5th lesson

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Theory

Věta 1. Let $f(x) = a_n x^n + \cdots + a_1 x + a_0$ be a function with $a_i \in \mathbb{Z}$, $i = 0, \ldots, n$. Then every rational root of f is of the following form

$$\frac{p}{q} = \pm \frac{\text{factor of } a_0}{\text{factor of } a_n}.$$

Exercises

1. Divide

(a) $\dfrac{x^2 - 4x - 12}{x - 6}$

(b) $\dfrac{2x^3 - 3x^2 - 19x + 30}{x - 3}$

(c) $\dfrac{2x^3 + 8x^2 - 3x - 12}{x + 4}$

(d) $\dfrac{2x^3 - 8x^2 + 9x - 2}{x - 2}$

(e) $\dfrac{8x^3 + 14x + 8}{2x + 1}$

(f) $\dfrac{27x^3 + 9x^2 - 3x - 10}{3x - 2}$

(g) $\dfrac{4x^3 - 8x^2 - 3x + 1}{x + \frac{1}{2}}$

(h) $\dfrac{x^4 + x^3 + 7x^2 - 6x + 8}{x^2 + 2x + 8}$

2. Divide

(a) $\dfrac{5x^3 - x^2 + 6}{x - 4}$

(b) $\dfrac{3x^4 - 5x^2 + 3}{x + 2}$

(c) $\dfrac{x^3 + 2x^2 - 3x + 4}{x - 7}$

(d) $\dfrac{2x^5 + x^4 - 6x + 9}{x^2 - 3x + 1}$

(e) $\dfrac{4x^3 - 3x - 2}{x + 1}$

(f) $\dfrac{6x^2 - x - 2}{x + 1}$

(g) $\dfrac{3x^4 + 2x^3 + x^2 + 4}{x^2 + 1}$

(h) $\dfrac{27x^3 + 9x^2 - 3x - 9}{3x - 2}$

Find all roots and factor polynomials

3. (a) $x^3 + 2x^2 - 11x - 12$

(b) $10x^4 - 3x^3 - 29x^2 + 5x + 12$

(c) $2x^3 - 13x^2 + 3x + 18$

(d) $x^4 - 3x^3 - 5x^2 + 3x + 4$

(e) $2x^4 - 7x^3 - 2x^2 + 28x - 24$

(f) $8x^5 + 36x^4 + 46x^3 + 7x^2 - 12x - 4$

4. You are designing a candle-making kit. Each kit will contain 25 cubic inches of candle wax and a mold for making a model of the pyramid-shaped building at

the Louvre Museum in Paris, France. You want the height of the candle to be 2 inches less than the length of each side of the candle's square base. What should the dimensions of your candle mold be?

Source:https://www.classzone.com/eservices/home/pdf/student/LA206FAD.pdf

Sections 6.3 Polynomial Long Division with no Remainder Textbook Pages # 314-320

Objective(s):

- Students will be able to divide polynomials using long division.

$$5\overline{)965} = 193$$
-5, 46, -45, 15, -15, ☺

Introduction: Divide each expression using long division. Show your work.

1. $360 \div 5$

72 R0
$5\overline{)360}$
-35
10
-10
0

*If you get a R=0, that means both #s are factors!!! 72 and 5 are factors of 360!

2. $1{,}234 \div 10$

123 R4
$10\overline{)1234}$
-10
23
-20
34
-30
4

$123\frac{4}{10}$
$=123.4$
$=123\frac{2}{5}$

3. $366 \div 11$

$33\frac{3}{11}$
$11\overline{)366}$
-33
36
-33
3

The Lesson: Polynomial Long Division

1a)

1. Divide $x^2 - 4x - 12$ by $x - 6$. $= \boxed{x+2}$

$x+2$ R0 Factors!
$x-6\overline{)x^2-4x-12}$
$-\ x^2-6x$
$2x-12$
$-2x-12$
0

$(x+2)(x-6)$

2. Divide $x^2 - 9x - 10$ by $x + 1$. $= \boxed{x-10}$

$x-10$ R0 Factors
$x+1\overline{)x^2-9x-10}$
$-(x^2+x)$
$-10x-10$
$-\ -10x-10$
0

3. Divide $2x^3 + 8x^2 - 3x - 12$ *by* $x + 4$. $= \boxed{2x^2 - 3}$

1c

Factors!

$2x^2 + 0x - 3$ RO

$x + 4 \overline{)\, 2x^3 + 8x^2 - 3x - 12}$

$- \; 2x^3 + 8x^2$

$0x^2 - 3x$

$- \; 0x^2 + 0x$

$-3x - 12$

$- \; -3x - 12$

0

4. Divide $2x^3 - 3x^2 - 19x + 30$ *by* $x - 3$.

1b

$2x^2 + 3x - 10$

$x - 3 \overline{)\, 2x^3 - 3x^2 - 19x + 30}$

$- \; 2x^3 - 6x^2$

$3x^2 - 19x$

$- \; 3x^2 - 9x$

$-10x + 30$

$- \; -10x - 30$

0

5. Divide $4x^3 - 8x^2 - 3x + 1$ *by* $x + \frac{1}{2}$.

$4x^2 - 10x + 2$

$x + \frac{1}{2} \overline{)\, 4x^3 - 8x^2 - 3x + 1}$

$- \; 4x^3 + 2x^2$

$-10x^2 - 3x$

$- \; -10x^2 - 5x$

$2x + 1$

$- \; 2x + 1$

0

Dividing Polynomials Using Long Division

Model Problems:

1d

Example 1: Divide $\dfrac{2x^3-8x^2+9x-2}{x-2}$ using long division.

$$x-2\overline{\smash{)}\,2x^3-8x^2+9x-2}$$

$x-2$ is called the divisor and $2x^3-8x^2+9x-2$ is called the dividend. The first step is to find what we need to multiply the first term of the divisor *(x)* by to obtain the first term of the dividend ($2x^3$). This is $2x^2$. We then multiply $x-2$ by $2x^2$ and put this expression underneath the dividend. The term $2x^2$ is part of the quotient, and is put on top of the horizontal line (above the $8x^2$). We then *subtract* $2x^3 - 4x^2$ from $2x^3 - 8x^2 + 9x - 2$.

$$\begin{array}{r l}
 & \;\;2x^2 \\
x-2 & \overline{\smash{)}\,2x^3-8x^2+9x-2} \\
 & \underline{-(2x^3-4x^2)} \\
 & \qquad\;\; -4x^2+9x-2
\end{array}$$

The same procedure is continued until an expression of lower degree than the divisor is obtained. This is called the remainder.

$$\begin{array}{r l}
 & \;\;2x^2 - 4x + 1 \\
x-2 & \overline{\smash{)}\,2x^3-8x^2+9x-2} \\
 & \;- \underline{(2x^3-4x^2)} \\
 & \qquad -4x^2+9x-2 \\
 & \qquad \underline{-(-4x^2+8x)} \\
 & \qquad\qquad\quad x-2 \\
 & \qquad\qquad\quad \underline{-(x-2)} \\
 & \qquad\qquad\qquad 0
\end{array}$$

We've found that $\dfrac{2x^3-8x^2+9x-2}{x-2} = \underline{\underline{2x^2-4x+1}}$

Example 2: $\frac{8t^3+14t+8}{2t+1}$

1e

1c

Since the dividend (the numerator) doesn't have a second-degree term, it is useful to use placeholders so that we do our subtraction correctly. The problem works out as follows:

$$2t + 12t + 1)\overline{8t^3 + 0t^2 + 14t + 8}$$

Dividing we get:

$$\begin{array}{rl}
 & \quad 4t^2 \quad -2t \quad +8 \\ \hline
2t+1) & \quad 8t^3 + 0t^2 + 14t + 8 \\
 & \underline{-(8t^3 \ + 4t^2)} \\
 & \qquad\quad -4t^2 + 14t \\
 & \qquad \underline{-(-4t^2 - \ 2t)} \\
 & \qquad\qquad\qquad +16t + 8 \\
 & \qquad\qquad \underline{-(+16t + 8)} \\
 & \qquad\qquad\qquad\qquad 0
\end{array}$$

PRACTICE:

1. $\dfrac{3x^3+5x^2-11x+3}{x+3}$ **2.** $\dfrac{4x^3+6x^2-10x+4}{2x-1}$ **3.** $\dfrac{x^3+1}{x-1}$

ANSWERS:

1. $3x^2-4x+1$ **2.** $2x^2+4x-3+\dfrac{1}{2x-1}$ **3.** $x^2+x+1+\dfrac{2}{x-1}$

3. Polynomial division

We now do the same process with algebra.

Example

Suppose we wish to find

11

$$\frac{27x^3 + 9x^2 - 3x - 10}{3x - 2}$$

The calculation is set out as we did before for long division of numbers:

$$3x-2\,\overline{)\,27x^3 + 9x^2 - 3x - 10}$$

The question we ask is 'how many times does $3x$, NOT $3x-2$, go into $27x^3$?'. The answer is $9x^2$ times. And we record this above the x^2 place, just as we did with the numbers:

$$\begin{array}{rrrrr}
 & & 9x^2 & & \\ \hline
3x-2\,\big) & 27x^3 & +\,9x^2 & -\,3x & -\,10
\end{array}$$

Just as we did with the numbers we need to find the remainder, and so we multiply $9x^2$ by $3x-2$ and write the answer down under $27x^3 + 9x^2$. Thus we get:

$$\begin{array}{rrrrr}
 & & 9x^2 & & \\ \hline
3x-2\,\big) & 27x^3 & +\,9x^2 & -\,3x & -\,10 \\
 & 27x^3 & -\,18x^2 & & \\ \cline{2-3}
\end{array}$$

To find out what is left we now subtract, and get $27x^2$, and as with the numbers we now bring down the next term, the $-3x$, and write it alongside the $27x^2$ to give:

$$\begin{array}{rrrrr}
 & & 9x^2 & & \\ \hline
3x-2\,\big) & 27x^3 & +\,9x^2 & -\,3x & -\,10 \\
 & 27x^3 & -\,18x^2 & & \\ \cline{2-3}
 & & 27x^2 & -\,3x &
\end{array}$$

The question to be asked now is, 'how many times does $3x$, NOT $3x-2$, go into $27x^2$?'. The answer is $9x$ and we write this above the $-3x$ term, i.e. above the x place:

$$\begin{array}{rrrrr}
 & & 9x^2 & +\,9x & \\ \hline
3x-2\,\big) & 27x^3 & +\,9x^2 & -\,3x & -\,10 \\
 & 27x^3 & -\,18x^2 & & \\ \cline{2-3}
 & & 27x^2 & -\,3x &
\end{array}$$

Again we want to know what is left over from the division, so we multiply $3x-2$ by the $9x$ and write the answer down so we can subtract it from $27x^2 - 3x$, giving us $15x$:

$$\begin{array}{rrrrr}
 & & 9x^2 & +\,9x & \\ \hline
3x-2\,\big) & 27x^3 & +\,9x^2 & -\,3x & -\,10 \\
 & 27x^3 & -\,18x^2 & & \\ \cline{2-3}
 & & 27x^2 & -\,3x & \\
 & & 27x^2 & -\,18x & \\ \cline{3-4}
 & & & 15x &
\end{array}$$

The process is now repeated for the third time; bring down the next term, -10, write it next to the $15x$, and ask 'how many times does $3x$, not $3x-2$, go into $15x$?'.

$$
\begin{array}{r|l}
\multicolumn{2}{r}{9x^2 + 9x \phantom{{}-10}} \\
3x-2 & 27x^3 + 9x^2 - 3x - 10 \\
& \underline{27x^3 - 18x^2} \\
& \phantom{27x^3+{}} 27x^2 - 3x \\
& \phantom{27x^3+{}} \underline{27x^2 - 18x} \\
& \phantom{27x^3+27x^2-{}} 15x - 10
\end{array}
$$

$3x$ divides into $15x$ five times, and so we record this above the number place, above the 10. We multiply $3x-2$ by 5 and write the answer down so we can subtract it from $15x-10$ and see what the remainder is:

$$
\begin{array}{r|l}
\multicolumn{2}{r}{9x^2 + 9x + 5} \\
3x-2 & 27x^3 + 9x^2 - 3x - 10 \\
& \underline{27x^3 - 18x^2} \\
& \phantom{27x^3+{}} 27x^2 - 3x \\
& \phantom{27x^3+{}} \underline{27x^2 - 18x} \\
& \phantom{27x^3+27x^2-{}} 15x - 10 \\
& \phantom{27x^3+27x^2-{}} \underline{15x - 10} \\
& \phantom{27x^3+27x^2-15x-1{}} 0
\end{array}
$$

And in this case there is no remainder. Thus

$$\frac{27x^3 + 9x^2 - 3x - 10}{3x-2} = 9x^2 + 9x + 5$$

Check back through the calculation again and compare it with 2675 divided by 25. You should see that effectively they are the same process.

Example

Suppose we wish to find

$$\frac{x^4 + x^3 + 7x^2 - 6x + 8}{x^2 + 2x + 8}$$

The calculation is set out as follows, and an explanation is given below.

$$
\begin{array}{r|l}
\multicolumn{2}{r}{x^2 - x + 1} \\
x^2+2x+8 & x^4 + x^3 + 7x^2 - 6x + 8 \\
& \underline{x^4 + 2x^3 + 8x^2} \\
& - x^3 - x^2 - 6x \\
& \underline{- x^3 - 2x^2 - 8x} \\
& \phantom{x^4 - x^3 - {}} x^2 + 2x + 8 \\
& \phantom{x^4 - x^3 - {}} \underline{x^2 + 2x + 8} \\
& \phantom{x^4 - x^3 - x^2 + 2x + {}} 0
\end{array}
$$

Work through the example. Your thinking should be moving along the lines:

'How many times does x^2 go into x^4 ?'. The answer is x^2 times.

Write x^2 above the x^2 place in $x^4 + x^3 + 7x^2 - 6x + 8$.

Multiply $x^2 + 2x + 8$ by x^2, write the answer down underneath $x^4 + x^3 + 7x^2$ and subtract to find the remainder - which is $-x^3 - x^2$.

Bring down the next term, $-6x$, to give $-x^3 - x^2 - 6x$.

'How many times does x^2 go into $-x^3$?'. The answer is $-x$ times.

Write $-x$ above the x place in $x^4 + x^3 + 7x^2 - 6x + 8$

3. Divide $2x^3 + 8x^2 - 3x - 12$ *by* $x + 4$. $= \boxed{2x^2-3}$

Factors!

$$\begin{array}{r} 2x^2 + 0x - 3 \quad R0 \\ x+4 \,\overline{)\, 2x^3 + 8x^2 - 3x - 12} \\ -\,(2x^3 + 8x^2) \qquad\qquad \\ \hline 0x^2 - 3x \qquad \\ -\,(0x^2 + 0x) \qquad \\ \hline -3x - 12 \\ -\,(-3x - 12) \\ \hline 0 \end{array}$$

4. Divide $2x^3 - 3x^2 - 19x + 30$ *by* $x - 3$.

$$\begin{array}{r} 2x^2 + 3x - 10 \\ x-3 \,\overline{)\, 2x^3 - 3x^2 - 19x + 30} \\ -\,(2x^3 - 6x^2) \qquad\qquad \\ \hline 3x^2 - 19x \qquad \\ -\,(3x^2 - 9x) \qquad \\ \hline -10x + 30 \\ -\,(-10x - 30) \\ \hline 0 \end{array}$$

5. Divide $4x^3 - 8x^2 - 3x + 1$ *by* $x + \frac{1}{2}$.

9

$$\begin{array}{r} 4x^2 - 10x + 2 \\ x+\frac{1}{2} \,\overline{)\, 4x^3 - 8x^2 - 3x + 1} \\ -\,(4x^3 + 2x^2) \qquad\qquad \\ \hline -10x^2 - 3x \qquad \\ -\,(-10x^2 - 5x) \qquad \\ \hline 2x + 1 \\ -\,(2x + 1) \\ \hline 0 \end{array}$$

$$
\begin{array}{r}
9x^2 + 9x \phantom{{}-10} \\
3x-2\,\big)\overline{27x^3 + 9x^2 - 3x - 10} \\
\underline{27x^3 - 18x^2} \phantom{{}- 3x - 10} \\
27x^2 - 3x \phantom{{}-10} \\
\underline{27x^2 - 18x} \phantom{{}-10} \\
15x - 10
\end{array}
$$

$3x$ divides into $15x$ five times, and so we record this above the number place, above the 10. We multiply $3x - 2$ by 5 and write the answer down so we can subtract it from $15x - 10$ and see what the remainder is:

$$
\begin{array}{r}
9x^2 + 9x + 5 \\
3x-2\,\big)\overline{27x^3 + 9x^2 - 3x - 10} \\
\underline{27x^3 - 18x^2} \phantom{{}- 3x - 10} \\
27x^2 - 3x \phantom{{}-10} \\
\underline{27x^2 - 18x} \phantom{{}-10} \\
15x - 10 \\
\underline{15x - 10} \\
0
\end{array}
$$

And in this case there is no remainder. Thus

$$\frac{27x^3 + 9x^2 - 3x - 10}{3x - 2} = 9x^2 + 9x + 5$$

Check back through the calculation again and compare it with 2675 divided by 25. You should see that effectively they are the same process.

Example

1b

Suppose we wish to find

$$\frac{x^4 + x^3 + 7x^2 - 6x + 8}{x^2 + 2x + 8}$$

The calculation is set out as follows, and an explanation is given below.

$$
\begin{array}{r}
x^2 - x + 1 \\
x^2+2x+8\,\big)\overline{x^4 + x^3 + 7x^2 - 6x + 8} \\
\underline{x^4 + 2x^3 + 8x^2} \phantom{{}- 6x + 8} \\
-x^3 - x^2 - 6x \phantom{{}+8} \\
\underline{-x^3 - 2x^2 - 8x} \phantom{{}+8} \\
x^2 + 2x + 8 \\
\underline{x^2 + 2x + 8} \\
0
\end{array}
$$

Work through the example. Your thinking should be moving along the lines:

'How many times does x^2 go into x^4 ?'. The answer is x^2 times.

Write x^2 above the x^2 place in $x^4 + x^3 + 7x^2 - 6x + 8$.

Multiply $x^2 + 2x + 8$ by x^2, write the answer down underneath $x^4 + x^3 + 7x^2$ and subtract to find the remainder - which is $-x^3 - x^2$.

Bring down the next term, $-6x$, to give $-x^3 - x^2 - 6x$.

'How many times does x^2 go into $-x^3$?'. The answer is $-x$ times.

Write $-x$ above the x place in $x^4 + x^3 + 7x^2 - 6x + 8$

Multiply $x^2 + 2x + 8$ by $-x$, write the answer underneath $-x^3 - x^2 - 6x$ and subtract to find the remainder, which is $x^2 + 2x$.

Bring down the next term, 8, to give $x^2 + 2x + 8$.

'How many times does x^2 go into x^2 ?' The answer is 1.

Write 1 above the number place in $x^4 + x^3 + 7x^2 - 6x + 8$.

Multiply $x^2 + 2x + 8$ by 1, write the answer down underneath $x^2 + 2x + 8$ and subtract to find the remainder, which is 0.

So we conclude that

$$\frac{x^4 + x^3 + 7x^2 - 6x + 8}{x^2 + 2x + 8} = x^2 - x + 1$$

Example

What happens if some of the terms are missing from the polynomial into which we are dividing? The answer is that we leave space for them when we set out the division and write in the answers to the various small divisions that we do in the places where they would normally go.

Suppose we wish to find

$$\frac{x^3 - 1}{x - 1}$$

The calculation is set out as follows:

$$\begin{array}{rrrrr}
 & & x^2 & +\ x & +\ 1 \\ \cline{2-5}
x - 1 \big) & x^3 & & & -\ 1 \\
 & x^3 & -\ x^2 & & \\ \cline{2-3}
 & & x^2 & & \\
 & & x^2 & -\ x & \\ \cline{3-4}
 & & & x & -\ 1 \\
 & & & x & -\ 1 \\ \cline{4-5}
 & & & & 0
\end{array}$$

We conclude that

$$\frac{x^3 - 1}{x - 1} = x^2 + x + 1$$

Example

What would we do if there was a final remainder, something other than 0 ?

For example, suppose we needed to find

$$\frac{27x^3 + 9x^2 - 3x - 9}{3x - 2}$$

We do exactly the same:

$$\begin{array}{rrrrr}
 & & 9x^2 & +\ 9x & +\ 5 \\ \cline{2-5}
3x - 2 \big) & 27x^3 & +\ 9x^2 & -\ 3x & -\ 9 \\
 & 27x^3 & -\ 18x^2 & & \\ \cline{2-3}
 & & 27x^2 & -\ 3x & \\
 & & 27x^2 & -\ 18x & \\ \cline{3-4}
 & & & 15x & -\ 9 \\
 & & & 15x & -\ 10 \\ \cline{4-5}
 & & & & 1
\end{array}$$

Section 5-1 : Dividing Polynomials

In this section we're going to take a brief look at dividing polynomials. This is something that we'll be doing off and on throughout the rest of this chapter and so we'll need to be able to do this.

2a

Let's do a quick example to remind us how long division of polynomials works.

Example 1 Divide $5x^3 - x^2 + 6$ by $x - 4$.

Solution
Let's first get the problem set up.

$$x-4\overline{\big)5x^3 - x^2 + 0x + 6}$$

Recall that we need to have the terms written down with the exponents in decreasing order and to make sure we don't make any mistakes we add in any missing terms with a zero coefficient.

Now we ask ourselves what we need to multiply $x - 4$ to get the first term in first polynomial. In this case that is $5x^2$. So multiply $x - 4$ by $5x^2$ and subtract the results from the first polynomial.

$$\begin{array}{r} 5x^2 \qquad\qquad\quad \\ x-4\overline{\big)5x^3 - x^2 + 0x + 6} \\ -\left(5x^3 - 20x^2\right) \qquad\quad \\ \hline 19x^2 + 0x + 6 \end{array}$$

The new polynomial is called the **remainder**. We continue the process until the degree of the remainder is less than the degree of the **divisor**, which is $x - 4$ in this case. So, we need to continue until the degree of the remainder is less than 1.

Recall that the **degree** of a polynomial is the highest exponent in the polynomial. Also, recall that a constant is thought of as a polynomial of degree zero. Therefore, we'll need to continue until we get a constant in this case.

Here is the rest of the work for this example.

$$
\begin{array}{r}
5x^2+19x+76 \\
x-4\,\overline{\big)\,5x^3-x^2+0x+6} \\
-\left(5x^3-20x^2\right) \\
\hline
19x^2+0x+6 \\
-\left(19x^2-76x\right) \\
\hline
76x+6 \\
-\left(76x-304\right) \\
\hline
310
\end{array}
$$

Okay, now that we've gotten this done, let's remember how we write the actual answer down. The answer is,

$$\frac{5x^3-x^2+6}{x-4}=5x^2+19x+76+\frac{310}{x-4}$$

There is actually another way to write the answer from the previous example that we're going to find much more useful, if for no other reason that it's easier to write down. If we multiply both sides of the answer by $x-4$ we get,

$$5x^3-x^2+6=(x-4)\left(5x^2+19x+76\right)+310$$

In this example we divided the polynomial by a linear polynomial in the form of $x-r$ and we will be restricting ourselves to only these kinds of problems. Long division works for much more general division, but these are the kinds of problems we are going to seeing the later sections.

In fact, we will be seeing these kinds of divisions so often that we'd like a quicker and more efficient way of doing them. Luckily there is something out there called **synthetic division** that works wonderfully for these kinds of problems. In order to use synthetic division we must be dividing a polynomial by a linear term in the form $x-r$. If we aren't then it won't work.

Let's redo the previous problem with synthetic division to see how it works.

Example 2 Use synthetic division to divide $5x^3-x^2+6$ by $x-4$.

Solution
Okay with synthetic division we pretty much ignore all the *x*'s and just work with the numbers in the polynomials.

First, let's notice that in this case *r=4*.

Now we need to set up the process. There are many different notations for doing this. We'll be using the following notation.

Section 5-1 : Dividing Polynomials

2b

1. Use long division to divide $3x^4 - 5x^2 + 3$ by $x + 2$.

Step 1

Let's first perform the long division. Just remember that we keep going until the remainder has degree that is strictly less that the degree of the polynomial we're dividing by, $x + 2$ in this case. The polynomial we're dividing by has degree one and so, in this case, we'll stop when the remainder is degree zero, *i.e.* a constant.

Here is the long division work for this problem.

$$
\begin{array}{r}
3x^3 - 6x^2 + 7x - 14 \\
x+2 \overline{\smash{\big)} \; 3x^4 - 5x^2 + 3} \\
-\underline{\left(3x^4 + 6x^3\right)} \\
-6x^3 - 5x^2 + 3 \\
-\underline{\left(-6x^3 - 12x^2\right)} \\
7x^2 + 3 \\
-\underline{\left(7x^2 + 14x\right)} \\
-14x + 3 \\
-\underline{\left(-14x - 28\right)} \\
31
\end{array}
$$

Step 2

We can put the answer in either of the two following forms.

$$\frac{3x^4 - 5x^2 + 3}{x + 2} = 3x^3 - 6x^2 + 7x - 14 + \frac{31}{x + 2}$$

$$3x^4 - 5x^2 + 3 = \left(x + 2\right)\left(3x^3 - 6x^2 + 7x - 14\right) + 31$$

Either answer is acceptable here although one may be more useful than the other depending on the application that is being done when you need to actually do the long division.

2c

2. Use long division to divide $x^3 + 2x^2 - 3x + 4$ by $x - 7$.

Step 1

2c

Let's first perform the long division. Just remember that we keep going until the remainder has degree that is strictly less that the degree of the polynomial we're dividing by, $x-7$ in this case. The polynomial we're dividing by has degree one and so, in this case, we'll stop when the remainder is degree zero, *i.e.* a constant.

Here is the long division work for this problem.

$$
\begin{array}{r}
x^2+9x+60 \\
x-7\overline{\big)\,x^3+2x^2-3x+4} \\
-\underline{\left(x^3-7x^2\right)} \qquad\quad \\
9x^2-3x+4 \\
-\underline{\left(9x^2-63x\right)} \quad \\
60x+4 \\
-\underline{\left(60x-420\right)} \\
424
\end{array}
$$

Step 2

We can put the answer in either of the two following forms.

$$\frac{x^3+2x^2-3x+4}{x-7}=x^2+9x+60+\frac{424}{x-7}$$

$$x^3+2x^2-3x+4=(x-7)\left(x^2+9x+60\right)+424$$

Either answer is acceptable here although one may be more useful than the other depending on the application that is being done when you need to actually do the long division.

2d

3. Use long division to divide $2x^5+x^4-6x+9$ by x^2-3x+1.

Step 1

Let's first perform the long division. Just remember that we keep going until the remainder has degree that is strictly less that the degree of the polynomial we're dividing by, x^2-3x+1 in this case. The polynomial we're dividing by has degree two and so, in this case, we'll stop when the remainder is degree one or zero.

Here is the long division work for this problem.

$$
\begin{array}{r}
2x^3+7x^2+19x+50 \\
x^2-3x+1 \overline{\big)\; 2x^5+x^4-6x+9} \\
-\left(2x^5-6x^4+2x^3\right) \\ \hline
7x^4-2x^3-6x+9 \\
-\left(7x^4-21x^3+7x^2\right) \\ \hline
19x^3-7x^2-6x+9 \\
-\left(19x^3-57x^2+19x\right) \\ \hline
50x^2-25x+9 \\
-\left(50x^2-150x+50\right) \\ \hline
125x-41
\end{array}
$$

Step 2

We can put the answer in either of the two following forms.

2c1

$$\frac{2x^5+x^4-6x+9}{x^2-3x+1}=2x^3+7x^2+19x+50+\frac{125x-41}{x^2-3x+1}$$

$$2x^5+x^4-6x+9=\left(x^2-3x+1\right)\left(2x^3+7x^2+19x+50\right)+125x-41$$

Either answer is acceptable here although one may be more useful than the other depending on the application that is being done when you need to actually do the long division.

---

4. Use synthetic division to divide x^3+x^2+x+1 by $x+9$.

Step 1

Here is the synthetic division. We'll leave it to you to check all the numbers.

$$
\begin{array}{r|rrrr}
-9 & 1 & 1 & 1 & 1 \\
 & & -9 & 72 & -657 \\ \hline
 & 1 & -8 & 73 & -656
\end{array}
$$

Step 2

The answer is then,

$$x^3+x^2+x+1=\left(x+9\right)\left(x^2-8x+73\right)-656$$

Example 2 (Continued):

Step 6: The process is complete at this point because we have a zero in the final row. From the long division table we see that $Q(x) = x + 5$ and $R(x) = 0$, so

$$3x^2 + 17x + 10 = (3x + 2)(x + 5) + 0$$

Note that since there is no remainder, this quotient could have been found by factoring and writing in lowest terms.

2c

Example 3: Find the quotient and remainder of $\frac{4x^3 - 3x - 2}{x+1}$ using long division.

Solution:

Step 1: Write the problem, making sure that both polynomials are written in descending powers of the variables. Add a term with 0 coefficient as a place holder for the missing x^2 term.

Missing term

$$x+1\overline{)4x^3 + 0x^2 - 3x - 2}$$

Step 2: Start with $\frac{4x^3}{x} = 4x^2$.

$$\begin{array}{rll} & 4x^2 & \leftarrow \frac{4x^3}{x} = 4x^2 \\ x+1 & \overline{)4x^3 + 0x^2 - 3x - 2} & \\ & \underline{4x^3 + 4x^2} & \leftarrow 4x^2(x+1) \end{array}$$

Step 3: Subtract by changing the signs on $4x^3 + 4x^2$ and adding. Then Bring down the next term.

$$\begin{array}{rll} & 4x^2 & \\ x+1 & \overline{)4x^3 + 0x^2 - 3x - 2} & \\ & \underline{4x^3 + 4x^2} & \\ & \quad -4x^2 - 3x & \leftarrow \text{Subtract and bring down } -3x \end{array}$$

Example 3 (Continued):

Step 4: Now continue with $\dfrac{-4x^2}{x} = -4x$.

$$
\begin{array}{rll}
 & 4x^2 - 4x & \leftarrow \dfrac{-4x^2}{x} = -4x \\
x+1 \big) & 4x^3 + 0x^2 - 3x - 2 & \\
 & \underline{4x^3 + 4x^2} & \\
 & -4x^2 - 3x & \\
 & \underline{-4x^2 - 4x} & \leftarrow -4x(x+1) \\
 & x - 2 & \leftarrow \text{Subtract and bring down } -2
\end{array}
$$

Step 5: Finally, $\dfrac{x}{x} = 1$.

$$
\begin{array}{rll}
 & 4x^2 - 4x + 1 & \leftarrow \dfrac{x}{x} = 1 \\
x+1 \big) & 4x^3 + 0x^2 - 3x - 2 & \\
 & \underline{4x^3 + 4x^2} & \\
 & -4x^2 - 3x & \\
 & \underline{-4x^2 - 4x} & \\
 & x - 2 & \\
 & \underline{x + 1} & \leftarrow 1(x+1) \\
 & -3 & \leftarrow \text{Subtract}
\end{array}
$$

Step 6: The process is complete at this point because –3 is of lesser degree than the divisor $x + 1$. Thus, the quotient is $4x^2 - 4x + 1$ and the remainder is –3, and

$$\frac{4x^3 - 3x - 2}{x+1} = 4x^2 - 4x + 1 + \frac{-3}{x+1}.$$

By: Crystal Hull

Factoring the quadratic expression enables us to write the cubic polynomial as a product of three linear factors:

$$2x^3 - 9x^2 + 7x + 6 = (2x + 1)(x^2 - 5x + 6) = (2x + 1)(x - 3)(x - 2)$$

Set the value of the polynomial equal to zero, $(2x + 1)(x - 3)(x - 2) = 0$, and solve for x. The zeros of the polynomial are $-\frac{1}{2}$, 2, and 3. In Example 1 and in the Your Turn, the remainder was 0. Sometimes there is a nonzero remainder (Example 2).

22

EXAMPLE 2 Dividing Polynomials Using Long Division; Nonzero Remainder

Divide $6x^2 - x - 2$ by $x + 1$.

Solution:

$$\begin{array}{r|l} & \ 6x - 7 \\ & x + 1\overline{)6x^2 - x - 2} \\ \text{Multiply } 6x(x + 1). & \quad -(6x^2 + 6x) \\ \text{Subtract and bring down } -2. & \qquad\quad -7x - 2 \\ \text{Multiply } -7(x + 1). & \qquad -(-7x - 7) \\ \text{Subtract and identify the remainder.} & \qquad\qquad +5 \end{array}$$

$$\underbrace{\frac{6x^2 - x - 2}{x + 1}}_{\text{Dividend / Divisor}} = \underbrace{6x - 7}_{\text{Quotient}} + \underbrace{\frac{5}{x + 1}}_{\text{Remainder / Divisor}} \qquad \boxed{x \neq -1}$$

Dividend, Quotient, Remainder, Divisor, Divisor

Check: Multiply the quotient and remainder by $x + 1$.

$$(6x - 7)(x + 1) + \frac{5}{(x + 1)} \cdot (x + 1)$$

$$= 6x^2 - x - 7 + 5$$

The result is the dividend.

$$= 6x^2 - x - 2 \quad ✓$$

■ **YOUR TURN** Divide $2x^3 + x^2 - 4x - 3$ by $x - 1$.

■ **Answer:** $2x^2 + 3x - 1$ R: -4 or $2x^2 + 3x - 1 - \dfrac{4}{x - 1}$

In general, when a polynomial is divided by another polynomial, we express the result in the following form:

$$\frac{P(x)}{d(x)} = Q(x) + \frac{r(x)}{d(x)}$$

where $P(x)$ is the **dividend**, $d(x) \neq 0$ is the divisor, $Q(x)$ is the quotient, and $r(x)$ is the remainder. Multiplying this equation by the divisor, $d(x)$, leads us to the division algorithm.

THE DIVISION ALGORITHM

If $P(x)$ and $d(x)$ are polynomials with $d(x) \neq 0$, and if the degree of $P(x)$ is greater than or equal to the degree of $d(x)$, then unique polynomials $Q(x)$ and $r(x)$ exist such that

$$P(x) = d(x) \cdot Q(x) + r(x)$$

If the remainder $r(x) = 0$, then we say that $d(x)$ divides $P(x)$ and that $d(x)$ and $Q(x)$ are factors of $P(x)$.

EXAMPLE 3 Long Division of Polynomials with "Missing" Terms

Divide $x^3 - 8$ by $x - 2$.

Solution:

$$\begin{array}{rl}
 & x^2 + 2x + 4 \\
x - 2\,\big) & x^3 + 0x^2 + 0x - 8 \\
 & -(x^3 - 2x^2) \\
 & \quad 2x^2 + 0x \\
 & \quad -(2x^2 - 4x) \\
 & \qquad 4x - 8 \\
 & \qquad -(4x - 8) \\
 & \qquad\quad 0
\end{array}$$

Insert $0x^2 + 0x$ for placeholders.

Multiply $x^2(x - 2) = x^3 - 2x^2$.

Subtract and bring down $0x$.

Multiply $2x(x - 2) = 2x^2 - 4x$.

Subtract and bring down -8.

Multiply $4(x - 2) = 4x - 8$.

Subtract and get remainder 0.

Since the remainder is 0, $x - 2$ is a factor of $x^3 - 8$.

$$\boxed{\frac{x^3 - 8}{x - 2} = x^2 + 2x + 4,\ x \neq 2}$$

Check: $x^3 - 8 = (x^2 + 2x + 4)(x - 2) = x^3 + 2x^2 + 4x - 2x^2 - 4x - 8 = x^3 - 8$ ✓

■ **YOUR TURN** Divide $x^3 - 1$ by $x - 1$.

■ **Answer:** $x^2 + x + 1$

EXAMPLE 4 Long Division of Polynomials

Divide $3x^4 + 2x^3 + x^2 + 4$ by $x^2 + 1$.

Solution:

$$\begin{array}{rl}
 & 3x^2 + 2x - 2 \\
x^2 + 0x + 1\,\big) & 3x^4 + 2x^3 + x^2 + 0x + 4 \\
 & -(3x^4 + 0x^3 + 3x^2) \\
 & \quad 2x^3 - 2x^2 + 0x \\
 & \quad -(2x^3 + 0x^2 + 2x) \\
 & \qquad -2x^2 - 2x + 4 \\
 & \qquad -(-2x^2 + 0x - 2) \\
 & \qquad\quad -2x + 6
\end{array}$$

Insert $0x$ as a placeholder in both the divisor and the dividend.

Multiply $3x^2(x^2 + 0x + 1)$.

Subtract and bring down $0x$.

Multiply $2x(x^2 + 0x + 1)$.

Subtract and bring down 4.

Multiply $-2(x^2 - 2x + 1)$.

Subtract and get remainder $-2x + 6$.

$$\boxed{\frac{3x^4 + 2x^3 + x^2 + 4}{x^2 + 1} = 3x^2 + 2x - 2 + \frac{-2x + 6}{x^2 + 1}}$$

■ **YOUR TURN** Divide $2x^5 + 3x^2 + 12$ by $x^3 - 3x - 4$.

■ **Answer:** $2x^2 + 6 + \dfrac{11x^2 + 18x + 36}{x^3 - 3x - 4}$

Multiply $x^2 + 2x + 8$ by $-x$, write the answer underneath $-x^3 - x^2 - 6x$ and subtract to find the remainder, which is $x^2 + 2x$.

Bring down the next term, 8, to give $x^2 + 2x + 8$.

'How many times does x^2 go into x^2 ?' The answer is 1.

Write 1 above the number place in $x^4 + x^3 + 7x^2 - 6x + 8$.

Multiply $x^2 + 2x + 8$ by 1, write the answer down underneath $x^2 + 2x + 8$ and subtract to find the remainder, which is 0.

So we conclude that

$$\frac{x^4 + x^3 + 7x^2 - 6x + 8}{x^2 + 2x + 8} = x^2 \;\; - \; x \; + 1$$

Example

What happens if some of the terms are missing from the polynomial into which we are dividing? The answer is that we leave space for them when we set out the division and write in the answers to the various small divisions that we do in the places where they would normally go.

Suppose we wish to find

$$\frac{x^3 - 1}{x - 1}$$

The calculation is set out as follows:

$$\begin{array}{r|llll}
 & & x^2 & + \; x & + 1 \\ \hline
x - 1 & x^3 & & & - 1 \\
 & x^3 & - x^2 & & \\ \cline{2-3}
 & & x^2 & & \\
 & & x^2 & - x & \\ \cline{3-4}
 & & & x & - 1 \\
 & & & x & - 1 \\ \cline{4-5}
 & & & & 0
\end{array}$$

We conclude that

$$\frac{x^3 - 1}{x - 1} = x^2 + \; x + 1$$

Example

What would we do if there was a final remainder, something other than 0 ?

For example, suppose we needed to find

2e

$$\frac{27x^3 + 9x^2 - 3x - 9}{3x - 2}$$

We do exactly the same:

$$\begin{array}{r|llll}
 & & 9x^2 & + 9x & + 5 \\ \hline
3x - 2 & 27x^3 + & 9x^2 & - 3x & - 9 \\
 & 27x^3 & - 18x^2 & & \\ \cline{2-3}
 & & 27x^2 & - 3x & \\
 & & 27x^2 & - 18x & \\ \cline{3-4}
 & & & 15x & - 9 \\
 & & & 15x & - 10 \\ \cline{4-5}
 & & & & 1
\end{array}$$

We now have a remainder of 1, which still has to be divided by $3x - 2$. Thus our final answer now is:

$$\frac{27x^3 + 9x^2 - 3x - 9}{3x - 2} = 9x^2 + 9x + 5 + \frac{1}{3x - 2}$$

2e

Exercises

Use polynomal division to simplify each of the following quotients.

a) $\dfrac{x^4 + 3x^3 - x^2 - x + 6}{x + 3}$	b) $\dfrac{2x^4 - 5x^3 + 2x^2 + 5x - 10}{x - 2}$	c) $\dfrac{7x^4 - 10x^3 + 3x^2 + 3x - 3}{x - 1}$
d) $\dfrac{2x^4 + 8x^3 - 5x^2 - 4x + 2}{x^2 + 4x - 2}$	e) $\dfrac{3x^4 - x^3 + 8x^2 + 5x + 3}{x^2 - x + 3}$	f) $\dfrac{3x^4 + 9x^3 - 5x^2 - 6x + 2}{3x^2 - 2}$
g) $\dfrac{x^3 - 2x^2 - 4}{x - 2}$	h) $\dfrac{x^3 - 4x^2 + 9}{x - 3}$	i) $\dfrac{x^4 - 13x - 42}{x^2 - x - 6}$

Answers

a) $x^3 - x + 2$	b) $2x^3 - x^2 + 5$	c) $7x^3 - 3x^2 + 3$
d) $2x^2 - 1$	e) $3x^2 + 2x + 1$	f) $x^2 + 3x - 1$
g) $x^2 + 2x + 2$	h) $x^2 - x - 3$	i) $x^2 + x + 7$

6.6 Finding Rational Zeros

What you should learn

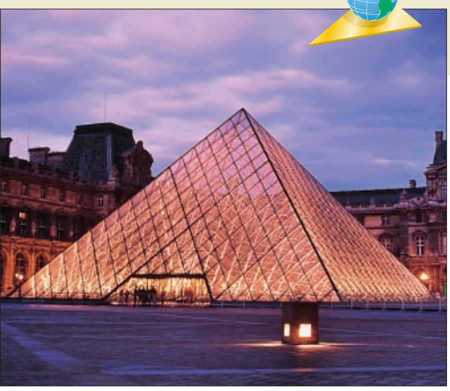
GOAL 1 Find the rational zeros of a polynomial function.

GOAL 2 Use polynomial equations to solve **real-life** problems, such as finding the dimensions of a monument in **Ex. 60**.

Why you should learn it

▼ To model **real-life** quantities, such as the volume of a representation of the Louvre pyramid in **Example 3**.

REAL LIFE



30

GOAL 1 USING THE RATIONAL ZERO THEOREM

The polynomial function

$$f(x) = 64x^3 + 120x^2 - 34x - 105$$

has $-\frac{3}{2}$, $-\frac{5}{4}$, and $\frac{7}{8}$ as its zeros. Notice that the numerators of these zeros (-3, -5, and 7) are factors of the constant term, -105. Also notice that the denominators (2, 4, and 8) are factors of the leading coefficient, 64. These observations are generalized by the *rational zero theorem.*

THE RATIONAL ZERO THEOREM

If $f(x) = a_nx^n + \cdots + a_1x + a_0$ has *integer* coefficients, then every rational zero of f has the following form:

$$\frac{p}{q} = \frac{\text{factor of constant term } a_0}{\text{factor of leading coefficient } a_n}$$

EXAMPLE 1 *Using the Rational Zero Theorem*

Find the rational zeros of $f(x) = x^3 + 2x^2 - 11x - 12$.

SOLUTION

List the possible rational zeros. The leading coefficient is 1 and the constant term is -12. So, the possible rational zeros are:

$$x = \pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{4}{1}, \pm\frac{6}{1}, \pm\frac{12}{1}$$

Test these zeros using synthetic division.

Test $x = 1$:

$$\begin{array}{r|rrrr} 1 & 1 & 2 & -11 & -12 \\ & & 1 & 3 & -8 \\ \hline & 1 & 3 & -8 & -20 \end{array}$$

Test $x = -1$:

$$\begin{array}{r|rrrr} -1 & 1 & 2 & -11 & -12 \\ & & -1 & -1 & 12 \\ \hline & 1 & 1 & -12 & 0 \end{array}$$

Since -1 is a zero of f, you can write the following:

$$f(x) = (x + 1)(x^2 + x - 12)$$

Factor the trinomial and use the factor theorem.

$$f(x) = (x + 1)(x^2 + x - 12) = (x + 1)(x - 3)(x + 4)$$

▶ The zeros of f are -1, 3, and -4.

In Example 1, the leading coefficient is 1. When the leading coefficient is not 1, the list of possible rational zeros can increase dramatically. In such cases the search can be shortened by sketching the function's graph—either by hand or by using a graphing calculator.

EXAMPLE 2 *Using the Rational Zero Theorem*

3b

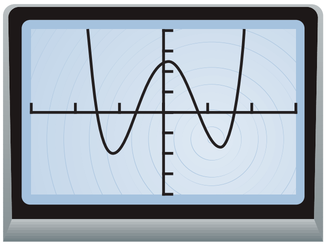
Find all real zeros of $f(x) = 10x^4 - 3x^3 - 29x^2 + 5x + 12$.

SOLUTION

List the possible rational zeros of f: $\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{4}{1}, \pm\frac{6}{1},$
$\pm\frac{12}{1}, \pm\frac{3}{2}, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{3}{5}, \pm\frac{6}{5}, \pm\frac{12}{5}, \pm\frac{1}{10}, \pm\frac{3}{10}, \pm\frac{12}{10}.$

Choose values to check.

With so many possibilities, it is worth your time to sketch the graph of the function. From the graph, it appears that some reasonable choices are $x = -\frac{3}{2}$, $x = -\frac{3}{5}$, $x = \frac{4}{5}$, and $x = \frac{3}{2}$.



Check the chosen values using synthetic division.

$$\begin{array}{r|rrrrr} -\frac{3}{2} & 10 & -3 & -29 & 5 & 12 \\ & & -15 & 27 & 3 & -12 \\ \hline & 10 & -18 & -2 & 8 & 0 \end{array}$$

$\leftarrow -\frac{3}{2}$ **is a zero.**

Factor out a binomial using the result of the synthetic division.

$f(x) = \left(x + \frac{3}{2}\right)(10x^3 - 18x^2 - 2x + 8)$ **Rewrite as a product of two factors.**

$= \left(x + \frac{3}{2}\right)(2)(5x^3 - 9x^2 - x + 4)$ **Factor 2 out of the second factor.**

$= (2x + 3)(5x^3 - 9x^2 - x + 4)$ **Multiply the first factor by 2.**

Repeat the steps above for $g(x) = 5x^3 - 9x^2 - x + 4$.

Any zero of g will also be a zero of f. The possible *rational* zeros of g are $x = \pm 1, \pm 2, \pm 4, \pm\frac{1}{5}, \pm\frac{2}{5}$, and $\pm\frac{4}{5}$. The graph of f shows that $\frac{4}{5}$ may be a zero.

$$\begin{array}{r|rrrr} \frac{4}{5} & 5 & -9 & -1 & 4 \\ & & 4 & -4 & -4 \\ \hline & 5 & -5 & -5 & 0 \end{array}$$

$\leftarrow \frac{4}{5}$ **is a zero.**

So $f(x) = (2x + 3)\left(x - \frac{4}{5}\right)(5x^2 - 5x - 5) = (2x + 3)(5x - 4)(x^2 - x - 1)$.

Find the remaining zeros of f by using the quadratic formula to solve $x^2 - x - 1 = 0$.

▶ The real zeros of f are $-\frac{3}{2}, \frac{4}{5}, \frac{1 + \sqrt{5}}{2}$, and $\frac{1 - \sqrt{5}}{2}$.

Section 5-4 : Finding Zeroes of Polynomials

1. Find all the zeroes of the following polynomial.

$$f\left(x\right)=2x^{3}-13x^{2}+3x+18$$

Step 1
We'll need all the factors of 18 and 2.

$$\begin{array}{ll} 18: & \pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18 \\ 2: & \pm 1,\ \pm 2 \end{array}$$

Step 2
Here is a list of all possible rational zeroes for the polynomial.

$$\frac{\pm 1}{\pm 1}=\pm 1 \qquad \frac{\pm 2}{\pm 1}=\pm 2 \qquad \frac{\pm 3}{\pm 1}=\pm 3 \qquad \frac{\pm 6}{\pm 1}=\pm 6 \qquad \frac{\pm 9}{\pm 1}=\pm 9 \qquad \frac{\pm 18}{\pm 1}=\pm 18$$

$$\frac{\pm 1}{\pm 2}=\frac{\pm 1}{\pm 2} \qquad \frac{\pm 2}{\pm 2}=\pm 1 \qquad \frac{\pm 3}{\pm 2}=\frac{\pm 3}{\pm 2} \qquad \frac{\pm 6}{\pm 2}=\pm 3 \qquad \frac{\pm 9}{\pm 2}=\frac{\pm 9}{\pm 2} \qquad \frac{\pm 18}{\pm 2}=\pm 9$$

So, we have a total of 18 possible zeroes for the polynomial.

Step 3
We now need to start the synthetic division work. We'll start with the "small" integers first.

$$\begin{array}{r|rrrr} & 2 & -13 & 3 & 18 \\ \hline -1 & 2 & -15 & 18 & 0 \end{array} \quad = f\left(-1\right)=0!!$$

Okay we now know that $x=-1$ is a zero and we can write the polynomial as,

$$f\left(x\right)=2x^{3}-13x^{2}+3x+18=\left(x+1\right)\left(2x^{2}-15x+18\right)$$

Step 4
We could continue with this process however, we have a quadratic for the second factor and we can just factor this so the fully factored form of the polynomial is,

$$f\left(x\right)=2x^{3}-13x^{2}+3x+18=\left(x+1\right)\left(2x-3\right)\left(x-6\right)$$

Step 5
From the fully factored form we get the following set of zeroes for the original polynomial.

$$\boxed{x = -1 \qquad\qquad x = \frac{3}{2} \qquad\qquad x = 6}$$

---

2. Find all the zeroes of the following polynomial.

3of

$$P(x) = x^4 - 3x^3 - 5x^2 + 3x + 4$$

Step 1
We'll need all the factors of 4 and 1.

$$\begin{array}{ll} 4: & \pm 1, \pm 2, \pm 4 \\ 1: & \pm 1 \end{array}$$

Step 2
Here is a list of all possible rational zeroes for the polynomial.

$$\frac{\pm 1}{\pm 1} = \pm 1 \qquad \frac{\pm 2}{\pm 1} = \pm 2 \qquad \frac{\pm 4}{\pm 1} = \pm 4$$

So, we have a total of 6 possible zeroes for the polynomial.

Step 3
We now need to start the synthetic division work. We'll start with the "small" integers first.

$$\begin{array}{r|rrrrr} & 1 & -3 & -5 & 3 & 4 \\ \hline -1 & 1 & -4 & -1 & 4 & 0 \end{array} \quad = P(-1) = 0!!$$

Okay we now know that $x = -1$ is a zero and we can write the polynomial as,

$$P(x) = x^4 - 3x^3 - 5x^2 + 3x + 4 = (x+1)(x^3 - 4x^2 - x + 4)$$

Step 4
So, now we need to continue the process using $Q(x) = x^3 - 4x^2 - x + 4$. The possible zeroes of this polynomial are the same as the original polynomial and so we won't write them back down.

Here's the synthetic division work for this $Q(x)$.

$$\begin{array}{r|rrrr} & 1 & -4 & -1 & 4 \\ \hline -1 & 1 & -5 & 4 & 0 \end{array} \quad = Q(-1) = 0!!$$

Therefore, $x = -1$ is also a zero of $Q(x)$ and the factored form of $Q(x)$ is,

$$Q(x) = x^3 - 4x^2 - x + 4 = (x+1)(x^2 - 5x + 4)$$

This also means that the factored form of the original polynomial is now,

$$P(x) = x^4 - 3x^3 - 5x^2 + 3x + 4 = (x+1)(x+1)(x^2 - 5x + 4) = (x+1)^2(x^2 - 5x + 4)$$

Step 5
We're down to a quadratic polynomial and so we can and we can just factor this to get the fully factored form of the original polynomial. This is,

$$P(x) = x^4 - 3x^3 - 5x^2 + 3x + 4 = (x+1)^2(x-4)(x-1)$$

Step 6
From the fully factored form we get the following set of zeroes for the original polynomial.

$$\boxed{x = -1 \ \text{(multiplicity 2)} \qquad\qquad x = 1 \qquad\qquad x = 4}$$

---

3. Find all the zeroes of the following polynomial.

3e

$$A(x) = 2x^4 - 7x^3 - 2x^2 + 28x - 24$$

Step 1
We'll need all the factors of -24 and 2.

$$\begin{array}{ll} -24: & \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24 \\ 2: & \pm 1, \pm 2 \end{array}$$

Step 2
Here is a list of all possible rational zeroes for the polynomial.

$$\frac{\pm 1}{\pm 1} = \pm 1 \qquad \frac{\pm 2}{\pm 1} = \pm 2 \qquad \frac{\pm 3}{\pm 1} = \pm 3 \qquad \frac{\pm 4}{\pm 1} = \pm 4 \qquad \frac{\pm 6}{\pm 1} = \pm 6 \qquad \frac{\pm 8}{\pm 1} = \pm 8$$

$$\frac{\pm 12}{\pm 1} = \pm 12 \qquad \frac{\pm 24}{\pm 1} = \pm 24$$

$$\frac{\pm 1}{\pm 2} = \frac{\pm 1}{\pm 2} \qquad \frac{\pm 2}{\pm 2} = \pm 1 \qquad \frac{\pm 3}{\pm 2} = \frac{\pm 3}{\pm 2} \qquad \frac{\pm 4}{\pm 2} = \pm 2 \qquad \frac{\pm 6}{\pm 2} = \pm 3 \qquad \frac{\pm 8}{\pm 2} = \pm 4$$

$$\frac{\pm 12}{\pm 2} = \pm 6 \qquad \frac{\pm 24}{\pm 2} = \pm 12$$

So, we have a total of 20 possible zeroes for the polynomial.

Step 3
We now need to start the synthetic division work. We'll start with the "small" integers first.

$$\begin{array}{r|rrrrr}
 & 2 & -7 & -2 & 28 & -24 \\
\hline
-1 & 2 & -9 & 7 & 21 & -45 \\
1 & 2 & -5 & -7 & 21 & -3 \\
-2 & 2 & -11 & 20 & -12 & 0
\end{array}
\begin{array}{l}
\\
= A(-1) \neq 0 \\
= A(1) \neq 0 \\
= A(-2) = 0!!
\end{array}$$

Okay we now know that $x = -2$ is a zero and we can write the polynomial as,

$$A(x) = 2x^4 - 7x^3 - 2x^2 + 28x - 24 = (x+2)(2x^3 - 11x^2 + 20x - 12)$$

Step 4
So, now we need to continue the process using $Q(x) = 2x^3 - 11x^2 + 20x - 12$. Here is a list of all possible zeroes of $Q(x)$.

$$\frac{\pm 1}{\pm 1} = \pm 1 \qquad \frac{\pm 2}{\pm 1} = \pm 2 \qquad \frac{\pm 3}{\pm 1} = \pm 3 \qquad \frac{\pm 4}{\pm 1} = \pm 4 \qquad \frac{\pm 6}{\pm 1} = \pm 6 \qquad \frac{\pm 12}{\pm 1} = \pm 12$$

$$\frac{\pm 1}{\pm 2} = \frac{\pm 1}{\pm 2} \qquad \frac{\pm 2}{\pm 2} = \pm 1 \qquad \frac{\pm 3}{\pm 2} = \frac{\pm 3}{\pm 2} \qquad \frac{\pm 4}{\pm 2} = \pm 2 \qquad \frac{\pm 6}{\pm 2} = \pm 3 \qquad \frac{\pm 12}{\pm 2} = \pm 6$$

So we have a list of 16 possible zeroes, but note that we've already proved that ± 1 can't be zeroes of the original polynomial and so can't be zeroes of $Q(x)$ either.

Here's the synthetic division work for this $Q(x)$.

$$\begin{array}{r|rrrr}
 & 2 & -11 & 20 & -12 \\
\hline
-2 & 2 & -15 & 50 & -112 \\
2 & 2 & -7 & 6 & 0
\end{array}
\begin{array}{l}
\\
= Q(-2) \neq 0 \\
= Q(2) = 0!!
\end{array}$$

Therefore, $x = 2$ is a zero of $Q(x)$ and the factored form of $Q(x)$ is,

$$Q(x) = 2x^3 - 11x^2 + 20x - 12 = (x-2)(2x^2 - 7x + 6)$$

This also means that the factored form of the original polynomial is now,

$$A(x) = 2x^4 - 7x^3 - 2x^2 + 28x - 24 = (x+2)(x-2)(2x^2 - 7x + 6)$$

Step 5

We're down to a quadratic polynomial and so we can and we can just factor this to get the fully factored form of the original polynomial. This is,

$$A(x) = 2x^4 - 7x^3 - 2x^2 + 28x - 24 = (x+2)(x-2)(x-2)(2x-3) = (x+2)(x-2)^2(2x-3)$$

Step 6

From the fully factored form we get the following set of zeroes for the original polynomial.

$$\boxed{x = -2 \qquad\qquad x = \frac{3}{2} \qquad\qquad x = 2 \;\; (\text{multiplicity } 2)}$$

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4. Find all the zeroes of the following polynomial.

32

$$g(x) = 8x^5 + 36x^4 + 46x^3 + 7x^2 - 12x - 4$$

Step 1

We'll need all the factors of -4 and 8.

$$\begin{array}{ll} -4: & \pm 1, \pm 2, \pm 4 \\ 8: & \pm 1, \pm 2, \pm 4, \pm 8 \end{array}$$

Step 2

Here is a list of all possible rational zeroes for the polynomial.

$$\begin{array}{lll} \frac{\pm 1}{\pm 1} = \pm 1 & \frac{\pm 2}{\pm 1} = \pm 2 & \frac{\pm 4}{\pm 1} = \pm 4 \\[2ex] \frac{\pm 1}{\pm 2} = \frac{\pm 1}{\pm 2} & \frac{\pm 2}{\pm 2} = \pm 1 & \frac{\pm 4}{\pm 2} = \pm 2 \\[2ex] \frac{\pm 1}{\pm 4} = \frac{\pm 1}{\pm 4} & \frac{\pm 2}{\pm 4} = \frac{\pm 1}{\pm 2} & \frac{\pm 4}{\pm 4} = \pm 1 \\[2ex] \frac{\pm 1}{\pm 8} = \frac{\pm 1}{\pm 8} & \frac{\pm 2}{\pm 8} = \frac{\pm 1}{\pm 4} & \frac{\pm 4}{\pm 8} = \frac{\pm 1}{\pm 2} \end{array}$$

So, we have a total of 12 possible zeroes for the polynomial.

Step 3

We now need to start the synthetic division work. We'll start with the "small" integers first.

$$\begin{array}{r|rrrrrrl} & 8 & 36 & 46 & 7 & -12 & -4 & \\ \hline -1 & 8 & 28 & 18 & -11 & -1 & -3 & = g(-1) \neq 0 \\ 1 & 8 & 44 & 90 & 97 & 85 & 81 & = g(1) \neq 0 \end{array}$$

Okay, notice that we have opposite signs for the two function evaluations listed above. Recall that means that we know we have a zero somewhere between them.

So, let's take a look at some of the fractions from our list and give them a try in the synthetic division table. We'll start with the fractions with the smallest denominators.

$$\begin{array}{r|rrrrrrl} & 8 & 36 & 46 & 7 & -12 & -4 & \\ \hline -\dfrac{1}{2} & 8 & 32 & 30 & -8 & -8 & 0 & = g(-1) = 0!! \end{array}$$

We now know that $x = -\frac{1}{2}$ is a zero and we can write the polynomial as,

$$g(x) = 8x^5 + 36x^4 + 46x^3 + 7x^2 - 12x - 4 = \left(x + \tfrac{1}{2}\right)\left(8x^4 + 32x^3 + 30x^2 - 8x - 8\right)$$

Step 4

So, now we need to continue the process using $Q(x) = 8x^4 + 32x^3 + 30x^2 - 8x - 8$. Here is a list of all possible zeroes of $Q(x)$.

$$\begin{array}{llll} \dfrac{\pm 1}{\pm 1} = \pm 1 & \dfrac{\pm 2}{\pm 1} = \pm 2 & \dfrac{\pm 4}{\pm 1} = \pm 4 & \dfrac{\pm 8}{\pm 1} = \pm 8 \\[2ex] \dfrac{\pm 1}{\pm 2} = \dfrac{\pm 1}{\pm 2} & \dfrac{\pm 2}{\pm 2} = \pm 1 & \dfrac{\pm 4}{\pm 2} = \pm 2 & \dfrac{\pm 8}{\pm 2} = \pm 4 \\[2ex] \dfrac{\pm 1}{\pm 4} = \dfrac{\pm 1}{\pm 4} & \dfrac{\pm 2}{\pm 4} = \dfrac{\pm 1}{\pm 2} & \dfrac{\pm 4}{\pm 4} = \pm 1 & \dfrac{\pm 8}{\pm 4} = \pm 2 \\[2ex] \dfrac{\pm 1}{\pm 8} = \dfrac{\pm 1}{\pm 8} & \dfrac{\pm 2}{\pm 8} = \dfrac{\pm 1}{\pm 4} & \dfrac{\pm 4}{\pm 8} = \dfrac{\pm 1}{\pm 2} & \dfrac{\pm 8}{\pm 8} = \pm 1 \end{array}$$

So we have a list of 14 possible zeroes (lots of repeats), but note that we've already proved that ± 1 can't be zeroes of the original polynomial and so can't be zeroes of $Q(x)$ either.

Here's the synthetic division work for this $Q(x)$.

$$\begin{array}{r|rrrrr} & 8 & 32 & 30 & -8 & -8 \\ \hline -2 & 8 & 16 & -2 & -4 & 0 \end{array} = Q(-2) = 0!!$$

Therefore, $x = -2$ is a zero of $Q(x)$ and the factored form of $Q(x)$ is,

$$Q(x) = 8x^4 + 32x^3 + 30x^2 - 8x - 8 = (x+2)\left(8x^3 + 16x^2 - 2x - 4\right)$$

This also means that the factored form of the original polynomial is now,

$$g(x) = 8x^5 + 36x^4 + 46x^3 + 7x^2 - 12x - 4 = \left(x + \tfrac{1}{2}\right)(x+2)\left(8x^3 + 16x^2 - 2x - 4\right)$$

Step 5

So, it looks like we need to continue with the synthetic division. This time we'll do it on the polynomial $P(x) = 8x^3 + 16x^2 - 2x - 4$.

The possible zeroes of this are the same as the original polynomial and so we won't write them down here. Again, however, we've already proved that ± 1 can't be zeroes of the original polynomial and so can't be zeroes of $P(x)$ either.

Here is the synthetic division for $P(x)$.

$$\begin{array}{r|rrrr} & 8 & 16 & -2 & -4 \\ \hline -2 & 8 & 0 & -2 & 0 \end{array} = P(-2) = 0!!$$

Therefore, $x = -2$ is a zero of $P(x)$ and the factored form of $P(x)$ is,

$$P(x) = 8x^3 + 16x^2 - 2x - 4 = (x+2)\left(8x^2 - 2\right) = 8(x+2)\left(x^2 - \tfrac{1}{4}\right)$$

We factored an 8 out of the quadratic to make it a little easier for the next step.

The factored form of the original polynomial is now,

$$g(x) = 8x^5 + 36x^4 + 46x^3 + 7x^2 - 12x - 4 = 8\left(x + \tfrac{1}{2}\right)(x+2)^2\left(x^2 - \tfrac{1}{4}\right)$$

Step 6

We're down to a quadratic polynomial and so we can and we can just factor this to get the fully factored form of the original polynomial. This is, 32

$$g(x) = 8x^5 + 36x^4 + 46x^3 + 7x^2 - 12x - 4 = 8\left(x+\tfrac{1}{2}\right)\left(x+2\right)^2\left(x+\tfrac{1}{2}\right)\left(x-\tfrac{1}{2}\right)$$
$$= 8\left(x+\tfrac{1}{2}\right)^2\left(x+2\right)^2\left(x-\tfrac{1}{2}\right)$$

Step 7
From the fully factored form we get the following set of zeroes for the original polynomial.

$$\boxed{x = -\frac{1}{2}\ \text{(multiplicity 2)} \qquad x = -2\ \text{(multiplicity 2)} \qquad x = \frac{1}{2}}$$

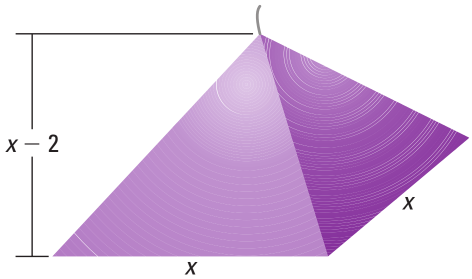
Note that this problem was VERY long and messy. The point of this problem was really just to illustrate just how long and messy these can get. The moral, if there is one, is that we generally sit back and really hope that we don't have to work these kinds of problems on a regular basis. They are just too long and it's too easy to make a mistake with them.

GOAL 2 SOLVING POLYNOMIAL EQUATIONS IN REAL LIFE

REAL LIFE Crafts

EXAMPLE 3 *Writing and Using a Polynomial Model*

You are designing a candle-making kit. Each kit will contain 25 cubic inches of candle wax and a mold for making a model of the pyramid-shaped building at the Louvre Museum in Paris, France. You want the height of the candle to be 2 inches less than the length of each side of the candle's square base. What should the dimensions of your candle mold be?



SOLUTION

The volume is $V = \frac{1}{3}Bh$ where B is the area of the base and h is the height.

PROBLEM SOLVING STRATEGY

VERBAL MODEL $\boxed{\text{Volume}} = \frac{1}{3} \cdot$ **Area of base** $\cdot$ **Height**

LABELS

Volume = **25**	(cubic inches)
Side of square base = $\boldsymbol{x}$	(inches)
Area of base = $\boldsymbol{x^2}$	(square inches)
Height = $\boldsymbol{x - 2}$	(inches)

ALGEBRAIC MODEL

$\mathbf{25} = \frac{1}{3}\boldsymbol{x^2}\,(\boldsymbol{x - 2})$ **Write algebraic model.**

$75 = x^3 - 2x^2$ **Multiply each side by 3 and simplify.**

$0 = x^3 - 2x^2 - 75$ **Subtract 75 from each side.**

The possible rational solutions are $x = \pm\frac{1}{1}, \pm\frac{3}{1}, \pm\frac{5}{1}, \pm\frac{15}{1}, \pm\frac{25}{1}, \pm\frac{75}{1}$.

Use the possible solutions. Note that in this case, it makes sense to test only positive x-values.

$$\begin{array}{r|rrrr} 1 & 1 & -2 & 0 & -75 \\ & & 1 & -1 & -1 \\ \hline & 1 & -1 & -1 & -76 \end{array} \qquad \begin{array}{r|rrrr} 3 & 1 & -2 & 0 & -75 \\ & & 3 & 3 & 9 \\ \hline & 1 & 1 & 3 & -66 \end{array}$$

$$\begin{array}{r|rrrr} 5 & 1 & -2 & 0 & -75 \\ & & 5 & 15 & 75 \\ \hline & 1 & 3 & 15 & 0 \end{array}$$ ← **5 is a solution.**

So $x = 5$ is a solution. The other two solutions, which satisfy $x^2 + 3x + 15 = 0$, are $x = \frac{-3 \pm i\sqrt{51}}{2}$ and can be discarded because they are imaginary.

▶ The base of the candle mold should be 5 inches by 5 inches. The height of the mold should be $5 - 2 = 3$ inches.

FOCUS ON PEOPLE

REAL LIFE **I.M. PEI** designed the pyramid at the Louvre. His geometric architecture can be seen in Boston, New York, Dallas, Los Angeles, Taiwan, Beijing, and Singapore.