6.6 Finding Rational Zeros

GOAL 1 USING THE RATIONAL ZERO THEOREM

What *you should learn*

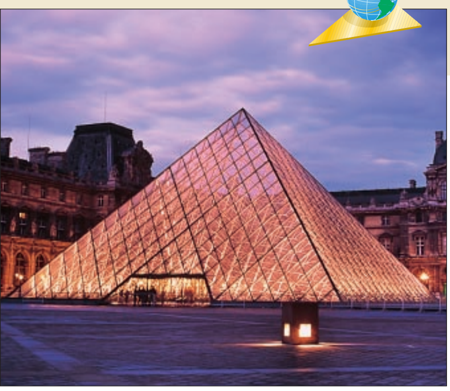
GOAL 1 Find the rational zeros of a polynomial function.

GOAL 2 Use polynomial equations to solve **real-life** problems, such as finding the dimensions of a monument in **Ex. 60**.

Why *you should learn it*

▼ To model **real-life** quantities, such as the volume of a representation of the Louvre pyramid in **Example 3**.

REAL LIFE



The polynomial function

$$f(x) = 64x^3 + 120x^2 - 34x - 105$$

has $-\frac{3}{2}$, $-\frac{5}{4}$, and $\frac{7}{8}$ as its zeros. Notice that the numerators of these zeros (−3, −5, and 7) are factors of the constant term, −105. Also notice that the denominators (2, 4, and 8) are factors of the leading coefficient, 64. These observations are generalized by the *rational zero theorem.*

THE RATIONAL ZERO THEOREM

If $f(x) = a_nx^n + \cdots + a_1x + a_0$ has *integer* coefficients, then every rational zero of f has the following form:

$$\frac{p}{q} = \frac{\text{factor of constant term } a_0}{\text{factor of leading coefficient } a_n}$$

EXAMPLE 1 *Using the Rational Zero Theorem*

Find the rational zeros of $f(x) = x^3 + 2x^2 - 11x - 12$.

SOLUTION

List the possible rational zeros. The leading coefficient is 1 and the constant term is −12. So, the possible rational zeros are:

$$x = \pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{4}{1}, \pm\frac{6}{1}, \pm\frac{12}{1}$$

Test these zeros using synthetic division.

Test $x = 1$:

1	1	2	−11	−12
		1	3	−8
	1	3	−8	−20

Test $x = -1$:

−1	1	2	−11	−12
		−1	−1	12
	1	1	−12	0

Since −1 is a zero of f, you can write the following:

$$f(x) = (x + 1)(x^2 + x - 12)$$

Factor the trinomial and use the factor theorem.

$$f(x) = (x + 1)(x^2 + x - 12) = (x + 1)(x - 3)(x + 4)$$

▶ The zeros of f are −1, 3, and −4.

In Example 1, the leading coefficient is 1. When the leading coefficient is not 1, the list of possible rational zeros can increase dramatically. In such cases the search can be shortened by sketching the function's graph—either by hand or by using a graphing calculator.

EXAMPLE 2 *Using the Rational Zero Theorem*

3b

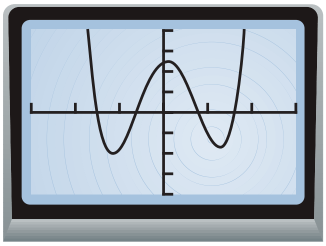
Find all real zeros of $f(x) = 10x^4 - 3x^3 - 29x^2 + 5x + 12$.

SOLUTION

List the possible rational zeros of f: $\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{4}{1}, \pm\frac{6}{1}, \pm\frac{12}{1}, \pm\frac{3}{2}, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{3}{5}, \pm\frac{6}{5}, \pm\frac{12}{5}, \pm\frac{1}{10}, \pm\frac{3}{10}, \pm\frac{12}{10}$.

Choose values to check.

With so many possibilities, it is worth your time to sketch the graph of the function. From the graph, it appears that some reasonable choices are $x = -\frac{3}{2}$, $x = -\frac{3}{5}$, $x = \frac{4}{5}$, and $x = \frac{3}{2}$.



Check the chosen values using synthetic division.

$-\frac{3}{2}$	10	−3	−29	5	12
		−15	27	3	−12
	10	−18	−2	8	0

← $-\frac{3}{2}$ **is a zero.**

Factor out a binomial using the result of the synthetic division.

$$f(x) = \left(x + \frac{3}{2}\right)(10x^3 - 18x^2 - 2x + 8)$$ **Rewrite as a product of two factors.**

$$= \left(x + \frac{3}{2}\right)(2)(5x^3 - 9x^2 - x + 4)$$ **Factor 2 out of the second factor.**

$$= (2x + 3)(5x^3 - 9x^2 - x + 4)$$ **Multiply the first factor by 2.**

Repeat the steps above for $g(x) = 5x^3 - 9x^2 - x + 4$.

Any zero of g will also be a zero of f. The possible *rational* zeros of g are $x = \pm 1, \pm 2, \pm 4, \pm\frac{1}{5}, \pm\frac{2}{5}$, and $\pm\frac{4}{5}$. The graph of f shows that $\frac{4}{5}$ may be a zero.

$\frac{4}{5}$	5	−9	−1	4
		4	−4	−4
	5	−5	−5	0

← $\frac{4}{5}$ **is a zero.**

So $f(x) = (2x + 3)\left(x - \frac{4}{5}\right)(5x^2 - 5x - 5) = (2x + 3)(5x - 4)(x^2 - x - 1)$.

Find the remaining zeros of f by using the quadratic formula to solve $x^2 - x - 1 = 0$.

▶ The real zeros of f are $-\frac{3}{2}, \frac{4}{5}, \frac{1 + \sqrt{5}}{2}$, and $\frac{1 - \sqrt{5}}{2}$.

Section 5-4 : Finding Zeroes of Polynomials

1. Find all the zeroes of the following polynomial.

$$f\left(x\right)=2x^3-13x^2+3x+18$$

Step 1

We'll need all the factors of 18 and 2.

$$\begin{array}{ll} 18: & \pm1,\pm2,\pm3,\pm6,\pm9,\pm18 \\ 2: & \pm1,\,\pm2 \end{array}$$

Step 2

Here is a list of all possible rational zeroes for the polynomial.

$$\frac{\pm1}{\pm1}=\pm1 \qquad \frac{\pm2}{\pm1}=\pm2 \qquad \frac{\pm3}{\pm1}=\pm3 \qquad \frac{\pm6}{\pm1}=\pm6 \qquad \frac{\pm9}{\pm1}=\pm9 \qquad \frac{\pm18}{\pm1}=\pm18$$

$$\frac{\pm1}{\pm2}=\frac{\pm1}{\pm2} \qquad \frac{\pm2}{\pm2}=\pm1 \qquad \frac{\pm3}{\pm2}=\frac{\pm3}{\pm2} \qquad \frac{\pm6}{\pm2}=\pm3 \qquad \frac{\pm9}{\pm2}=\frac{\pm9}{\pm2} \qquad \frac{\pm18}{\pm2}=\pm9$$

So, we have a total of 18 possible zeroes for the polynomial.

Step 3

We now need to start the synthetic division work. We'll start with the "small" integers first.

$$\begin{array}{r|rrrr} & 2 & -13 & 3 & 18 \\ \hline -1 & 2 & -15 & 18 & 0 \end{array} \quad = f\left(-1\right)=0!!$$

Okay we now know that $x=-1$ is a zero and we can write the polynomial as,

$$f\left(x\right)=2x^3-13x^2+3x+18=\left(x+1\right)\left(2x^2-15x+18\right)$$

Step 4

We could continue with this process however, we have a quadratic for the second factor and we can just factor this so the fully factored form of the polynomial is,

$$f\left(x\right)=2x^3-13x^2+3x+18=\left(x+1\right)\left(2x-3\right)\left(x-6\right)$$

Step 5

From the fully factored form we get the following set of zeroes for the original polynomial.

$$\boxed{x = -1 \qquad\qquad x = \frac{3}{2} \qquad\qquad x = 6}$$

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2. Find all the zeroes of the following polynomial.

$$P(x) = x^4 - 3x^3 - 5x^2 + 3x + 4$$

Step 1
We'll need all the factors of 4 and 1.

$$\begin{array}{ll} 4: & \pm 1, \pm 2, \pm 4 \\ 1: & \pm 1 \end{array}$$

Step 2
Here is a list of all possible rational zeroes for the polynomial.

$$\frac{\pm 1}{\pm 1} = \pm 1 \qquad \frac{\pm 2}{\pm 1} = \pm 2 \qquad \frac{\pm 4}{\pm 1} = \pm 4$$

So, we have a total of 6 possible zeroes for the polynomial.

Step 3
We now need to start the synthetic division work. We'll start with the "small" integers first.

$$\begin{array}{r|rrrrr} & 1 & -3 & -5 & 3 & 4 \\ \hline -1 & 1 & -4 & -1 & 4 & 0 \end{array} \; = P(-1) = 0!!$$

Okay we now know that $x = -1$ is a zero and we can write the polynomial as,

$$P(x) = x^4 - 3x^3 - 5x^2 + 3x + 4 = (x+1)(x^3 - 4x^2 - x + 4)$$

Step 4
So, now we need to continue the process using $Q(x) = x^3 - 4x^2 - x + 4$. The possible zeroes of this polynomial are the same as the original polynomial and so we won't write them back down.

Here's the synthetic division work for this $Q(x)$.

$$\begin{array}{r|rrrr} & 1 & -4 & -1 & 4 \\ \hline -1 & 1 & -5 & 4 & 0 \end{array} \; = Q(-1) = 0!!$$

Therefore, $x = -1$ is also a zero of $Q(x)$ and the factored form of $Q(x)$ is,

$$Q(x) = x^3 - 4x^2 - x + 4 = (x+1)(x^2 - 5x + 4)$$

This also means that the factored form of the original polynomial is now,

$$P(x) = x^4 - 3x^3 - 5x^2 + 3x + 4 = (x+1)(x+1)(x^2 - 5x + 4) = (x+1)^2(x^2 - 5x + 4)$$

Step 5

We're down to a quadratic polynomial and so we can and we can just factor this to get the fully factored form of the original polynomial. This is,

$$P(x) = x^4 - 3x^3 - 5x^2 + 3x + 4 = (x+1)^2(x-4)(x-1)$$

Step 6

From the fully factored form we get the following set of zeroes for the original polynomial.

$$\boxed{x = -1 \text{ (multiplicity 2)} \qquad x = 1 \qquad x = 4}$$

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3. Find all the zeroes of the following polynomial.

$$A(x) = 2x^4 - 7x^3 - 2x^2 + 28x - 24$$

Step 1

We'll need all the factors of -24 and 2.

$$\begin{array}{ll} -24: & \pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24 \\ 2: & \pm1, \pm2 \end{array}$$

Step 2

Here is a list of all possible rational zeroes for the polynomial.

$$\frac{\pm1}{\pm1} = \pm1 \qquad \frac{\pm2}{\pm1} = \pm2 \qquad \frac{\pm3}{\pm1} = \pm3 \qquad \frac{\pm4}{\pm1} = \pm4 \qquad \frac{\pm6}{\pm1} = \pm6 \qquad \frac{\pm8}{\pm1} = \pm8$$

$$\frac{\pm12}{\pm1} = \pm12 \qquad \frac{\pm24}{\pm1} = \pm24$$

$$\frac{\pm1}{\pm2} = \frac{\pm1}{\pm2} \qquad \frac{\pm2}{\pm2} = \pm1 \qquad \frac{\pm3}{\pm2} = \frac{\pm3}{\pm2} \qquad \frac{\pm4}{\pm2} = \pm2 \qquad \frac{\pm6}{\pm2} = \pm3 \qquad \frac{\pm8}{\pm2} = \pm4$$

$$\frac{\pm12}{\pm2} = \pm6 \qquad \frac{\pm24}{\pm2} = \pm12$$

So, we have a total of 20 possible zeroes for the polynomial.

Step 3
We now need to start the synthetic division work. We'll start with the "small" integers first.

$$\begin{array}{r|rrrrr l} & 2 & -7 & -2 & 28 & -24 & \\ \hline -1 & 2 & -9 & 7 & 21 & -45 & = A(-1) \neq 0 \\ 1 & 2 & -5 & -7 & 21 & -3 & = A(1) \neq 0 \\ -2 & 2 & -11 & 20 & -12 & 0 & = A(-2) = 0!! \end{array}$$

Okay we now know that $x = -2$ is a zero and we can write the polynomial as,

$$A(x) = 2x^4 - 7x^3 - 2x^2 + 28x - 24 = (x+2)(2x^3 - 11x^2 + 20x - 12)$$

Step 4
So, now we need to continue the process using $Q(x) = 2x^3 - 11x^2 + 20x - 12$. Here is a list of all possible zeroes of $Q(x)$.

$$\frac{\pm 1}{\pm 1} = \pm 1 \qquad \frac{\pm 2}{\pm 1} = \pm 2 \qquad \frac{\pm 3}{\pm 1} = \pm 3 \qquad \frac{\pm 4}{\pm 1} = \pm 4 \qquad \frac{\pm 6}{\pm 1} = \pm 6 \qquad \frac{\pm 12}{\pm 1} = \pm 12$$

$$\frac{\pm 1}{\pm 2} = \frac{\pm 1}{\pm 2} \qquad \frac{\pm 2}{\pm 2} = \pm 1 \qquad \frac{\pm 3}{\pm 2} = \frac{\pm 3}{\pm 2} \qquad \frac{\pm 4}{\pm 2} = \pm 2 \qquad \frac{\pm 6}{\pm 2} = \pm 3 \qquad \frac{\pm 12}{\pm 2} = \pm 6$$

So we have a list of 16 possible zeroes, but note that we've already proved that ± 1 can't be zeroes of the original polynomial and so can't be zeroes of $Q(x)$ either.

Here's the synthetic division work for this $Q(x)$.

$$\begin{array}{r|rrrr l} & 2 & -11 & 20 & -12 & \\ \hline -2 & 2 & -15 & 50 & -112 & = Q(-2) \neq 0 \\ 2 & 2 & -7 & 6 & 0 & = Q(2) = 0!! \end{array}$$

Therefore, $x = 2$ is a zero of $Q(x)$ and the factored form of $Q(x)$ is,

$$Q(x) = 2x^3 - 11x^2 + 20x - 12 = (x-2)(2x^2 - 7x + 6)$$

This also means that the factored form of the original polynomial is now,

$$A(x) = 2x^4 - 7x^3 - 2x^2 + 28x - 24 = (x+2)(x-2)(2x^2 - 7x + 6)$$

Step 5
We're down to a quadratic polynomial and so we can and we can just factor this to get the fully factored form of the original polynomial. This is,

$$A\left(x\right)=2x^4-7x^3-2x^2+28x-24=\left(x+2\right)\left(x-2\right)\left(x-2\right)\left(2x-3\right)=\left(x+2\right)\left(x-2\right)^2\left(2x-3\right)$$

Step 6
From the fully factored form we get the following set of zeroes for the original polynomial.

$$\boxed{x=-2 \qquad x=\frac{3}{2} \qquad x=2\ \ (\text{multiplicity } 2)}$$

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4. Find all the zeroes of the following polynomial.

$$g\left(x\right)=8x^5+36x^4+46x^3+7x^2-12x-4$$

Step 1
We'll need all the factors of -4 and 8.

$$\begin{array}{ll} -4: & \pm1,\pm2,\pm4 \\ 8: & \pm1,\pm2,\pm4,\pm8 \end{array}$$

Step 2
Here is a list of all possible rational zeroes for the polynomial.

$$\begin{array}{lll} \frac{\pm1}{\pm1}=\pm1 & \frac{\pm2}{\pm1}=\pm2 & \frac{\pm4}{\pm1}=\pm4 \\ \frac{\pm1}{\pm2}=\frac{\pm1}{\pm2} & \frac{\pm2}{\pm2}=\pm1 & \frac{\pm4}{\pm2}=\pm2 \\ \frac{\pm1}{\pm4}=\frac{\pm1}{\pm4} & \frac{\pm2}{\pm4}=\frac{\pm1}{\pm2} & \frac{\pm4}{\pm4}=\pm1 \\ \frac{\pm1}{\pm8}=\frac{\pm1}{\pm8} & \frac{\pm2}{\pm8}=\frac{\pm1}{\pm4} & \frac{\pm4}{\pm8}=\frac{\pm1}{\pm2} \end{array}$$

So, we have a total of 12 possible zeroes for the polynomial.

Step 3

We now need to start the synthetic division work. We'll start with the "small" integers first.

$$\begin{array}{r|rrrrrr} & 8 & 36 & 46 & 7 & -12 & -4 \\ \hline -1 & 8 & 28 & 18 & -11 & -1 & -3 \\ 1 & 8 & 44 & 90 & 97 & 85 & 81 \end{array} \begin{array}{l} \\ = g(-1) \neq 0 \\ = g(1) \neq 0 \end{array}$$

Okay, notice that we have opposite signs for the two function evaluations listed above. Recall that means that we know we have a zero somewhere between them.

So, let's take a look at some of the fractions from our list and give them a try in the synthetic division table. We'll start with the fractions with the smallest denominators.

$$\begin{array}{r|rrrrrr} & 8 & 36 & 46 & 7 & -12 & -4 \\ \hline -\frac{1}{2} & 8 & 32 & 30 & -8 & -8 & 0 \end{array} = g(-1) = 0!!$$

We now know that $x = -\frac{1}{2}$ is a zero and we can write the polynomial as,

$$g(x) = 8x^5 + 36x^4 + 46x^3 + 7x^2 - 12x - 4 = \left(x + \tfrac{1}{2}\right)\left(8x^4 + 32x^3 + 30x^2 - 8x - 8\right)$$

Step 4

So, now we need to continue the process using $Q(x) = 8x^4 + 32x^3 + 30x^2 - 8x - 8$. Here is a list of all possible zeroes of $Q(x)$.

$$\frac{\pm 1}{\pm 1} = \pm 1 \qquad \frac{\pm 2}{\pm 1} = \pm 2 \qquad \frac{\pm 4}{\pm 1} = \pm 4 \qquad \frac{\pm 8}{\pm 1} = \pm 8$$

$$\frac{\pm 1}{\pm 2} = \frac{\pm 1}{\pm 2} \qquad \frac{\pm 2}{\pm 2} = \pm 1 \qquad \frac{\pm 4}{\pm 2} = \pm 2 \qquad \frac{\pm 8}{\pm 2} = \pm 4$$

$$\frac{\pm 1}{\pm 4} = \frac{\pm 1}{\pm 4} \qquad \frac{\pm 2}{\pm 4} = \frac{\pm 1}{\pm 2} \qquad \frac{\pm 4}{\pm 4} = \pm 1 \qquad \frac{\pm 8}{\pm 4} = \pm 2$$

$$\frac{\pm 1}{\pm 8} = \frac{\pm 1}{\pm 8} \qquad \frac{\pm 2}{\pm 8} = \frac{\pm 1}{\pm 4} \qquad \frac{\pm 4}{\pm 8} = \frac{\pm 1}{\pm 2} \qquad \frac{\pm 8}{\pm 8} = \pm 1$$

So we have a list of 14 possible zeroes (lots of repeats), but note that we've already proved that ± 1 can't be zeroes of the original polynomial and so can't be zeroes of $Q(x)$ either.

Here's the synthetic division work for this $Q(x)$.

$$\begin{array}{r|rrrrr} & 8 & 32 & 30 & -8 & -8 \\ \hline -2 & 8 & 16 & -2 & -4 & 0 \end{array} = Q(-2) = 0!!$$

Therefore, $x=-2$ is a zero of $Q(x)$ and the factored form of $Q(x)$ is,

$$Q(x) = 8x^4 + 32x^3 + 30x^2 - 8x - 8 = (x+2)(8x^3 + 16x^2 - 2x - 4)$$

This also means that the factored form of the original polynomial is now,

$$g(x) = 8x^5 + 36x^4 + 46x^3 + 7x^2 - 12x - 4 = (x + \tfrac{1}{2})(x+2)(8x^3 + 16x^2 - 2x - 4)$$

Step 5

So, it looks like we need to continue with the synthetic division. This time we'll do it on the polynomial $P(x) = 8x^3 + 16x^2 - 2x - 4$.

The possible zeroes of this are the same as the original polynomial and so we won't write them down here. Again, however, we've already proved that ± 1 can't be zeroes of the original polynomial and so can't be zeroes of $P(x)$ either.

Here is the synthetic division for $P(x)$.

$$\begin{array}{r|rrrr} & 8 & 16 & -2 & -4 \\ \hline -2 & 8 & 0 & -2 & 0 \end{array} = P(-2) = 0!!$$

Therefore, $x=-2$ is a zero of $P(x)$ and the factored form of $P(x)$ is,

$$P(x) = 8x^3 + 16x^2 - 2x - 4 = (x+2)(8x^2 - 2) = 8(x+2)(x^2 - \tfrac{1}{4})$$

We factored an 8 out of the quadratic to make it a little easier for the next step.

The factored form of the original polynomial is now,

$$g(x) = 8x^5 + 36x^4 + 46x^3 + 7x^2 - 12x - 4 = 8(x + \tfrac{1}{2})(x+2)^2(x^2 - \tfrac{1}{4})$$

Step 6

We're down to a quadratic polynomial and so we can and we can just factor this to get the fully factored form of the original polynomial. This is,

$$g(x) = 8x^5 + 36x^4 + 46x^3 + 7x^2 - 12x - 4 = 8\left(x+\tfrac{1}{2}\right)\left(x+2\right)^2\left(x+\tfrac{1}{2}\right)\left(x-\tfrac{1}{2}\right)$$
$$= 8\left(x+\tfrac{1}{2}\right)^2\left(x+2\right)^2\left(x-\tfrac{1}{2}\right)$$

Step 7
From the fully factored form we get the following set of zeroes for the original polynomial.

$$\boxed{x = -\frac{1}{2} \text{ (multiplicity 2)} \qquad x = -2 \text{ (multiplicity 2)} \qquad x = \frac{1}{2}}$$

Note that this problem was VERY long and messy. The point of this problem was really just to illustrate just how long and messy these can get. The moral, if there is one, is that we generally sit back and really hope that we don't have to work these kinds of problems on a regular basis. They are just too long and it's too easy to make a mistake with them.

6th lesson

https://www2.karlin.mff.cuni.cz/~kuncova/en/teachIM.php, kunck6am@natur.cuni.cz

Theory

Věta 1. Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ be a function with $a_i \in \mathbb{Z}$, $i = 0, \ldots, n$. Then every rational root of f is of the following form

$$\frac{p}{q} = \pm \frac{\text{factor of } a_0}{\text{factor of } a_n}.$$

Exercises

1. Find all roots and factor polynomials

   (a) $x^3 + 2x^2 - 11x - 12$

   (b) $10x^4 - 3x^3 - 29x^2 + 5x + 12$

   (c) $2x^3 - 13x^2 + 3x + 18$

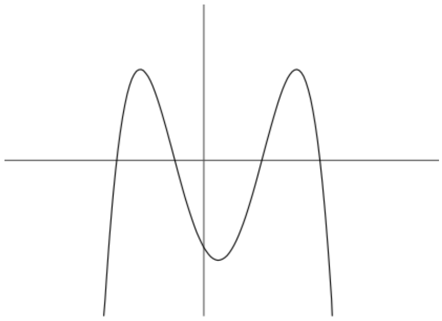
   (d) $x^4 - 3x^3 - 5x^2 + 3x + 4$

   (e) $2x^4 - 7x^3 - 2x^2 + 28x - 24$

   (f) $8x^5 + 36x^4 + 46x^3 + 7x^2 - 12x - 4$

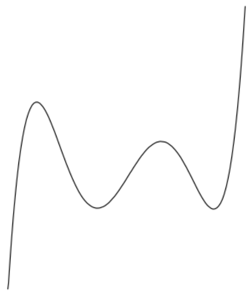
Main source: `http://mathquest.carroll.edu/precalc.html`

2. The sketched polynomial is of:
   (a) odd degree, lead coefficient negative
   (b) odd degree, lead coefficient positive
   (c) even degree, lead coefficient negative
   (d) even degree, lead coefficient positive



Solution: (c)

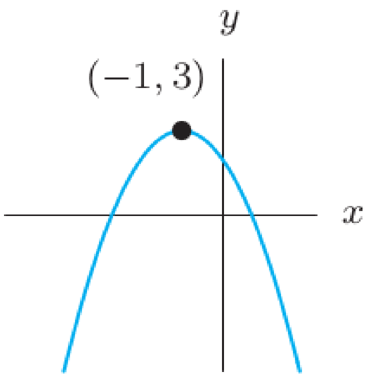
3. The sketched polynomial is of:
   (a) odd degree, lead coefficient negative
   (b) odd degree, lead coefficient positive
   (c) even degree, lead coefficient negative
   (d) even degree, lead coefficient positive



Solution: (b)

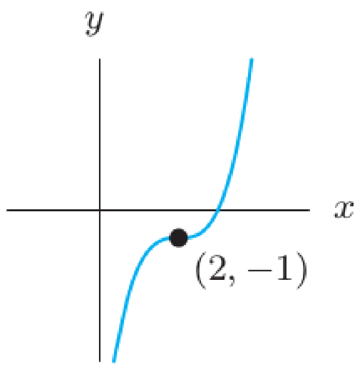
Source for 4,5: Calculus: Single and Multivariable, 6th Edition, Deborah Hughes-Hallett, Andrew M. Gleason, William G. McCallum

4. The sketched polynomial is:
   (a) $(x-1)^2+3$
   (b) $-(x+3)^2-1$
   (c) $(x-3)^2+1$
   (d) $(x+3)^2-1$
   (e) $-(x+1)^2+3$



Solution: (e)

5. The sketched polynomial is:



   (a) $(x-2)^3 - 1$
   (b) $(x+2)^3 - 1$
   (c) $(x+2)^3 + 1$
   (d) $(x-2)^3 + 1$
   (e) $-(2-x)^3 - 1$

   Solution: (a), (e)

6. What is the degree of the polynomial $y = x(2x+1)^3(x-4)^2(5-x)^5$?

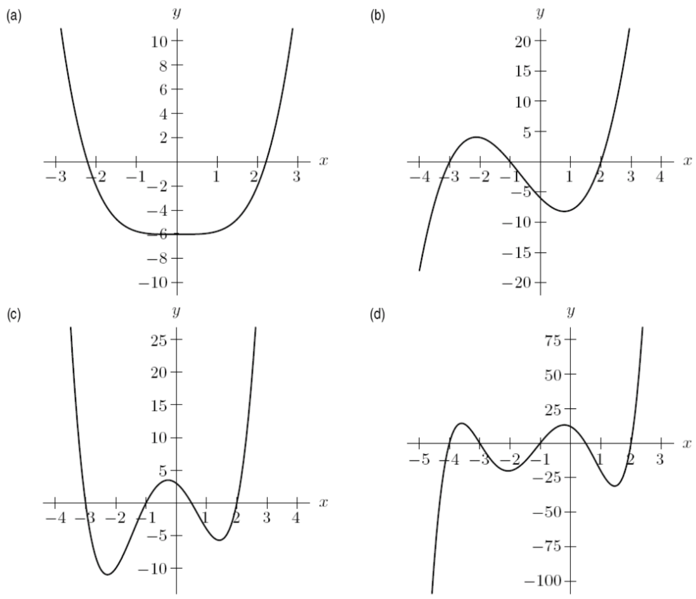
   Solution: 11

7. Find the polynomial with the smallest possible degree, with zeros at $x = 1$, $x = 2$ and $x = 3$ such that $f(5) = 8$.

   (a) $(x-1)(x-2)(x-3)$
   (b) $(x-1)(x-2)(x-3)(x-5)$
   (c) $8(x-1)(x-2)(x-3)$
   (d) $8(x-1)(x-2)(x-3)(x-5)$
   (e) $\frac{1}{3}(x-1)(x-2)(x-3)$
   (f) $\frac{1}{42}(x-1)(x-2)(x-3)$

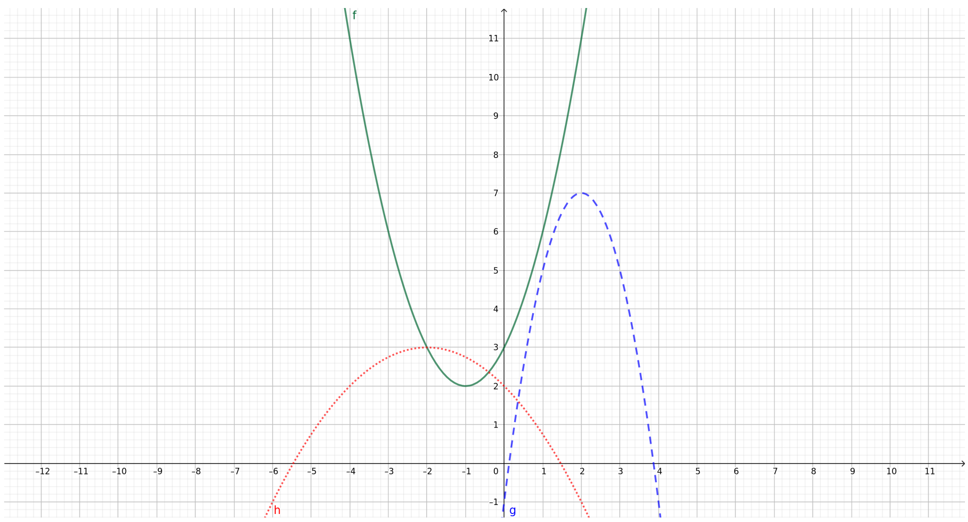
   Solution: (e)

8. Find the graph of the function $y = x^3 + 2x^2 - 5x - 6$



Solution: (b)

9. Find the formula for the quadratic functions:



Solution: green: $(x+1)^2+2$, red dotted: $\frac{-(x+2)^2+12}{4}$, blue dashed: $-2(x-2)^2+7$

10. Decide

TRUE If $f(x)$ is a polynomial such that $f(c) = 0$ for $c \in \mathbb{R}$, then $f(x)$ can be written as $(x - c)g(x)$ for some polynomial $g(x)$.

FALSE A polynomial function may have a horizontal asymptote.

FALSE A polynomial function may have a vertical asymptote.

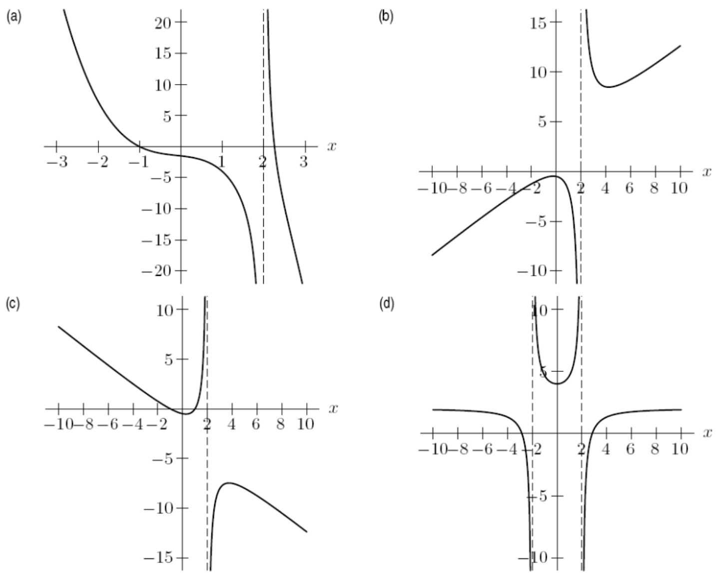
FALSE For $x \in \mathbb{R}$ we have: $x \leq x^2$.

FALSE Every polynomial of even degree is an odd function and every polynomial of odd degree is even function.

FALSE Every polynomial of even degree is an even function and every polynomial of odd degree is odd function.

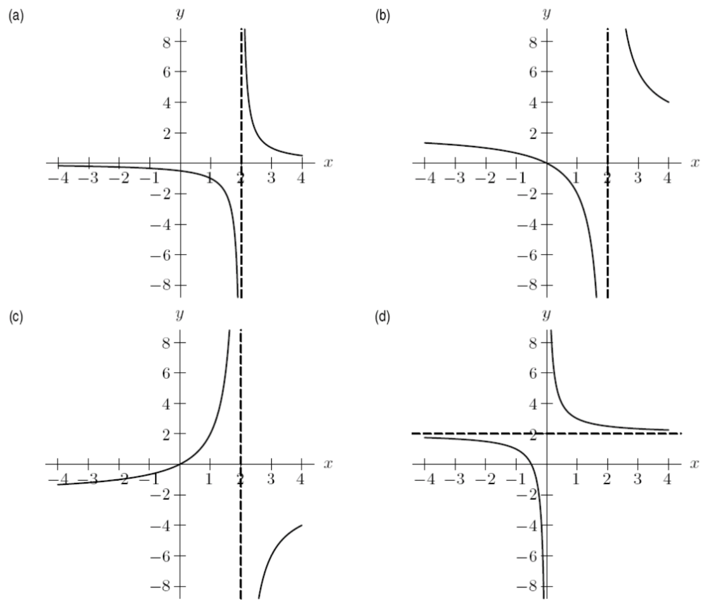
FALSE Let $f(x) = \frac{x^2-1}{x+1}$, $g(x) = x - 1$. Then $f(x) = g(x)$.

11. Find the graph of the function $y = \frac{1-x^2}{x-2}$



Solution: (c)

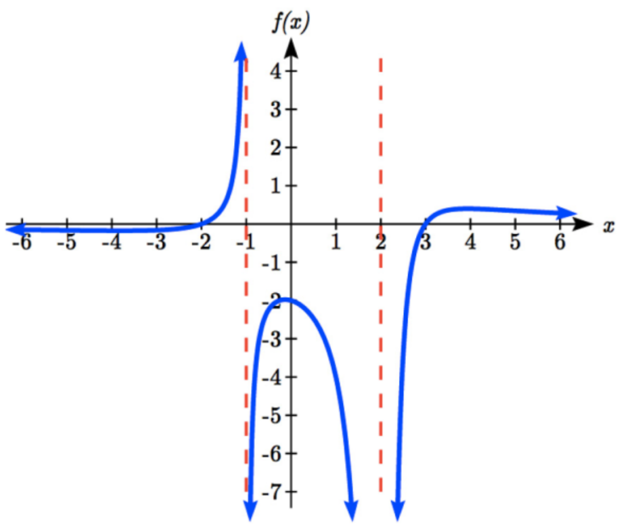
12. Find the graph of the function $y = \frac{2x}{x-2}$



Solution: (c)

Source:`http://www.opentextbookstore.com/precalc/2/Precalc3-7.pdf`

13. Find the possible formulas for graphed functions.



$$\frac{4(x+2)(x-3)}{3(x+1)(x-2)^2}$$

Try it Now

5. Given the function $f(x)=\dfrac{(x+2)^2(x-2)}{2(x-1)^2(x-3)}$, use the characteristics of polynomials and rational functions to describe its behavior and sketch the function.

Since a rational function written in factored form will have a horizontal intercept where each factor of the numerator is equal to zero, we can form a numerator that will pass through a set of horizontal intercepts by introducing a corresponding set of factors. Likewise, since the function will have a vertical asymptote where each factor of the denominator is equal to zero, we can form a denominator that will produce the vertical asymptotes by introducing a corresponding set of factors.

Writing Rational Functions from Intercepts and Asymptotes

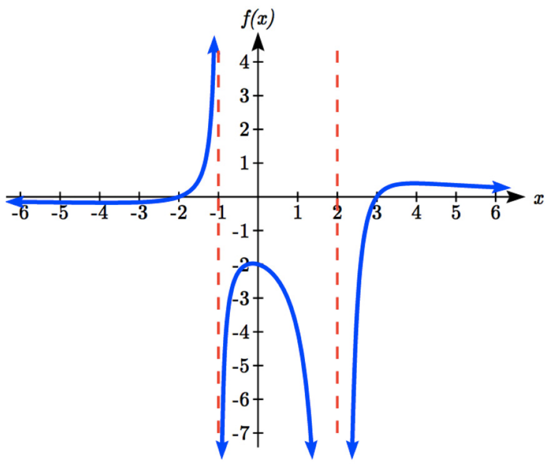
If a rational function has horizontal intercepts at $x=x_1, x_2, \ldots, x_n$, and vertical asymptotes at $x=v_1, v_2, \ldots, v_m$ then the function can be written in the form

$$f(x)=a\frac{(x-x_1)^{p_1}(x-x_2)^{p_2}\cdots(x-x_n)^{p_n}}{(x-v_1)^{q_1}(x-v_2)^{q_2}\cdots(x-v_m)^{q_n}}$$

where the powers p_i or q_i on each factor can be determined by the behavior of the graph at the corresponding intercept or asymptote, and the stretch factor a can be determined given a value of the function other than the horizontal intercept, or by the horizontal asymptote if it is nonzero.

Example 10

Write an equation for the rational function graphed here.



The graph appears to have horizontal intercepts at $x = -2$ and $x = 3$. At both, the graph passes through the intercept, suggesting linear factors.

The graph has two vertical asymptotes. The one at $x = -1$ seems to exhibit the basic behavior similar to $\dfrac{1}{x}$, with the graph heading toward positive infinity on one side and heading toward negative infinity on the other.

The asymptote at $x = 2$ is exhibiting a behavior similar to $\frac{1}{x^2}$, with the graph heading toward negative infinity on both sides of the asymptote.

Utilizing this information indicates an function of the form
$$f(x) = a\frac{(x+2)(x-3)}{(x+1)(x-2)^2}$$

To find the stretch factor, we can use another clear point on the graph, such as the vertical intercept (0,-2):
$$-2 = a\frac{(0+2)(0-3)}{(0+1)(0-2)^2}$$
$$-2 = a\frac{-6}{4}$$
$$a = \frac{-8}{-6} = \frac{4}{3}$$

This gives us a final function of $f(x) = \frac{4(x+2)(x-3)}{3(x+1)(x-2)^2}$

Oblique Asymptotes

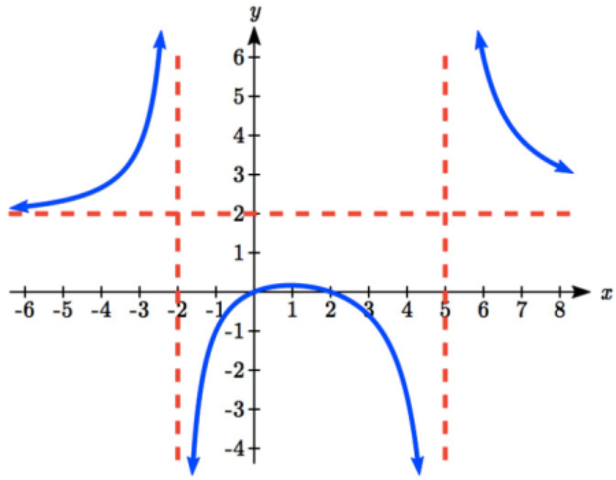
Earlier we saw graphs of rational functions that had no horizontal asymptote, which occurs when the degree of the numerator is larger than the degree of the denominator. We can, however, describe in more detail the long-run behavior of a rational function.

Example 11

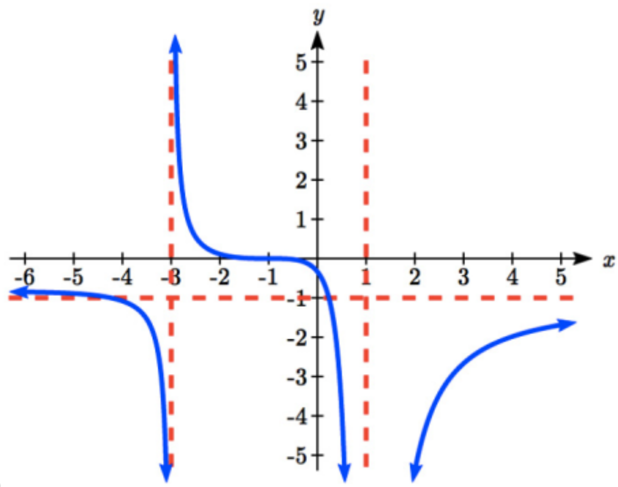
Describe the long-run behavior of $f(x) = \frac{3x^2+2}{x-5}$

Earlier we explored this function when discussing horizontal asymptotes. We found the long-run behavior is $f(x) \approx \frac{3x^2}{x} = 3x$, meaning that $x \to \pm\infty$, $f(x) \to \pm\infty$, respectively, and there is no horizontal asymptote.

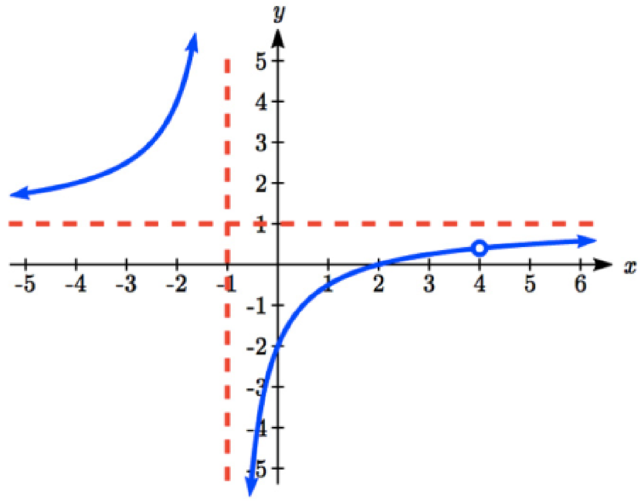
If we were to do polynomial long division, we could get a better understanding of the behavior as $x \to \pm\infty$.



$$\frac{2x(x-2)}{(x+2)(x-5)}$$



$$\frac{-(x+1)^3}{(x+3)(x-1)^2}$$



$$\frac{(x-2)(x-4)}{(x+1)(x-4)}$$