11. cvičení – Fourierova transformace

https://www2.karlin.mff.cuni.cz/~kuncova/vyuka.php, kuncova@karlin.mff.cuni.cz

Příklady

1. Určete Fourierovu transformaci následujících funkcí

   (a)
$$f(t) = \begin{cases} 0; & t < 0 \\ \frac{1}{2}; & t = 0; \\ e^{-t}; & t > 0, \end{cases}$$

Řešení: Protože hodnota v bodě $t = 0$ nemá na integrál vliv, můžeme psát

$$\sqrt{2\pi}\ F(u) = \int_{-\infty}^{\infty} f(t)e^{-iut}\,dt = \int_{0}^{\infty} e^{-t}e^{-iut}\,dt = \int_{0}^{\infty} e^{-(1+iu)t}\,dt = \left[\frac{e^{-(1+iu)t}}{-(1+iu)}\right]_0^{\infty}$$
$$= \frac{1}{1+iu}.$$

Tedy Fourierova transformace funkce $f(t)$ je

$$F(u) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{1+iu}.$$

   (b) $f(t) = \begin{cases} a+t, & t \in (-a, 0), \\ 0, & \text{jinak}, \end{cases} \qquad a > 0$

Řešení: Protože $f(t) = 0$ mimo $(-a, 0)$, píšeme

$$\begin{aligned} \sqrt{2\pi}F(u) &= \int_{-a}^{0} (a+t)e^{-iut}\,dt = a\int_{-a}^{0} e^{-iut}\,dt + \int_{-a}^{0} te^{-iut}\,dt \\ &= a\left[\frac{e^{-iut}}{-iu}\right]_{-a}^{0} + \left[t\frac{e^{-iut}}{-iu}\right]_{-a}^{0} + \frac{1}{iu}\int_{-a}^{0} e^{-iut}\,dt \\ &= a\left[\frac{e^{-iut}}{-iu}\right]_{-a}^{0} + \left[t\frac{e^{-iut}}{-iu}\right]_{-a}^{0} + \left[\frac{e^{-iut}}{-(iu)^2}\right]_{-a}^{0} \\ &= \frac{a(1-e^{iua})}{-iu} + \frac{ae^{iua}}{-iu} + \frac{1-e^{iua}}{-(iu)^2} = \frac{aiu+1-e^{iau}}{u^2} \end{aligned}$$

Tedy $F(u) = \frac{aiu+1-e^{iau}}{\sqrt{2\pi}u^2}$.
Pro $u = 0$ spočteme zvlášť:

$$\sqrt{2\pi}F(0) = \int_{-a}^{0} (a+t) = \left[at + \frac{1}{2}t^2\right]_{-a}^{0} = \frac{a^2}{2}$$

   (c) $f(t) = \begin{cases} \sin t, & t \in (0, \pi), \\ 0, & \text{jinak}, \end{cases}$

Řešení: Protože $f(t)=0$ mimo $(0,\pi)$, tedy máme

$$\begin{aligned}
\sqrt{2\pi}F(u) &= \int_0^\pi \sin t\, e^{-iut}\,dt = \frac{1}{2i}\int_0^\pi \big(e^{it}-e^{-it}\big)e^{-iut}\,dt\\
&= \frac{1}{2i}\int_0^\pi \left(e^{i(1-u)t}-e^{-i(1+u)t}\right)dt\\
&= \frac{1}{2i}\left[\frac{e^{i(1-u)t}}{i(1-u)}+\frac{e^{-i(1+u)t}}{i(1+u)}\right]_0^\pi\\
&= -\frac{1}{2}\left[\frac{e^{i(1-u)t}}{1-u}+\frac{e^{-i(1+u)t}}{1+u}\right]_0^\pi\\
&= -\frac{1}{2}\left(\frac{e^{i(1-u)\pi}-1}{1-u}+\frac{e^{-i(1+u)\pi}-1}{1+u}\right)
\end{aligned}$$

Dále využijeme

$$e^{i(1-u)\pi}=-e^{-iu\pi},\qquad e^{-i(1+u)\pi}=-e^{-iu\pi}.$$

$$\begin{aligned}
\sqrt{2\pi}F(u) &= -\frac{1}{2}\left(\frac{-e^{-iu\pi}-1}{1-u}+\frac{-e^{-iu\pi}-1}{1+u}\right)\\
&= \frac{1}{2}(e^{-iu\pi}+1)\left(\frac{1}{1-u}+\frac{1}{1+u}\right)\\
&= \frac{1}{2}(e^{-iu\pi}+1)\frac{2}{1-u^2}\\
&= \frac{1+e^{-iu\pi}}{1-u^2}.
\end{aligned}$$

Tedy

$$\sqrt{2\pi}F(u)=\frac{1+e^{-iu\pi}}{1-u^2}.$$

Nyní zvlášť spočítáme hodnoty v $u=1$ a $u=-1$.
Pro $u=1$

$$\sqrt{2\pi}F(1)=\int_0^\pi \sin t\, e^{-it}\,dt.$$

Protože $e^{-it}=\cos t - i\sin t$,

$$\sin t\, e^{-it}=\sin t\cos t - i\sin^2 t.$$

Dále

$$\int_0^\pi \sin t\cos t\,dt=\frac{1}{2}\int_0^\pi \sin 2t\,dt=0,$$

$$\int_0^\pi \sin^2 t\,dt=\frac{\pi}{2}.$$

Tedy
$$\sqrt{2\pi}F(1) = 0 - i\frac{\pi}{2} = -\frac{i\pi}{2}.$$
Pro $u = -1$
$$\sqrt{2\pi}F(-1) = \int_0^\pi \sin t\, e^{it}\, dt.$$
Protože $e^{it} = \cos t + i \sin t$,
$$\sin t\, e^{it} = \sin t \cos t + i \sin^2 t.$$
První integrál je opět nulový, druhý dává $\pi/2$, takže
$$\sqrt{2\pi}F(-1) = i\frac{\pi}{2}.$$

(d)
$$f(t) = \chi_{(0,1)}(t)$$

Řešení:
$$\begin{aligned}\sqrt{2\pi}F(u) &= \int_{-\infty}^{\infty} \chi_{(0,1)}(t)\, e^{-iut}\, dt \\ &= \int_0^1 e^{-iut}\, dt \\ &= \left[\frac{e^{-iut}}{-iu}\right]_0^1 \\ &= \frac{1-e^{-iu}}{iu}.\end{aligned}$$
Pro $u = 0$ máme
$$\sqrt{2\pi}F(0) = \int_{-\infty}^{\infty} \chi_{(0,1)}(t)\, \mathrm{d}t = 1$$

(e)
$$f(t) = \chi_{(4,6)}(t)$$

Řešení:
$$\begin{aligned}\chi_{(4,6)}(u) &= \chi_{(0,2)}(u-4) \\ \sqrt{2\pi}F(u) &= e^{-4iu}\widehat{\chi_{(0,2)}(t)} \\ &= e^{-4iu}\widehat{\chi_{(0,1)}(\frac{1}{2}t)} \\ &= e^{-4iu}2\widehat{\chi_{(0,1)}}(2u) \\ &= e^{-4iu}\frac{1-e^{-2iu}}{iu}.\end{aligned}$$
Pro $u = 0$ z definice
$$\sqrt{2\pi}F(0) = 2.$$

(f)
$$f(t) = t\chi_{(-1,1)}(t)$$

Řešení:

$$\begin{aligned}
\sqrt{2\pi}F(u) &= \int_{-\infty}^{\infty} t\,\chi_{(-1,1)}(t)\,e^{-iut}\,dt \\
&= \int_{-1}^{1} t\,e^{-iut}\,dt \\
&= \left[\frac{t\,e^{-iut}}{-iu}\right]_{-1}^{1} - \int_{-1}^{1} \frac{1}{-iu}\,e^{-iut}\,dt \\
&= \frac{1\cdot e^{-iu} - (-1)\cdot e^{iu}}{-iu} + \frac{1}{iu}\int_{-1}^{1} e^{-iut}\,dt \\
&= \frac{e^{-iu} + e^{iu}}{-iu} + \frac{1}{iu}\left[\frac{e^{-iut}}{-iu}\right]_{-1}^{1} \\
&= -\frac{2\cos u}{iu} + \frac{e^{-iu} - e^{iu}}{-(iu)^2} \\
&= -\frac{2\cos u}{iu} + \frac{-2i\sin u}{-(iu)^2} \\
&= -\frac{2\cos u}{iu} + \frac{2i\sin u}{(i^2u^2)} \\
&= -\frac{2\cos u}{iu} + \frac{2\sin u}{u^2}.
\end{aligned}$$

Pro $u = 0$ máme

$$\sqrt{2\pi}F(0) = \int_{-1}^{1} t\,\mathrm{d}t = 0$$

(g) $f(t) = e^t$
Řešení: Funkce $f \notin L^1$. Když zkusíme spočítat Fourierovu transformaci, tak dostaneme

$$\sqrt{2\pi}F(u) = \int_{-\infty}^{\infty} e^t e^{-iut}\,dt = \int_{-\infty}^{\infty} e^{-iut+t}\,dt = \left[\frac{e^{(1-iu)t}}{1-iu}\right]_{-\infty}^{\infty} = \infty.$$

Tedy Fourierova transformace není definována.

(h) $f(t) = \begin{cases} 0, & t < 0, \\ e^{-at}, & \text{jinak}, \end{cases} \qquad a > 0$

Řešení: $\sqrt{2\pi}F(u) = \int_0^{\infty} e^{-at}e^{-iut}\,dt = \int_0^{\infty} e^{-(a+iu)t}\,dt = \left[\frac{e^{-(a+iu)t}}{-(a+iu)}\right]_0^{\infty} = \frac{1}{a+iu}.$

(i)
$$f(t) = \begin{cases} 0, & |t| \geq \frac{\pi}{2}, \\ \cos t, & |t| < \frac{\pi}{2}, \end{cases}$$

Řešení:

$$
\begin{aligned}
\sqrt{2\pi}F(u) &= \int_{-\pi/2}^{\pi/2} \cos t\, e^{-iut}\, dt \\
&= \int_{-\pi/2}^{\pi/2} \frac{e^{it}+e^{-it}}{2}\, e^{-iut}\, dt \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(e^{-i(u-1)t} + e^{-i(u+1)t}\right) dt \\
&= \frac{1}{2}\left[\frac{e^{-i(u-1)t}}{-i(u-1)} + \frac{e^{-i(u+1)t}}{-i(u+1)}\right]_{-\pi/2}^{\pi/2} \\
&= \frac{\sin((u-1)\pi/2)}{u-1} + \frac{\sin((u+1)\pi/2)}{u+1} \\
&= \frac{-2\cos(\pi u/2)}{u^2-1}.
\end{aligned}
$$

Pro $u = 1$ máme

$$
\begin{aligned}
\sqrt{2\pi}F(1) = \int_{-\pi/2}^{\pi/2} \cos t\, e^{-it}\, dt &= \int_{-\pi/2}^{\pi/2} \frac{e^{it}+e^{-it}}{2} e^{-it}\, dt \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(e^{-it} + e^{-2it}\right) dt \\
&= \frac{1}{2}\left[\frac{e^{-it}}{-i} + \frac{e^{-2it}}{-2i}\right]_{-\pi/2}^{\pi/2} \\
&= \frac{1}{2}\left(\frac{e^{-i\pi/2} - e^{i\pi/2}}{-i} + \frac{e^{-i\pi} - e^{i\pi}}{-2i}\right) \\
&= \frac{1}{2}\left(\frac{-i-i}{-i} + 0\right) \\
&= 1.
\end{aligned}
$$

Pro $u = -1$ máme

$$
\begin{aligned}
\sqrt{2\pi}F(-1) = \int_{-\pi/2}^{\pi/2} \cos t\, e^{it}\, dt &= \int_{-\pi/2}^{\pi/2} \frac{e^{it}+e^{-it}}{2}\, e^{it}\, dt \\
&= \frac{1}{2}\int_{-\pi/2}^{\pi/2} \left(e^{2it} + 1\right) dt \\
&= \frac{1}{2}\left[\frac{e^{2it}}{2i} + t\right]_{-\pi/2}^{\pi/2} \\
&= \frac{1}{2}\left(\frac{e^{i\pi} - e^{-i\pi}}{2i} + \pi/2 - (-\pi/2)\right) \\
&= \frac{1}{2}(0+\pi) \\
&= \frac{\pi}{2}.
\end{aligned}
$$

(j)

$$f(t) = e^{-t^2}$$

Řešení:

$$\begin{aligned}
\sqrt{2\pi}F(u) &= \int_{-\infty}^{\infty} e^{-t^2} e^{-iut}\,dt \\
&= \int_{-\infty}^{\infty} e^{-t^2-iut}\,dt \\
&= \int_{-\infty}^{\infty} e^{-(t+iu/2)^2} e^{-u^2/4}\,dt \\
&= e^{-u^2/4} \int_{-\infty}^{\infty} e^{-(t+iu/2)^2}\,dt \\
&= e^{-u^2/4} \int_{-\infty}^{\infty} e^{-s^2}\,ds \quad (s = t + iu/2) \\
&= \sqrt{\pi}\, e^{-u^2/4}.
\end{aligned}$$

2. Nechť $\mathcal{F}[f(t)] = F(u)$ s definicí

$$\mathcal{F}[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iut}\,dt.$$

Dokažte následující vlastnosti:

(a) $\mathcal{F}[f(at)] = \frac{1}{a} F\left(\frac{u}{a}\right), \quad a > 0$
Řešení:

$$\begin{aligned}
\mathcal{F}[f(at)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(at) e^{-iut}\,dt \\
&\text{(substituce } s = at,\ ds = a\,dt) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) e^{-i(u/a)s} \frac{1}{a} ds \\
&= \frac{1}{a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) e^{-i(u/a)s}\,ds \\
&= \frac{1}{a} F\left(\frac{u}{a}\right).
\end{aligned}$$

(b) $\mathcal{F}[f(-t)] = F(-u)$
Řešení:

$$\begin{aligned}
\mathcal{F}[f(-t)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-t) e^{-iut}\,dt \\
&\text{(substituce } s = -t,\ ds = -dt) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) e^{ius}\,ds \\
&= F(-u).
\end{aligned}$$

(c) Pokud $f(t)$ je sudá ($f(-t) = f(t)$), pak pomocí substituce $s = -t$

$$\begin{aligned} F(u) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iut} dt = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} -f(-s) e^{ius} ds \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) e^{ius} ds = F(-u), \end{aligned}$$

tedy $F(u)$ je sudá.
Pokud $f(t)$ je lichá ($f(-t) = -f(t)$), pak analogicky

$$\begin{aligned} F(u) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iut} dt = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} -f(-s) e^{ius} ds \\ &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) e^{ius} ds = -F(-u), \end{aligned}$$

tedy $F(u)$ je lichá.

(d) $\mathcal{F}[f(t-a)] = e^{-iau} F(u)$
Řešení:

$$\begin{aligned} \mathcal{F}[f(t-a)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t-a) e^{-iut}\, dt \\ &\text{(substituce } s = t-a,\ ds = dt) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) e^{-iu(s+a)}\, ds \\ &= e^{-iua} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s) e^{-ius}\, ds \\ &= e^{-iua} F(u). \end{aligned}$$