(1) $y' = \frac{2\sqrt{y}}{x}$ $\qquad g(y) = 2\sqrt{y} \quad h(x) = \frac{1}{x}$

- $y_0 \equiv 0 \qquad x \in \underbrace{(-\infty, 0)}_{I_1}, \underbrace{(0, \infty)}_{I_2}$

- $y \in (0, \infty) = J$

- $\int \frac{1}{2\sqrt{y}}\, dy = \int \frac{1}{x}\, dx$

$$\underbrace{\sqrt{y}}_{G(y)} = \underbrace{\log|x| + c}_{H(x)+c}$$

- $y \in J \quad x \in I_1$

$G(J) = (0, \infty)$

$$\log|x| + c > 0$$
$$\log(-x) > -c$$
$$-x > e^{-c}$$
$$\boxed{-e^{-c} > x}$$

$y_1 = (\log|x| + c)^2$

- $y \in J \quad x \in I_2$

$G(J) = (0, \infty)$

$$\log x + c > 0$$
$$\log x > -c$$
$$\boxed{x > e^{-c}}$$

$y_2 = (\log|x| + c)^2$

- Resumé

$y_0 \equiv 0 \quad x \in (-\infty, 0)$

$\overline{y_0} \equiv 0 \quad x \in (0, \infty)$

$y_1 = (\log|x| + c)^2 \quad x \in (-\infty, -e^{-c})$

$y_2 = (\log|x| + c)^2 \quad x \in (e^{-c}, \infty)$

$$\lim_{x \to -e^{-c}} y_1 = \lim_{x \to -e^{-c}} (\log|x| + c)^2 = (\log e^{-c} + c)^2 = 0$$

analogicky

$$\lim_{x \to e^{-c}} y_2 = 0$$

lze slepit z věty o lepení (g spoj. na I_1, h spoj. na I_1, I_2)

<u>Závěr:</u>

$$y_0 \equiv 0 \qquad x \in (-\infty, 0)$$

$$y_0 \equiv 0 \qquad x \in (0, \infty)$$

$$y_3 = \begin{cases} 0 & x \in (0, e^{-c}] \\ (\log|x| + c)^2 & x \in (e^{-c}, \infty) \end{cases}$$

$$y_4 = \begin{cases} (\log|x| + c)^2 & x \in (-\infty, -e^{-c}) \\ 0 & x \in [-e^{-c}, \infty) \end{cases}$$

(2) $y' - \frac{y}{x+2} = (x+2)^2$ $\qquad y(0) = 12$

$\frac{-1}{x+2}$, $(x+2)^2$ spoj.

na $(-\infty,-2)$, $(-2,\infty)$

- $y' = \frac{y}{x+2}$

$$\int \frac{1}{y}\, dy = \int \frac{1}{x+2}\, dx$$

$$\log|y| = \log|x+2| + c$$

$$|y| = |x+2|\, e^c$$

$$\underline{y = k(x+2)} \qquad k \in \mathbb{R}$$

- $y' = k'(x+2) + k$

$$k'(x+2) + k - \frac{k(x+2)}{x+2} = (x+2)^2$$

$$k' = x+2$$

$$k = \frac{x^2}{2} + 2x + d$$

$$y = \underline{\left(\frac{x^2}{2} + 2x + d\right)}(x+2) \qquad x \in \underline{(-\infty,-2), (-2,\infty)}$$

$y(0) = 12 \rightarrow$ $2d = 12$, $d = 6$ $\qquad \underline{y = \left(\frac{x^2}{2} + 2x + 6\right)(x+2) \quad x \in (-2,\infty)}$

(3) $y'' - 4y = -8e^{2x}$

- $\lambda^2 - 4 = 0$

$\lambda = \pm 2$ $\qquad \underline{y_H = c_1 e^{2x} + c_2 e^{-2x}}$ $\qquad c_{1,2}, x \in \mathbb{R}$

- $-8e^{2x} = e^{2x}(-8\cos 0x + 0 \sin 0x)$ $\qquad$ $2+0i$ je 1-nás. kořen

$$y_p = Ae^{2x} x^1$$

$$y_p' = Ae^{2x} + xA2e^{2x} = Ae^{2x}(2x+1)$$

$$y_p'' = 2Ae^{2x}(2x+1) + 2Ae^{2x}$$

dosazení

$$2Ae^{2x}(2x+2) - 4Ae^{2x}x = -8e^{2x}$$

$$4A\cancel{x}e^{2x} - 4A\cancel{x}e^{2x} + 4Ae^{2x} = -8e^{2x}$$

$$A = -2$$

<u>Závěr</u> $y = c_1e^{2x} + c_2e^{-2x} - 2xe^{2x}$ $\quad x, c_1, c_2 \in \mathbb{R}$

(4) (a) f je lipschitzovská $\Rightarrow$ f je spoj.

chceme: f je spoj: $\forall a \in X \ \forall \varepsilon > 0 \ \exists \delta \ \forall x \in B(a,\delta) \quad f(x) \in B(f(a),\varepsilon)$

máme: $\exists k > 0 \ \forall x,y \in X \quad |f(x) - f(y)| < k\,\rho(x,y)$.

zvolme $a \in X$ a $\varepsilon > 0$. Položme $\delta = \frac{\varepsilon}{k}$.

Pak pro $x \in B(a,\delta)$: $\rho(x,a) < \delta = \frac{\varepsilon}{k}$ máme

$$|f(x) - f(a)| < k\,\rho(x,a) < k \cdot \frac{\varepsilon}{k} = \varepsilon$$

neboli $f(x) \in B(f(a),\varepsilon)$

Platí.

(b) f spoj $\Rightarrow$ f lipschitzovská

Neplatí. Uvažujme $f = \sqrt{x}$ na $X = [0,\infty)$

položme $x = 0$ a $y_n = \frac{1}{n}$ $n \in \mathbb{N}$

Pak $|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = |0 - \sqrt{\tfrac{1}{n}}| = \sqrt{\frac{1}{n}} = \frac{\sqrt{n}}{n} = \sqrt{n}\,|x-y|$

Tedy pro $\forall k > 0$ najdeme y_n $(n > k^2)$ taž, že

$$|f(x) - f(y)| = \sqrt{n}\,\rho(x,y) > k\,\rho(x,y).$$

Tedy f není lipschitzovská.