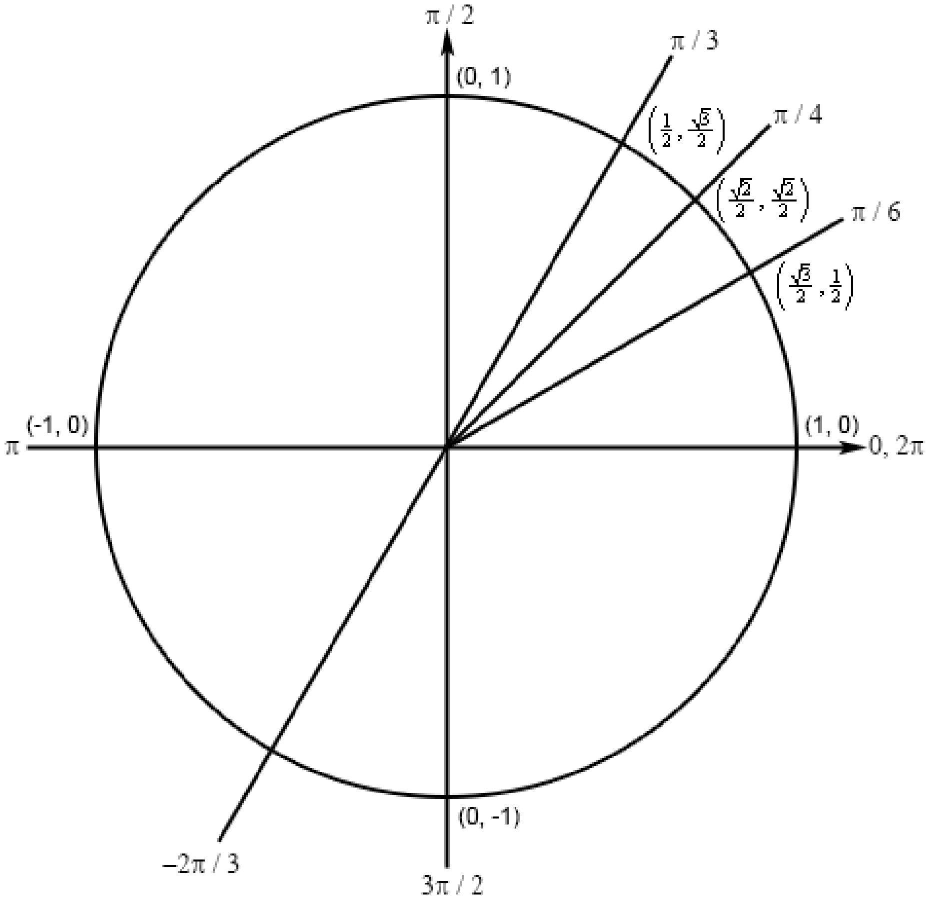
4. Determine the exact value of $\cos\left(-\frac{2\pi}{3}\right)$ without using a calculator.

Hint 1 : Sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1

First we can notice that $-\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$ so (recalling that negative angles rotate clockwise and positive angles rotation counter clockwise) the terminal line for $-\frac{2\pi}{3}$ will form an angle of $\frac{\pi}{3}$ with the negative *x*-axis in the third quadrant and we'll have the following unit circle for this problem.



Hint 2 : Given the obvious symmetry in the unit circle relate the coordinates of the line representing $-\frac{2\pi}{3}$ to the coordinates of the line representing $\frac{\pi}{3}$ and use those to answer the question.

Step 2

The line representing $-\frac{2\pi}{3}$ is a mirror image of the line representing $\frac{\pi}{3}$ and so the coordinates for $-\frac{2\pi}{3}$ will be the same as the coordinates for $\frac{\pi}{3}$ except that both coordinates will now be negative. So, our new coordinates will then be $\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$ and so the answer is,

$$\cos\left(-\frac{2\pi}{3}\right)=-\frac{1}{2}$$

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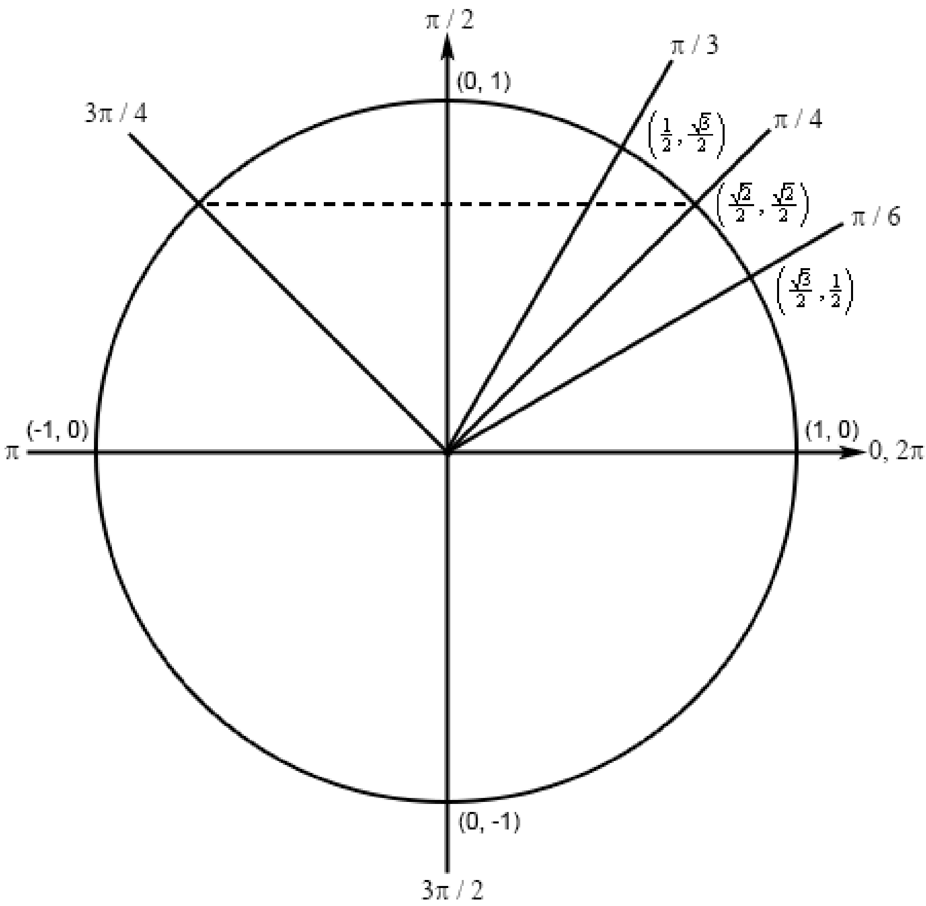
5. Determine the exact value of $\tan\left(\frac{3\pi}{4}\right)$ without using a calculator.

Hint 1 : Even though a unit circle only tells us information about sine and cosine it is still useful for tangents so sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1

First we can notice that $\pi - \frac{\pi}{4} = \frac{3\pi}{4}$ and so (remembering that negative angles are rotated clockwise) we can see that the terminal line for $\frac{3\pi}{4}$ will form an angle of $\frac{\pi}{4}$ with the negative *x*-axis in the second quadrant and we'll have the following unit circle for this problem.



Hint 2 : Given the obvious symmetry in the unit circle relate the coordinates of the line representing $\frac{3\pi}{4}$ to the coordinates of the line representing $\frac{\pi}{4}$ and and then recall how tangent is defined in terms of sine and cosine to answer the question.

Step 2

The coordinates of the line representing $\frac{3\pi}{4}$ will be the same as the coordinates of the line representing $\frac{\pi}{4}$ except that the *x* coordinate will now be negative. So, our new coordinates will then be $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and so the answer is,

$$\tan\left(\frac{3\pi}{4}\right) = \frac{\sin\left(\frac{3\pi}{4}\right)}{\cos\left(\frac{3\pi}{4}\right)} = \frac{\sqrt{2}/2}{-\sqrt{2}/2} = -1$$

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6. Determine the exact value of $\sec\left(-\frac{11\pi}{6}\right)$ without using a calculator.

Hint 1 : Even though a unit circle only tells us information about sine and cosine it is still useful for secant so sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1

First, we can notice that $\frac{\pi}{6} - 2\pi = -\frac{11\pi}{6}$ and so (remembering that negative angles are rotated clockwise) we can see that the terminal line for $-\frac{11\pi}{6}$ will form an angle of $\frac{\pi}{6}$ with the positive *x*-axis in the first quadrant. In other words, $-\frac{11\pi}{6}$ and $\frac{\pi}{6}$ represent the same angle. So, we'll have the following unit circle for this problem.

Example 2 Evaluating Trigonometric Functions

Evaluate the six trigonometric functions at $t = -\dfrac{\pi}{3}$.

Solution

Moving *clockwise* around the unit circle, it follows that $t = -\pi/3$ corresponds to the point $(x, y) = \left(1/2, -\sqrt{3}/2\right)$.

$$\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} \qquad \csc\left(-\frac{\pi}{3}\right) = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

$$\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2} \qquad \sec\left(-\frac{\pi}{3}\right) = 2$$

$$\tan\left(-\frac{\pi}{3}\right) = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3} \qquad \cot\left(-\frac{\pi}{3}\right) = \frac{1/2}{-\sqrt{3}/2} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

CHECK*Point* Now try Exercise 33.

Domain and Period of Sine and Cosine

The *domain* of the sine and cosine functions is the set of all real numbers. To determine the *range* of these two functions, consider the unit circle shown in Figure 4.24. By definition, $\sin t = y$ and $\cos t = x$. Because (x, y) is on the unit circle, you know that $-1 \le y \le 1$ and $-1 \le x \le 1$. So, the values of sine and cosine also range between -1 and 1.

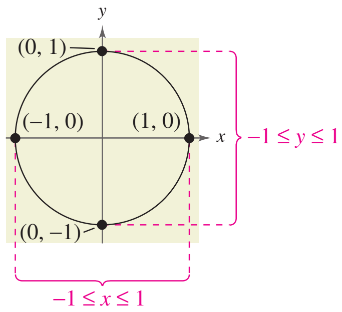

FIGURE 4.24

$$\begin{array}{ccc} -1 \le \quad y \quad \le 1 & & -1 \le \quad x \quad \le 1 \\ & \text{and} & \\ -1 \le \sin t \le 1 & & -1 \le \cos t \le 1 \end{array}$$

Adding 2π to each value of t in the interval $[0, 2\pi]$ completes a second revolution around the unit circle, as shown in Figure 4.25. The values of $\sin(t + 2\pi)$ and $\cos(t + 2\pi)$ correspond to those of $\sin t$ and $\cos t$. Similar results can be obtained for repeated revolutions (positive or negative) on the unit circle. This leads to the general result

$$\sin(t + 2\pi n) = \sin t$$

and

$$\cos(t + 2\pi n) = \cos t$$

for any integer n and real number t. Functions that behave in such a repetitive (or cyclic) manner are called **periodic.**

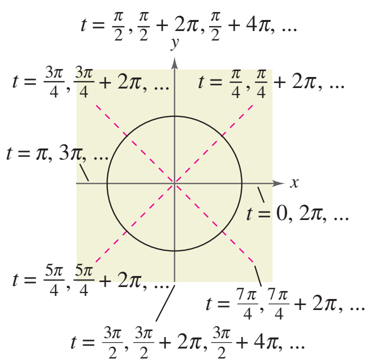

FIGURE 4.25

Definition of Periodic Function

A function f is **periodic** if there exists a positive real number c such that

$$f(t + c) = f(t)$$

for all t in the domain of f. The smallest number c for which f is periodic is called the **period** of f.

$$\tan\left(-\frac{\pi}{3}\right) = \frac{\sin\left(-\frac{\pi}{3}\right)}{\cos\left(-\frac{\pi}{3}\right)} = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3}$$

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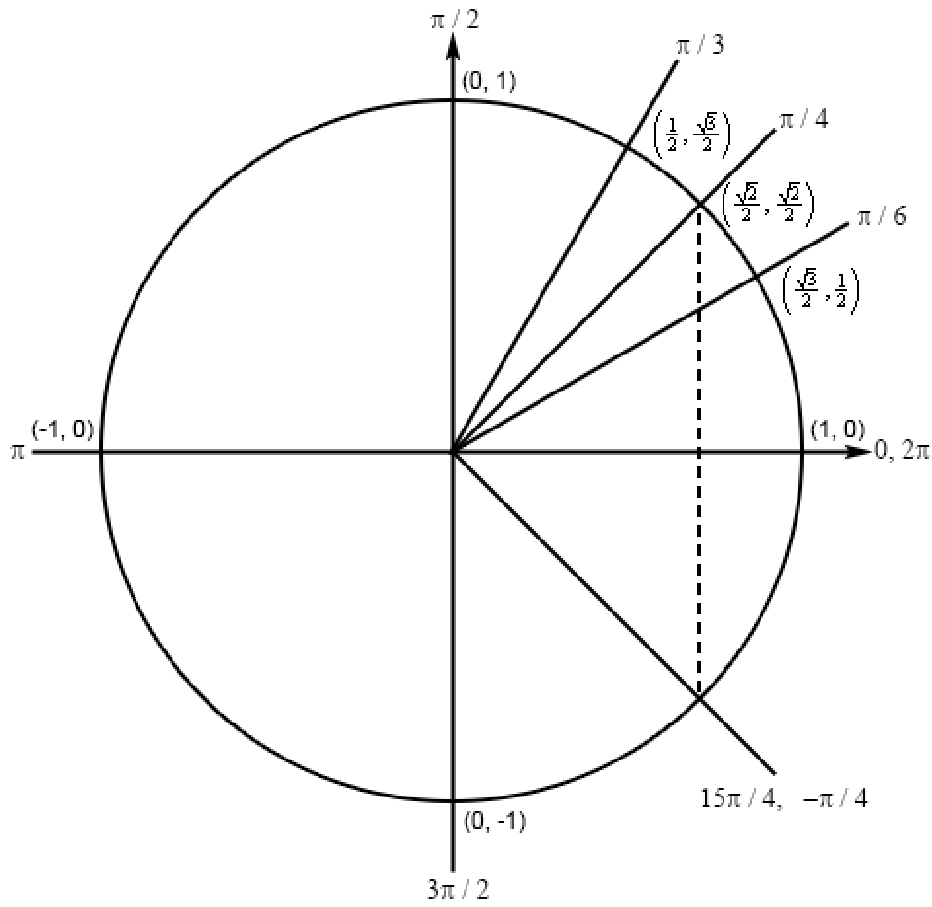
9. Determine the exact value of $\tan\left(\frac{15\pi}{4}\right)$ without using a calculator.

Hint 1 : Even though a unit circle only tells us information about sine and cosine it is still useful for tangents so sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1

First we can notice that $4\pi - \frac{\pi}{4} = \frac{15\pi}{4}$ and also note that 4π is two complete revolutions so the terminal line for $\frac{15\pi}{4}$ and $-\frac{\pi}{4}$ represent the same angle. Also note that $-\frac{\pi}{4}$ will form an angle of $\frac{\pi}{4}$ with the positive *x*-axis in the fourth quadrant and we'll have the following unit circle for this problem.



Hint 2 : Given the obvious symmetry in the unit circle relate the coordinates of the line representing $\frac{15\pi}{4}$ to the coordinates of the line representing $\frac{\pi}{4}$ and the definition of tangent in terms of sine and cosine to answer the question.

Step 2

The coordinates of the line representing $\frac{15\pi}{4}$ will be the same as the coordinates of the line representing $\frac{\pi}{4}$ except that the *y* coordinate will now be negative. So, our new coordinates will then be $\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$ and so the answer is,

$$\tan\left(\frac{15\pi}{4}\right)=\frac{\sin\left(\frac{15\pi}{4}\right)}{\cos\left(\frac{15\pi}{4}\right)}=\frac{-\sqrt{2}/2}{\sqrt{2}/2}=-1$$

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10. Determine the exact value of $\sin\left(-\frac{11\pi}{3}\right)$ without using a calculator.

Hint 1 : Sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1

First we can notice that $\frac{\pi}{3}-4\pi=-\frac{11\pi}{3}$ and note that 4π is two complete revolutions (also, remembering that negative angles are rotated clockwise) we can see that the terminal line for $-\frac{11\pi}{3}$ and $\frac{\pi}{3}$ are the same angle and so we'll have the following unit circle for this problem.

Example 2 Evaluating Trigonometric Functions

Evaluate the six trigonometric functions at $t = -\dfrac{\pi}{3}$.

Solution

Moving *clockwise* around the unit circle, it follows that $t = -\pi/3$ corresponds to the point $(x, y) = \left(1/2, -\sqrt{3}/2\right)$.

$$\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2} \qquad \csc\left(-\frac{\pi}{3}\right) = -\frac{2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$$

$$\cos\left(-\frac{\pi}{3}\right) = \frac{1}{2} \qquad \sec\left(-\frac{\pi}{3}\right) = 2$$

$$\tan\left(-\frac{\pi}{3}\right) = \frac{-\sqrt{3}/2}{1/2} = -\sqrt{3} \qquad \cot\left(-\frac{\pi}{3}\right) = \frac{1/2}{-\sqrt{3}/2} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

CHECK*Point* Now try Exercise 33.

Domain and Period of Sine and Cosine

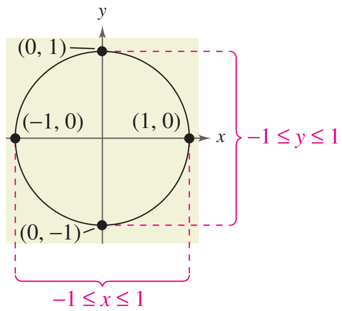

FIGURE 4.24

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$$-1 \le y \le 1 \qquad \text{and} \qquad -1 \le x \le 1$$

$$-1 \le \sin t \le 1 \qquad\qquad -1 \le \cos t \le 1$$

Adding 2π to each value of t in the interval $[0, 2\pi]$ completes a second revolution around the unit circle, as shown in Figure 4.25. The values of $\sin(t + 2\pi)$ and $\cos(t + 2\pi)$ correspond to those of $\sin t$ and $\cos t$. Similar results can be obtained for repeated revolutions (positive or negative) on the unit circle. This leads to the general result

$$\sin(t + 2\pi n) = \sin t$$

and

$$\cos(t + 2\pi n) = \cos t$$

for any integer n and real number t. Functions that behave in such a repetitive (or cyclic) manner are called **periodic.**

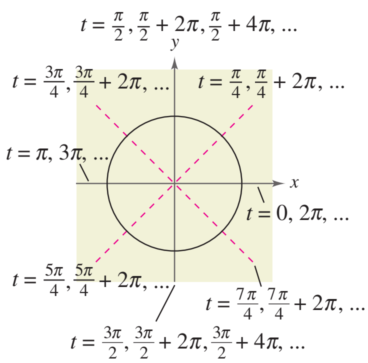


FIGURE 4.25

Definition of Periodic Function

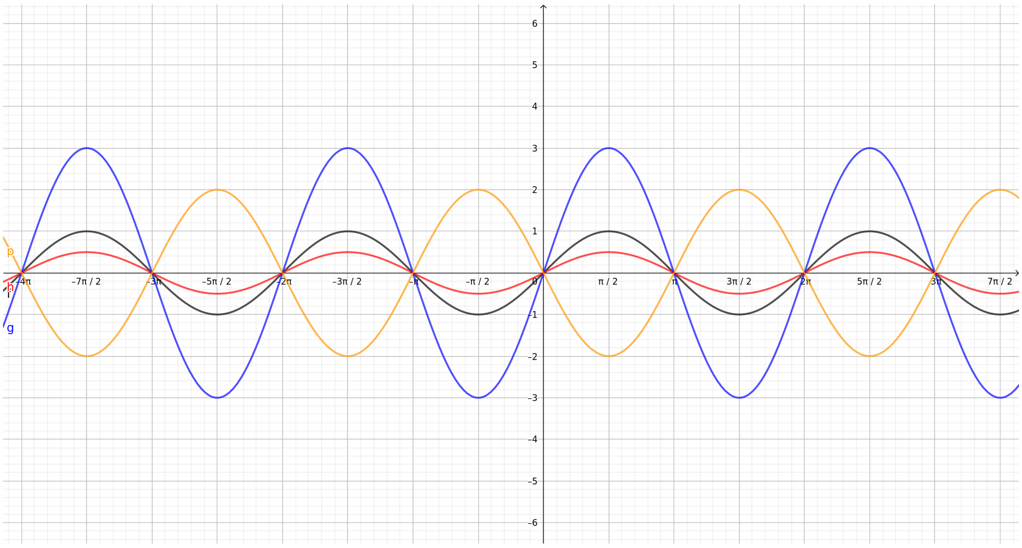
A function f is **periodic** if there exists a positive real number c such that

$$f(t + c) = f(t)$$

for all t in the domain of f. The smallest number c for which f is periodic is called the **period** of f.

2. Find the graph of

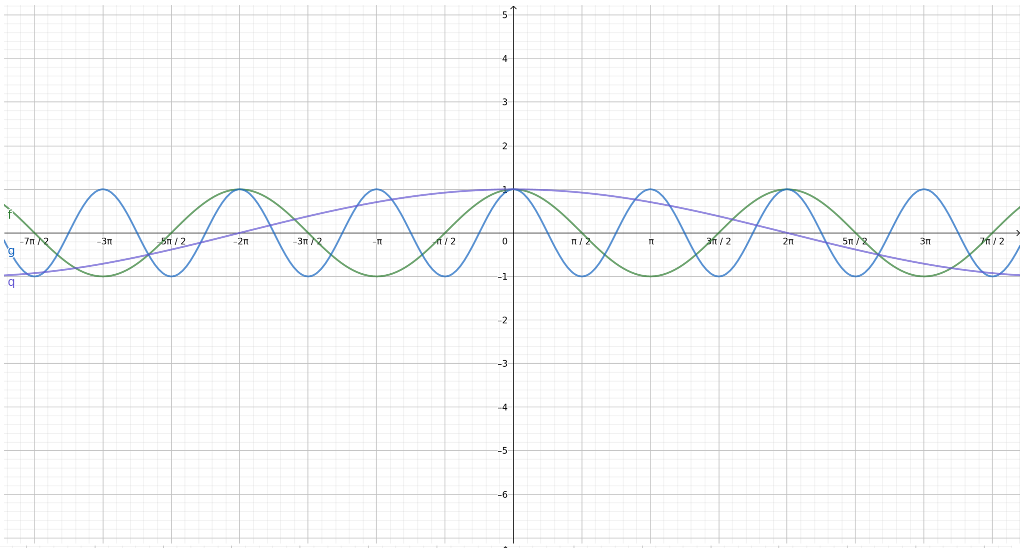
(a) $\sin x$ (b) $3\sin x$ (c) $-2\sin x$ (d) $\frac{1}{2}\sin x$

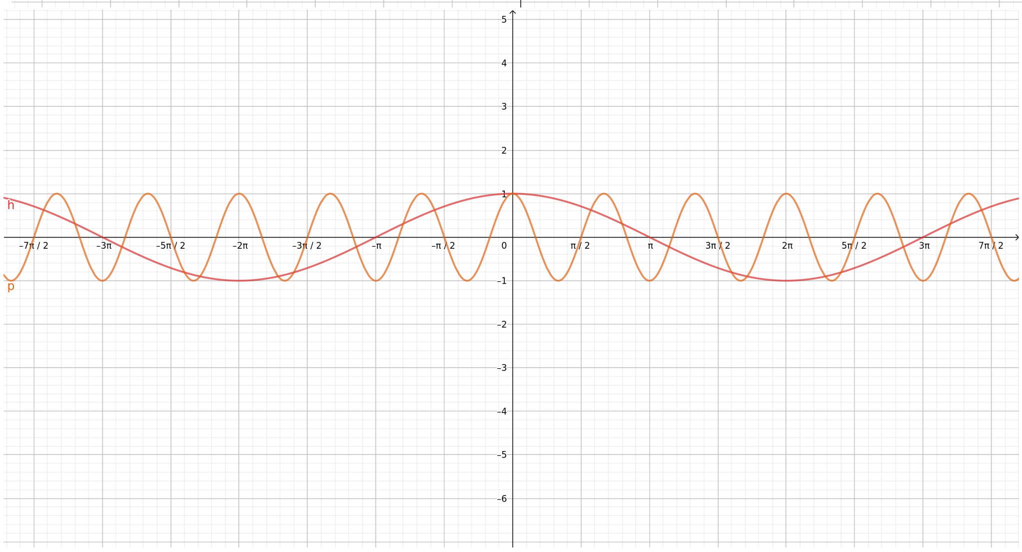


Solution: `https://www.geogebra.org/calculator/wgqev63a`

3. Find the graph of

(a) $\cos x$ (b) $\cos(2x)$ (c) $\cos(3x)$ (d) $\cos(x/2)$ (e) $\cos(x/4)$

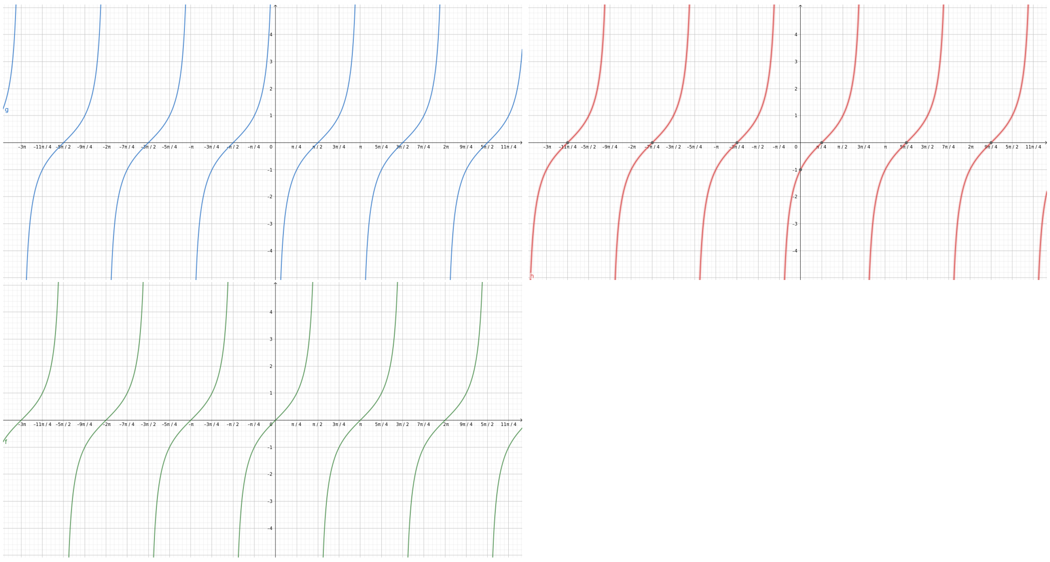




Solution: `https://www.geogebra.org/calculator/tugd5yvv`

4. Find the graph of

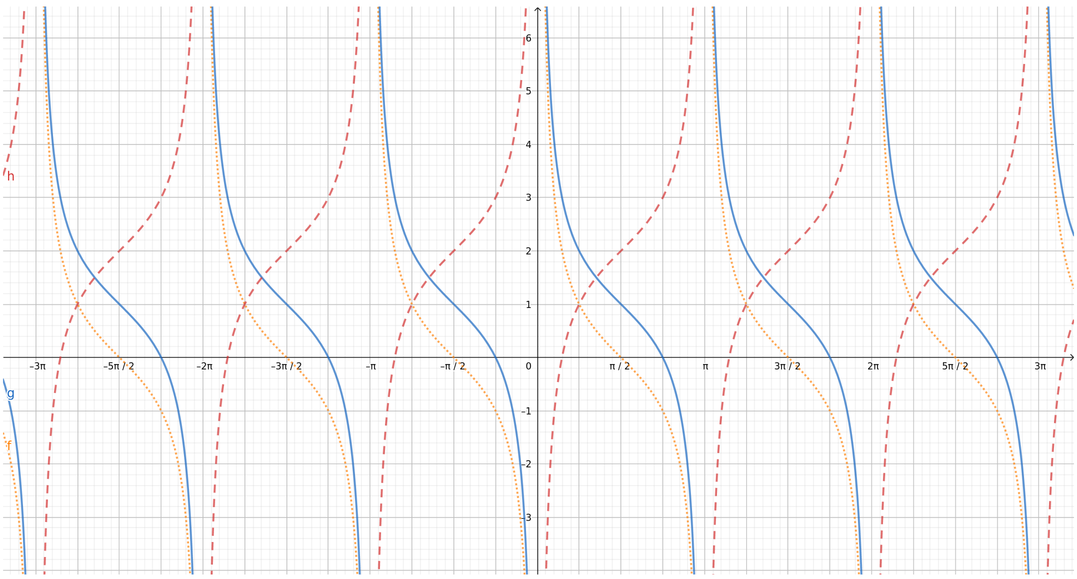
   (a) $\tan x$ (b) $\tan\left(x+\frac{\pi}{2}\right)$ (c) $\tan\left(x-\frac{\pi}{4}\right)$



Solution: `https://www.geogebra.org/calculator/c4wycc7g`

5. Find the graph of

   (a) $\cot x$ (b) $1+\cot x$ (c) $2-\cot x$



Solution: `https://www.geogebra.org/calculator/n57s6qpa`

6. **Solution:**

   (a) $\sin(x + \frac{\pi}{2}) = \cos x$

   (b) $\sin(x - \frac{\pi}{2}) = -\cos(x)$

   (c) $\cos(x + \frac{\pi}{2}) = -\sin x$

   (d) $\cos(x - \frac{\pi}{2}) = \sin x$

7. Sketch the following graphs (at first, try without computer):

   (a) $3\cos(2x - \frac{\pi}{3}) = 3\cos(2(x - \frac{\pi}{6}))$

   (b) $\sin(2\pi x)$

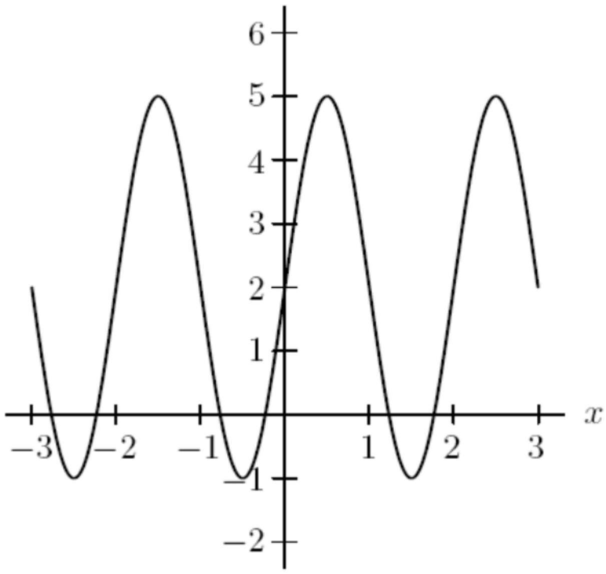
   (c) $-\frac{1}{2}\cot(2x)$

   (d) $\tan\,(\frac{\pi}{6} - \frac{x}{2})$

   Solution: `https://www.geogebra.org/calculator/ayfbdtpc`

   Source of a lot of the following questions: ***http://mathquest.carroll.edu/libraries/PRE.student.01.05.pdf***

8. Find the formula



   A $3\sin(2x) + 2$

   B $3\cos(2x) + 2$

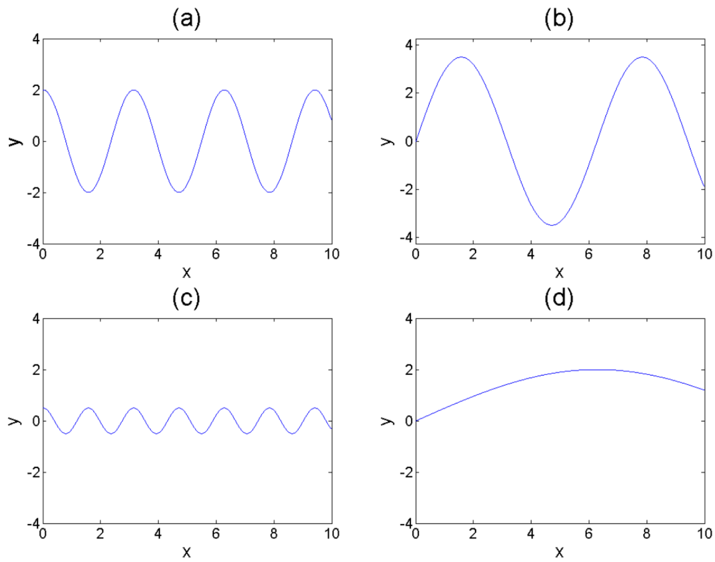
   C $3\sin(\pi x) + 2$

   D $3\cos(\pi x) + 2$

   E $3\sin\left(\frac{x}{\pi}\right) + 2$

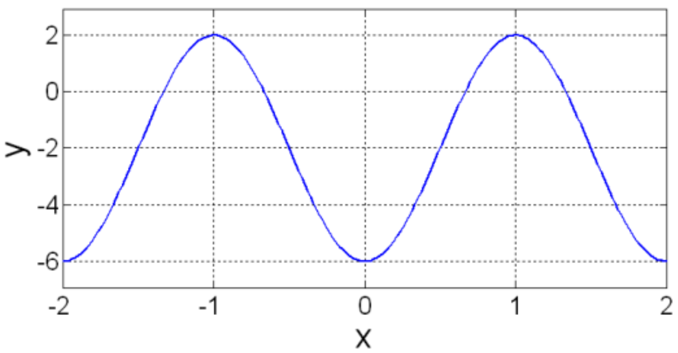
   C

9. There is a function of the form $y = A\sin(Bx + C)$, where $A, B, C \in \mathbb{R}$. Which function has the largest value of B?



C

10. Find the formulae



A $4\sin\left(\pi x - \frac{\pi}{2}\right) - 2$

B $-4\sin\left(\pi x + \frac{\pi}{2}\right) - 2$

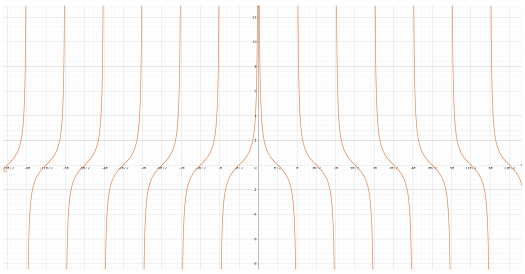
C $-4\cos\left(\pi x\right) - 2$

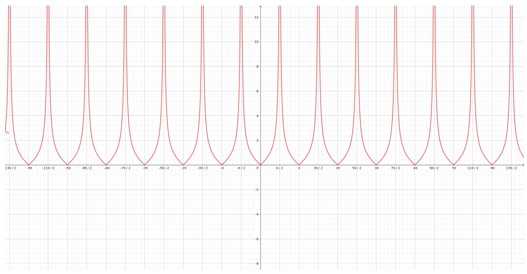
D $4\cos\left(\pi x + \pi\right) - 2$

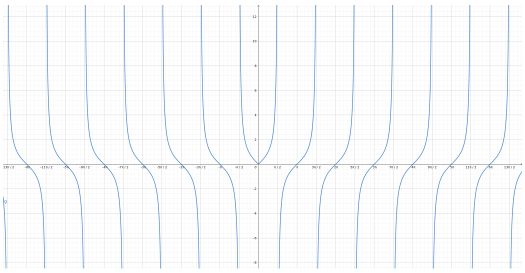
A, B, C, D

11. Find the formulae:

A $\tan|x|$ B $|\tan x|$ C $\cot|x|$ D $|\cot x|$







C, B, A, D

The coordinates of the line representing $\frac{8\pi}{3}$ will be the same as the coordinates of the line representing $\frac{\pi}{3}$ except that the *x* coordinate will now be negative. So, our new coordinates will then be $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ and so the answer is,

$$\cos\left(\frac{8\pi}{3}\right)=-\frac{1}{2}$$

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8. Determine the exact value of $\tan\left(-\frac{\pi}{3}\right)$ without using a calculator.

Hint 1 : Even though a unit circle only tells us information about sine and cosine it is still useful for tangents so sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1

To do this problem all we need to notice is that $-\frac{\pi}{3}$ will form an angle of $\frac{\pi}{3}$ with the positive *x*-axis in the fourth quadrant and we'll have the following unit circle for this problem.